A short proof of Fermat's Last Theorem based on the difference in volume of two cubes

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Abstract

Over the centuries, numerous mathematicians have tried to proof Fermat's Last Theorem. In the year 1994, Fermat's Last Theorem in the form of $a^m + b^m \neq c^m$ with a, b and c being natural numbers and m being a natural number > 2 was shown to be correct In this publication I demonstrate that the difference in volume of two cubes having different side lengths cannot be a cube in itself with a side length having the value of a natural number.This also holds for cubes having higher dimensions than three, since the surfaces of these cubes all consist of three-dimensional cubes,

Proof for Three-Dimensional Cubes

In the year 1994. the equation

$$a^m + b^m = c^m \quad \textbf{(A1a)} \text{ or}$$

was proven not to have a solution for m > 2 and element of naturals, when a, b and c are all natural numbers, i.e.

$$a^m + b^m \neq c^m \quad \textbf{(A1b)}$$

(Fermat's Last Theorem), on over 90 pages. It is known that Fermat himself envisaged a short proof which, however, has never been found in his records.

In the following, I present a short proof of his last theorem based on the difference in volume of two cubes having different side lengths

We rearrange **(A1a) to**

$$a^m - c^m = b^m \quad \textbf{(A2a)}$$

and show

$$a^m - c^m \neq b^m \quad \textbf{(A2b)}$$

with $m > 2$ being an element of naturals and also a, b and c being all natural numbers,

With a and $c>a$ being naturals **(I)** in equation **(A2a),** the following can be defined:

$a^3 = A$ (volume of a cube with a being the side length) and
$c^3 = C$ (volume of a larger cube with c being the side length)

Since $c > a$, c can be expressed as follows (see Fig. 1 below):

$c = a + x$, **(II)** with $x < c$, namely $x = c – a$ and element of the naturals.

Then we get

$c^3 = (a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3 = C$ **(III)** and
$\sqrt[3]{C} = c = \sqrt[3]{(a^3 + 3a^2x + 3ax^2 + x^3)}$,

and furthermore

$c^3 – a^3 = B$ **(IV),**

wherein B is the difference of the volumes of cube C and cube A.

We then define

$\sqrt[3]{B} = b$, with b being the side of cube B,

and thus

$b^3 = B$

We then can write:

$c^3 – a^3 = b^3 = 3a^2x + 3ax^2 + x^3$ **(V),** which follows from **(III)** and **(IV).**

Obviously,:

$b^3 > x^3$ and $b > x$ (see also Fig. 1 below)

Accordingly

$b – x > 0$

We now define

$b - x = y$ and thus
$b = x + y$ **(VI)**, wherein y is at least a positive real number..

Accordingly,

$b^3 = B = (x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3$ **(VII)**

On the other hand,

$b^3 = 3a^2x + 3ax^2 + x^3$ **(V)**

We now can set up the equation **(VII) = (V)**

$x^3 + 3x^2y + 3x y^2 + y^3 = 3a^2x + 3ax^2 + x^3$ **(VIII)**

and examine, if $b = x + y$ can be a natural.

If b is to be a natural, also y has to be a natural, since x according to **(II)** is a natural.

Conversion of **(VIII)** delivers:

$3x^2y + 3x y^2 + y^3 = 3a^2x + 3ax^2$
$y^3 + 3x^2y + 3xy^2 - 3x^2a - 3xa^2 = 0$ **(IX)**;

This is a polynomial of third degree, which is notoriously difficult to solve.

However, **(IX)** is also a quadratic equation of x, which is considerably easier to solve than a polynomial of third degree.

(IX) solved for x gives:

$(3a-3y)x^2 + (3a^2-3y^2)x - y^3 = 0$ **(X)**
$(a-y)x^2 + (a^2-y^2)x - y^3/3 = 0$
$x_1, x_2 = [-(a^2-y^2) +/- \sqrt{((a^2-y^2)^2 - 4(a-y)(-y^3/3))}]/2(a-y)$ **(XI)**
$= [-(a^2-y^2) +/- \sqrt{(a^4 - 2a^2y^2 + y^4 - 4(a-y)(-y^3/3))}]/2(a-y)$
$= [-(a^2-y^2) +/- \sqrt{(a^4 - 2a^2y^2 + y^4 - 4y^4/3 + 4ay^3/3)}]/2(a-y)$

or further converted

$= [-(a^2-y^2) +/- \sqrt{(a^4 + y^4 - 4y^4/3 + 4ay^3/3 - 2a^2y^2)}/ 2(a-y)$ **(XII)**
$= [-(a^2-y^2) +/- \sqrt{(a^4 - y^2/3(y^2 - 4ay + 6a^2))}/ 2(a-y)$ **(XIII)**

x in **(XI)** und **(XII)** is expressed as a function of y, which according to **(II)** has to deliver x as a natural, if equation **(VIII)** were to yield $x+y = b$ with b being the side of a cube as a natural. This means that the function for x may in no case contain an irrational or complex number, and more specifically that the expression under the square root as a whole may not yield an irrational or complex number, nor y as an Irrational or complex number.

The total expression under the square root can be natural or rational only in three instances:

1) If the total expression could be converted to $((s^2a^2-t^2y^2)^2$ with s and t being optional fractions; this is obviously not the case.

2) If the total expression under the square root could be converted to $(sa+ty)^4$, with s and t having the same meanings as above. This, too, is obviously not the case, since $y^4/3$ is not the fourth potency of a natural or rational number.

3) If the expression $-y^2/3(y^2 + 4ay-6a^2)$ were set to zero, since then only $\sqrt{a^4}$ remains. This can be done in two ways. One is to set y to 0, but this contradicts prerequisite **(VI).** The other one is to set

$(y^2 - 4ay + 6a^2) = 0$

With this we get

$y_{1,2} = [4a +/- \sqrt{(16a^2 – 24a^2)}]/2 =2a +/- a\sqrt{-2}$

Thus, the square root yields an natural, but at the cost of y being a complex number. If this solution for y is put into

$x_{1,2} = [-(a^2-y^2) +/- \sqrt{(a^4 -y^2/3(y^2 - 4ay + 6a^2)}]/ 2(a-y)$ (**XIII**)

we get:

$x_{1,2} = [-(a^2-y^2) +/- \sqrt{a^4}]/2(a-y)$,

and with

$y_{1,2} = 2a +/- a\sqrt{-2}$

we get

$x_{1,2} = [-(a+2a +/- a\sqrt{-2})(a-(2a +/- a\sqrt{-2})) +/- a^2]/2(a-2a +/- a\sqrt{-2})$
$= -(a+2a +/- a\sqrt{-2})/2 +/- a^2/2(a-2a +/- a\sqrt{-2})$
$= -a(1+2 +/- \sqrt{-2})/2 +/- a^2/2a(1-2 +/- \sqrt{-2})$
$= (-a/2)(1+2 +/- \sqrt{-2}) +/- a/[2(1-2 +/- \sqrt{-2})]$
$= (-a/2)(3 +/- \sqrt{-2}) +/- a/[2(-1 +/- \sqrt{-2})]$

= -3a/2 +/- a√-2/2) +/- a/[-2 +/- 2√-2]
= -3a/2 +/- (a√2)i/2) +/- a/[-2 +/- (2√2)i]

Thus, $x_{1,2}$ are also complex numbers, when $y_{1,2}$ are complex numbers.

According to the above, we showed that there is no solution for b in **(VI)**, wherein b = x + y **(VI)** is a natural, and accordingly there is no solution for **(A1a)** or **(A2a)**, in which m = 3 and all three of a, b and c are naturals,

Since the surfaces of all cubes of dimensions higher than three consist of three-dimensional cubes, the above also proves Fermat's Last Theorem for all m>3.

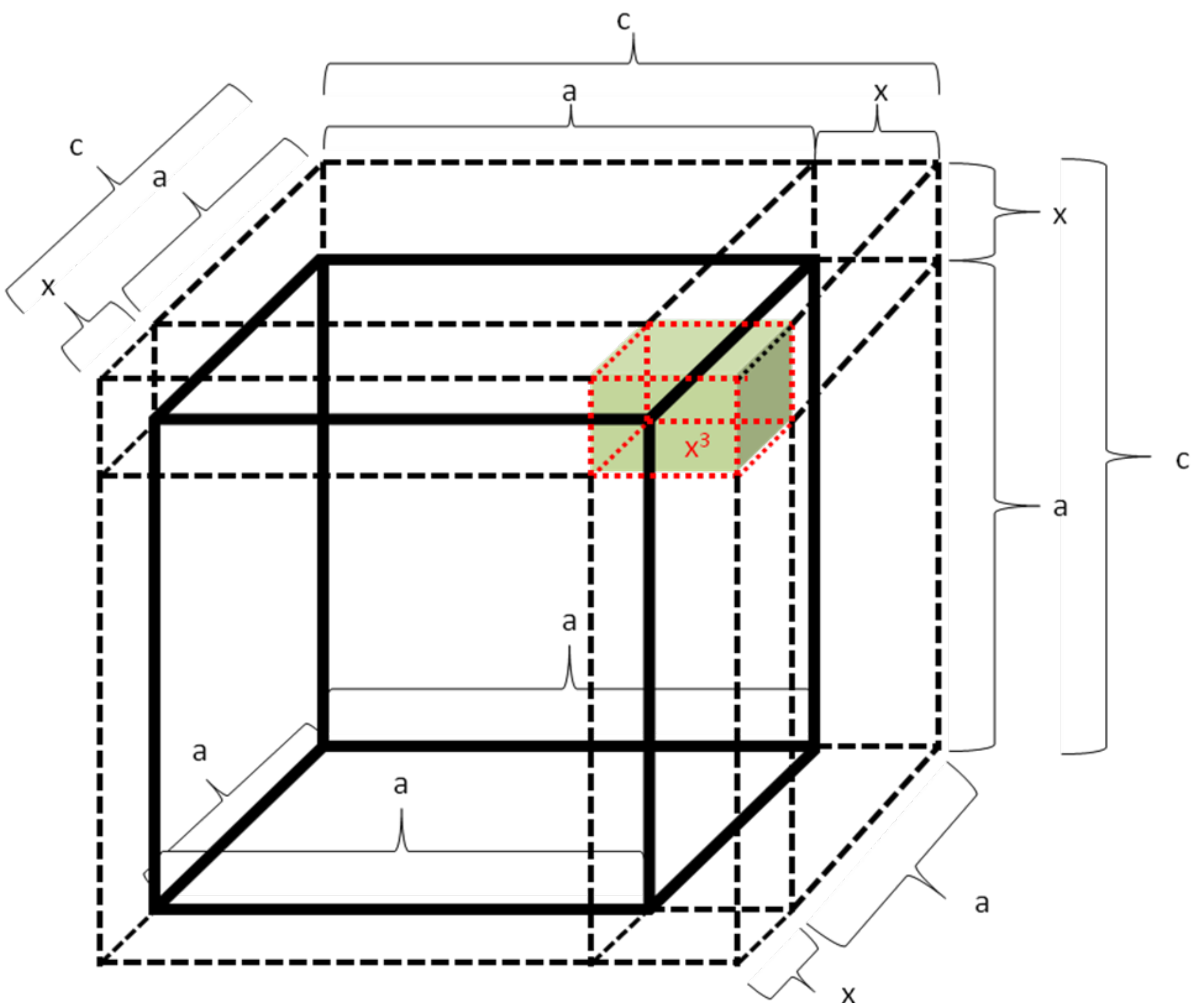

Fig. 1