On the generalization of Glaisher–Kinkelin‘s constant and its applications

Ri[1*]Son Hyang, Yun [2] Il , Ji [3]Chol Ho Ri [4]Hong Rok

Faculty of Mathematics , **Kim Il Sung** University, Taesong District, Pyongyang, DPR Korea

*Corresponding Author E-mail address CHH.JI1013@star-co.net.kp;

Corresponding Author E-mail address yi0101@star-co.net.kp;

Abstract. In this paper, we have found the exquisite relation between Stirling's f ormula and interesting expression for Glaisher–Kinkelin's constant, Bendersky–Ada mchik's constant, and have introduced new constants as their generalization. As w e calculate the singular integral or sum of series in form of closed expression by introducing constant π, Euler constant, Glaisher-Kinkelin's constant, Bendersky–Adamchik's constant, new constants will play important role in calculating of the sum of multi-series which has the multi-power in general term. To demonstrate th e superiority of our new constants in several calculations, we propose an example We show that our new constants generalize Glaisher–Kinkelin's constant and

Bendersky–Adamchik's constant

Mathematics Subject Classification. 11Y60, 11A55, 41A25.

Keywords: Stirling‘s formula, Glaisher-Kinkelin's constant, Bendersky–Adamchik's constants, Generalized Glaisher-Kinkelin's constant

Declarations

Funding: The authors did not receive any support from any organization for the submitted work.

Conflicts of interest/Competing interests: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and material: The data that supports the findings of this study are available within the article and its supplementary material.

Authors' contributions
Conceptualization: Son Hyang Ri, Hyong Rok Ri ;
Methodology: Il Yun , Chol Ho Ji ;
Formal analysis and investigation: Chol Ho Ji, Son Hyang Ri ;
Resources: Il Yun
Writing: Son Hyang Ri, Hyong Rok Ri ; Editing: Il Yun
Consent to participate
All authors read and approved the final manuscript.

1. Introduction

we can calculate easily the singular integral or sum of series in form of closed expression by introducing constant π, Euler constant, Glaisher-Kinkelin's constant.

But multiple Hardy series

$$\sum_{n_1,n_2,\cdots,n_k=1}^{\infty}(-1)^{n_1+n_2+\cdots+n_k}(H_{n_1+n_2+\cdots+n_k}-\ln(n_1+n_2+\cdots+n_k)-\gamma),$$

where γ is Euler's constant, $H_n=\sum_{k=1}^{n}\frac{1}{k}$.

can't calculate by the constants already well known .

We calculate the multiple Hardy series by new constants researched in this paper.

To research new the constants, we firstly study the exquisite relation between some constants.

Stirling's formula is one of the most famous formulas and already well known [2, 6]. As you can see, taking logarithm about both sides of the Stirling's formula

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}} \quad (0<\theta<1)$$

and calculating a little, we obtain the following expression

$$\frac{1}{2}\ln(2\pi)=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\ln k-\left(n+\frac{1}{2}\right)\ln n+n\right). \tag{1.1}$$

Also, taking logarithm about both sides in the definition of Glaisher-Kinkelin constant A

$$A = \lim_{n\to\infty} n^{-\frac{n^2}{2}-\frac{n}{2}-\frac{1}{12}} e^{\frac{n^2}{4}} \prod_{k=1}^{n} k^k,$$

we can obtain [1-3,6,9,12-13]

$$\ln A = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}\right). \tag{1.2}$$

These constants play important role in indicating simply singular integral or sum of series to closed expression type.[2,6,8,12-13].

Comparing (1.1) with (1.2), we can see the interesting type of the sequences

$$\sum_{k=1}^{n} k^{p-1}\ln k - X_p(n)\ln n + Y_p(n),$$

where $X_p(n)$ and $Y_p(n)$ are p*th*- polynomials.

Motivated by (1.1) and (1.2), a natural question arises, that is, which kinds of the sequences containing p*th*- polynomial $X_p(n), Y_p(n)$ might converge to any number $p \in N$.

Some constants are defined by limit of the following type of the sequences(see [1-3,9,12-13]):

$$\ln L_1 = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\ln k - \left(n+\frac{1}{2}\right)\ln n + n\right] = 0.918938533204672741780...\left(=\ln\sqrt{2\pi}\right),$$

$$\ln L_2 = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}\right] = 0.248754477033784262547...(=\ln A),$$

$$\ln L_3 = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^2\ln k - \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\ln n + \left(\frac{n^3}{9}-\frac{n}{12}\right)\right] =$$
$$= 0.03044845705839...(=\ln B), \tag{1.3}$$

$$\ln L_4 = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}\right)\ln n + \left(\frac{n^4}{16}-\frac{n^2}{12}\right)\right] =$$
$$= -0.020656354135552078922\ldots(= \ln C), \qquad (1.4)$$

where $\ln L_1$ and $\ln L_2$ are well known constants in Stirling's formula and Glaisher–Kinkelin's constant, and $\ln L_3$ and $\ln L_4$ are Bendersky-Adamchik's constants[1, 3,12-13].

In [9], it is also called Choi-Srivastava's constant.

We provide the method finding more general constants, that is, generalized Glaisher –Ki nkelin's constant

$$\ln L_p = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^{p-1}\ln k - X_p(n)\ln n + Y_p(n)\right] \quad , p\in N$$

2. An example concerning with new constants

While we are solving some open problems proposed by the professor Ovidiu Furdui[6], we realize that we must generalize Glaisher–Kinkelin's constant and propose an inter esting example.

Example. Calculate in form of closed expression the multi- series

$$\sum_{n_1,n_2,n_3,n_4,n_5=1}^{\infty}(-1)^{n_1+n_2+n_3+n_4+n_5}\left(H_{n_1+n_2+n_3+n_4+n_5} - \ln(n_1+n_2+n_3+n_4+n_5)-\gamma\right), \qquad (2.1)$$

where γ is Euler's constant, $H_n = \sum_{k=1}^{n}\frac{1}{k}$.

Solution. using the Abel's summation formula, we rewrite (2.1) as follows;

$$S_5 = \sum_{n=5}^{\infty}\binom{n-1}{4}\left[\sum_{l=0}^{5}(-1)^{l+1}\binom{5}{l}\ln(2n-l) + \frac{24}{(2n-4)(2n-3)(2n-2)(2n-1)2n}\right]. \qquad (2.2)$$

We calculate the $n+2$ th partial sum of (2.2).

$$S_5(n)=\sum_{k=5}^{n+2}\binom{k-1}{4}\left[\sum_{l=0}^{5}(-1)^{l+1}\binom{5}{l}\ln(2k-l)+\frac{24}{(2k-4)(2k-3)(2k-2)(2k-1)2k}\right]. \tag{2.3}$$

In (2.3), we calculate odd-th sum and even-th sum separately,

$$\begin{aligned}
S_5(n)&=\sum_{k=5}^{n+2}\sum_{l=0}^{5}(-1)^{l+1}\binom{k-1}{4}\binom{5}{l}\ln(2k-l)+\sum_{k=5}^{n+2}\frac{24\cdot\binom{k-1}{4}}{(2k-4)(2k-3)(2k-2)(2k-1)2k}\\
&=\sum_{l=0}^{2}\binom{5}{2l+1}\sum_{k=5}^{n+2}\binom{k-1}{4}\ln(2k-2l-1)-\sum_{l=0}^{2}\binom{5}{2l}\sum_{k=5}^{n+2}\binom{k-1}{4}\ln(2k-2l)+\sum_{k=5}^{n+2}\frac{k^2-7k+12}{8k(2k-3)(2k-1)}\\
&=\sum_{l=0}^{2}\binom{5}{2l+1}\left[\sum_{k=5}^{n+l}\binom{k-1}{4}\ln(2k-2l-1)+\sum_{k=n+l+1}^{n+2}\binom{k-1}{4}\ln(2k-2l-1)\right]-\\
&-\sum_{l=0}^{2}\binom{5}{2l}\left[\sum_{k=5}^{n+l}\binom{k-1}{4}\ln(2k-2l)+\sum_{k=n+l+1}^{n+2}\binom{k-1}{4}\ln(2k-2l)\right]+\frac{1}{32}\sum_{k=5}^{n+2}\left(\frac{16}{k}-\frac{35}{2k-1}+\frac{5}{2k-3}\right).
\end{aligned} \tag{2.4}$$

In (2.4), we let

$$A(n)=\sum_{l=0}^{2}\binom{5}{2l+1}\sum_{k=n+l+1}^{n+2}\binom{k-1}{4}\ln(2k-2l-1)-\sum_{l=0}^{2}\binom{5}{2l}\sum_{k=n+l+1}^{n+2}\binom{k-1}{4}\ln(2k-2l),$$

then we have

$$\begin{aligned}
A(n)&=5\binom{n}{4}\ln(2n+1)+5\binom{n+1}{4}\ln(2n+3)+10\binom{n+1}{4}\ln(2n+1)\\
&-\binom{n}{4}\ln(2n+2)-\binom{n+1}{4}\ln(2n+4)-10\binom{n+1}{4}\ln(2n+2)\\
&=\ln(2n)\left(\frac{n^4}{3}-\frac{4n^3}{3}+\frac{5n^2}{3}-\frac{2n}{3}\right)+\left(\frac{n^3}{12}-\frac{5n^2}{12}+\frac{197n}{288}-\frac{415}{1152}\right).
\end{aligned} \tag{2.5}$$

Thus, (2.4) is equal to the

$$\begin{aligned}
S_5(n)&=\sum_{l=0}^{2}\binom{5}{2l+1}\left[\sum_{k=1}^{n}\binom{k+l-1}{4}\ln(2k-1)\right]-\sum_{l=0}^{2}\binom{5}{2l}\left[\sum_{k=1}^{n}\binom{k+l-1}{4}\ln(2k)\right]+A(n)\\
&+\frac{1}{2}\left(\ln n+\gamma-\frac{25}{12}\right)-\frac{35}{32}\left(\ln 2+\frac{1}{2}\ln n+\frac{\gamma}{2}-\frac{176}{105}\right)+\frac{5}{32}\left(\ln 2+\frac{1}{2}\ln n+\frac{\gamma}{2}-\frac{23}{15}\right)+O\left(\frac{1}{n}\right).
\end{aligned}$$

Using the simple substitution , we get that

$$S_5(n)=\sum_{l=0}^{2}\binom{5}{2l+1}\left[\sum_{k=1}^{2n}\binom{\frac{k}{2}+l-\frac{1}{2}}{4}\ln k-\sum_{k=1}^{n}\binom{k+\frac{2l+1}{2}-1}{4}\ln(2k)\right]$$

$$-\sum_{l=0}^{2}\binom{5}{2l}\left[\sum_{k=1}^{n}\binom{k+\frac{2l}{2}-1}{4}\ln(2k)\right]+A(n)+\frac{1}{32}\ln n+\frac{\gamma}{32}-\frac{15}{16}\ln 2+\frac{53}{96}+O\left(\frac{1}{n}\right)$$

$$=\sum_{k=1}^{2n}\left(\frac{k^4}{24}-\frac{5k^3}{12}+\frac{35k^2}{24}-\frac{25k}{12}+1\right)\ln k-\sum_{k=1}^{n}\left(\frac{4k^4}{3}-\frac{20k^3}{3}+\frac{35k^2}{3}-\frac{25k}{3}+2\right)\ln k$$

$$-\left(\frac{4n^5}{15}-n^4+n^3-\frac{4n}{15}\right)\ln 2+A(n)+\frac{1}{32}\ln n+\frac{\gamma}{32}-\frac{15}{16}\ln 2+\frac{53}{96}+O\left(\frac{1}{n}\right).$$

From (1.1) - (1.4), we have

$$S_5(n)=\frac{1}{24}\left(\sum_{k=1}^{2n}k^4\ln k-\left(\frac{(2n)^5}{5}+\frac{(2n)^4}{2}+\frac{(2n)^3}{3}-\frac{(2n)}{30}\right)\ln(2n)+\left(\frac{(2n)^5}{25}-\frac{(2n)^3}{12}+\frac{13(2n)}{360}\right)\right)$$

$$-\frac{4}{3}\left(\sum_{k=1}^{n}k^4\ln k-\left(\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}\right)\ln n+\left(\frac{n^5}{25}-\frac{n^3}{12}+\frac{13n}{360}\right)\right)- \qquad (2.6)$$

$$-\frac{5\ln C}{12}+\frac{35\ln B}{24}-\frac{25\ln A}{12}+\ln\sqrt{2\pi}+\frac{20\ln C}{3}-\frac{35\ln B}{3}+\frac{25\ln A}{3}-2\ln\sqrt{2\pi}+$$

$$-\frac{415}{1152}+\frac{\gamma}{32}-\frac{175}{288}\ln 2+\frac{53}{96}+O\left(\frac{1}{n}\right).$$

To calculate the limit of (2.6), we must use our new constant

$$\ln L_5=\lim_{n\to\infty}\left[\sum_{k=1}^{n}k^4\ln k-\left(\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}\right)\ln n+\left(\frac{n^5}{25}-\frac{n^3}{12}+\frac{13n}{360}\right)\right]=$$

$$=-\zeta'(-4)=-0.00798381145026862428070\cdots.$$

With this constant, we finish the example and the sum is

$$S=-\frac{31}{24}\ln L_5+\frac{25\ln C}{4}-\frac{245\ln B}{24}+\frac{25\ln A}{4}-\ln\sqrt{2\pi}-\frac{175}{288}\ln 2+\frac{221}{1152}+\frac{\gamma}{32}.$$

Motivated by this example, we study the method for investigating new constant as the generalization of Glaisher-Kinkelin's constant

3. Main result

Theorem For any natural number $p>1$, let $X_p(n),\ Y_p(n)$ be p th- polynomials.

Then sequence

$$\ln L_p(n)=\sum_{k=1}^{n}k^{p-1}\ln k-X_p(n)\ln n+Y_p(n),\quad Y_p(0)=0 \tag{3.1}$$

converge if

$$X_p(n)=\sum_{k=1}^{n}k^{p-1}-\left(\frac{1}{p(p+1)}-\frac{1}{2p}+\sum_{k=2}^{p-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{p-k+1}\right), \tag{3.2}$$

$$Y_p(n)=\frac{n^p}{p^2}+\sum_{k=2}^{p-1}n^{p-k}\left(\frac{\binom{p-1}{k-1}}{p}\frac{B_k}{k}+\frac{1}{p-k}\left(\frac{1}{p(k+1)}-\frac{1}{2k}+\sum_{i=2}^{k}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k-i+1}\right)\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{pk}-\frac{1}{2(k-1)}+\sum_{i=2}^{k-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k-i}\right)+\sum_{j=2}^{k-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\frac{B_{k-j+1}}{k-j+1}\binom{p-j}{k-j}\right) \tag{3.3}$$

where B_k are the Bernoulli's numbers defined as([3, 8])

$$B_1=-\frac{1}{2},B_2=\frac{1}{6},B_3=0,B_4=-\frac{1}{30},B_5=0,B_6=\frac{1}{42},\dots .$$

If p=1, then (See (1.1))

$$\ln L_1(n)=\sum_{k=1}^{n}\ln k-\left(n+\frac{1}{2}\right)\ln n+n.$$

Proof. For any natural number $p>1$, we are going to find $X_p(n),\ Y_p(n)$ so that sequence (3.1) converge

We firstly calculate as follows;

$$\ln L_p(n)-\ln L_p(n-1)=\left(\sum_{k=1}^{n}k^{p-1}\ln k - X_p(n)\ln n + Y_p(n)\right)-\left(\sum_{k=1}^{n-1}k^{p-1}\ln k - X_p(n-1)\ln(n-1)+Y_p(n-1)\right)$$
$$= n^{p-1}\ln n - X_p(n)\ln n + X_p(n-1)\ln(n-1)+Y_p(n)-Y_p(n-1)$$
$$=\left(n^{p-1}-X_p(n)+X_p(n-1)\right)\ln n+\left(X_p(n-1)\ln\left(1-\frac{1}{n}\right)+Y_p(n)-Y_p(n-1)\right).$$

Now, we get $X_p(n)$ so that

$$n^{p-1}-X_p(n)+X_p(n-1)=0.$$

Since $X_p(n)-X_p(n-1)=n^{p-1}$, using the formula

$$\sum_{k=1}^{n}k^{p-1}=\frac{n^p}{p}+\frac{n^{p-1}}{2}+\sum_{k=2}^{p-1}\frac{B_k}{k}\binom{p-1}{k-1}n^{p-k},\qquad p>1 \tag{3.4}$$

we have

$$X_p(n)=\sum_{k=1}^{n}\left(X_p(k)-X_p(k-1)\right)+X_p(0)=\sum_{k=1}^{n}k^{p-1}+X_p(0)=\frac{n^p}{p}+\frac{n^{p-1}}{2}+\sum_{k=2}^{p-1}\frac{B_k}{k}\binom{p}{k-1}n^{p-k}+X_p(0) \tag{3.5}$$

Let

$$V(n)=X_p(n-1)\ln\left(1-\frac{1}{n}\right)+Y_p(n)-Y_p(n-1),$$

then, from the relation

$$-\ln\left(1-\frac{1}{n}\right)=\sum_{k=1}^{\infty}\frac{1}{kn^k},$$

we get that

$$-\ln\left(1-\frac{1}{n}\right)>\sum_{k=1}^{p+3}\frac{1}{kn^k}, \tag{3.6}$$

and

$$-\ln\left(1-\frac{1}{n}\right)<\sum_{k=1}^{p+3}\frac{1}{kn^k}+\frac{1/(p+4)}{n^{p+4}}\sum_{k=0}^{\infty}\frac{1}{n^k}$$
$$=\sum_{k=1}^{p+3}\frac{1}{kn^k}+\frac{1/(p+4)}{n^{p+4}}\cdot\frac{1}{1-\frac{1}{n}}=\sum_{k=1}^{p+3}\frac{1}{kn^k}+\frac{1/(p+4)}{n^{p+3}(n-1)}<\sum_{k=1}^{p+2}\frac{1}{kn^k}+\frac{1}{n^{p+3}}. \tag{3.7}$$

Moreover, we have

$$X_p(n-1)=X_p(n)-n^{p-1}=\frac{n^p}{p}-\frac{n^{p-1}}{2}+\sum_{k=2}^{p-1}\frac{B_k}{k}\binom{p-1}{k-1}n^{p-k}+X_p(0),$$

and for sufficiently large *n*,

$$X_p(n-1)=\sum_{k=1}^{n-1}k^{p-1}+X_p(0)$$

may be a positive number. Using

$$-\ln\left(1-\frac{1}{n}\right)=\sum_{k=1}^{\infty}\frac{1}{kn^k}=\left(\frac{1}{n}+\frac{1}{2n^2}+\ldots+\frac{1}{(p+3)n^{p+3}}\right)+O\left(\frac{1}{n^{p+4}}\right),$$

and (3.5), we have

$$-X_p(n-1)\ln\left(1-\frac{1}{n}\right)$$

$$=\left(\frac{n^p}{p}-\frac{n^{p-1}}{2}+\sum_{k=2}^{p-1}\frac{B_k}{k}\binom{p-1}{k-1}n^{p-k}+X_p(0)\right)\left(\frac{1}{n}+\frac{1}{2n^2}+\ldots+\frac{1}{(p+3)n^{p+3}}+O\left(\frac{1}{n^{p+4}}\right)\right).$$

We expand this and rewrite it with respect to polynomial on n, sum on $\frac{1}{n}$ and sum on $\frac{1}{n^2}$, etc. as follows;

$$-X_p(n-1)\ln\left(1-\frac{1}{n}\right)=$$

$$=\left(\frac{n^{p-1}}{p}+\sum_{j=2}^{p}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{k=2}^{j-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{j-k}\right)n^{p-j}\right)+\frac{1}{n}\left(\frac{1}{p(p+1)}-\frac{1}{2p}+\sum_{k=2}^{p-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{p-k+1}+X_p(0)\right)$$

$$+\frac{1}{n^2}\left(\frac{1}{p(p+2)}-\frac{1}{2(p+1)}+\sum_{k=2}^{p-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{p-k+2}+\frac{X_p(0)}{2}\right)+O\left(\frac{1}{n^3}\right).$$

And let

$$X(n)=\frac{n^{p-1}}{p}+\sum_{j=2}^{p}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{k=2}^{j-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{j-k}\right)n^{p-j},$$

$$c_1=\frac{1}{p(p+1)}-\frac{1}{2p}+\sum_{k=2}^{p-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{p-k+1}+X_p(0),$$

$$c_2=\frac{1}{p(p+2)}-\frac{1}{2(p+1)}+\sum_{k=2}^{p-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{p-k+2}+\frac{X_p(0)}{2},$$

then, we have

$$-X_p(n-1)\ln\left(1-\frac{1}{n}\right)=X(n)+\frac{c_1}{n}+\frac{c_2}{n^2}+O\left(\frac{1}{n^3}\right). \tag{3.8}$$

Using (3.6) and (3.7) and Estimating (3.8), we get that

$$-X_p(n-1)\ln\left(1-\frac{1}{n}\right)>X_p(n-1)\left(\sum_{k=1}^{p+2}\frac{1}{kn^k}+\frac{1}{(p+3)n^{p+3}}\right)=X(n)+\frac{c_1}{n}+\frac{c_2}{n^2}+\sum_{k=3}^{p+3}\frac{a_k}{n^k} \tag{3.9}$$

$$-X_p(n-1)\ln\left(1-\frac{1}{n}\right)<X_p(n-1)\left(\sum_{k=1}^{p+2}\frac{1}{kn^k}+\frac{1}{n^{p+3}}\right)=X(n)+\frac{c_1}{n}+\frac{c_2}{n^2}+\sum_{k=3}^{p+3}\frac{b_k}{n^k} \tag{3.10}$$

Using (3.9), (3.10) and definition of $V(n)$, we have

$$-\frac{c_2}{n^2}-\sum_{k=3}^{p+3}\frac{b_k}{n^k}<V(n)<-\frac{c_2}{n^2}-\sum_{k=3}^{p+3}\frac{a_k}{n^k},$$

For sufficiently large n, $V(n)$ is monotonic (increasing ($c_2<0$), decreasing ($c_2>0$) . and since

$$\ln L_p(n)=\sum_{k=2}^{n}\left[\ln L_p(k)-\ln L_p(k-1)\right]+\ln L_p(1)=\sum_{k=2}^{n}V(k)+\ln L_p(1),$$

we get that

$$\sum_{k=2}^{n}\left(\frac{c}{k^2}+\sum_{l=3}^{p+3}\frac{a_l}{k^l}\right)+Y_p(1)<\ln L_p(n)<\sum_{k=2}^{n}\left(\frac{c}{k^2}+\sum_{l=3}^{p+3}\frac{b_l}{k^l}\right)+Y_p(1),$$

Thus in case of $Y_p(n)-Y_p(n-1)-X(n)=0,$

$$c_1=\frac{1}{p(p+1)}-\frac{1}{2p}+\sum_{k=2}^{p-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{p-k+1}+X_p(0)=0, \qquad (3.11)$$

sequence $\ln L_p(n)$ converges. From (3.11) shows that

$$X_p(0)=-\frac{1}{p(p+1)}+\frac{1}{2p}-\sum_{k=2}^{p-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{p-k+1}. \qquad (3.12)$$

Combining (3.12) with (3.5), we have (3.2).

On the otherhand, using (3.4) and formula

$$Y_p(n)-Y_p(n-1)=X(n)=\frac{n^{p-1}}{p}+\sum_{j=2}^{p}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{k=2}^{j-1}\frac{B_k}{k}\frac{\binom{p-1}{k-1}}{j-k}\right)n^{p-j},$$

we can get $Y_p(n)$ as follows;

$$Y_p(n)=\sum_{k=1}^{n}\left[Y_p(k)-Y_p(k-1)\right]+Y_p(0)=\sum_{k=1}^{n}\left(\frac{k^{p-1}}{p}+\sum_{j=2}^{p}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)k^{p-j}\right)$$

$$=\frac{1}{p}\sum_{k=1}^{n}k^{p-1}+\sum_{j=2}^{p-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\sum_{k=1}^{n}k^{p-j}+\left(\frac{1}{p^2}-\frac{1}{2(p-1)}+\sum_{i=2}^{p-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{p-i}\right)n$$

$$=\frac{1}{p}\left(\frac{n^p}{p}+\frac{n^{p-1}}{2}+\sum_{i=2}^{p-1}\frac{B_i}{i}\binom{p-1}{i-1}n^{p-i}\right)+\sum_{j=2}^{p-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)$$

$$\left(\frac{n^{p-j+1}}{p-j+1}+\frac{n^{p-j}}{2}+\sum_{l=2}^{p-j}\frac{B_l}{l}\binom{p-i}{l-1}n^{p-j-l+1}\right)+\left(\frac{1}{p^2}-\frac{1}{2(p-1)}+\sum_{i=2}^{p-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{p-i}\right)n. \tag{3.13}$$

We straighten (3.13) term by term as follows;

$$\frac{1}{p}\left(\frac{n^p}{p}+\frac{n^{p-1}}{2}+\sum_{i=2}^{p-1}\frac{B_i}{i}\binom{p-1}{i-1}n^{p-i}\right)\overset{i=k}{=}\left(\frac{n^p}{p^2}+\frac{n^{p-1}}{2p}+\sum_{k=2}^{p-1}\frac{B_k}{pk}\binom{p-1}{k-1}n^{p-k}\right),$$

$$\sum_{j=2}^{p-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\frac{n^{p-j+1}}{p-j+1}+\left(\frac{1}{p^2}-\frac{1}{2(p-1)}+\sum_{i=2}^{p-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{p-i}\right)n$$

$$\overset{j=k+1}{=}\sum_{k=1}^{p-2}\left(\frac{1}{p(k+1)}-\frac{1}{2k}+\sum_{i=2}^{k}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k+1-i}\right)\frac{n^{p-k}}{p-k}+\left(\frac{1}{p^2}-\frac{1}{2(p-1)}+\sum_{i=2}^{p-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{p-i}\right)n$$

$$=\left(\frac{1}{2p}-\frac{1}{2}\right)\frac{n^{p-1}}{p-1}+\sum_{k=2}^{p-1}\left(\frac{1}{p(k+1)}-\frac{1}{2k}+\sum_{i=2}^{k}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k+1-i}\right)\frac{n^{p-k}}{p-k},$$

$$\sum_{j=2}^{p-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{C_{p-1}^{i-1}}{j-i}\right)\frac{n^{p-j}}{2}\overset{j=k}{=}\sum_{k=2}^{p-1}\left(\frac{1}{pk}-\frac{1}{2(k-1)}+\sum_{i=2}^{k-1}\frac{B_i}{i}\frac{C_{p-1}^{i-1}}{k-i}\right)\frac{n^{p-k}}{2},$$

and

$$\sum_{j=2}^{p-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\sum_{l=2}^{p-j}\frac{B_l}{l}\binom{p-j}{l-1}n^{p-j-l+1}=$$

$$\overset{l=k-j+1}{=}\sum_{j=2}^{p-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\sum_{k=j+1}^{p-1}\frac{B_{k-j+1}}{k-j+1}\binom{p-j}{k-j}n^{p-k}$$

$$=\sum_{j=2}^{p-1}\sum_{k=j+1}^{p-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\frac{B_{k-j+1}}{k-j+1}\binom{p-j}{k-j}n^{p-k}$$

$$=\sum_{k=3}^{p-1}n^{p-k}\sum_{j=2}^{k-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\frac{B_{k-j+1}}{k-j+1}\binom{p-j}{k-j}.$$

We substitute these terms for (3-13) and get that

$$Y_p(n)=\frac{n^p}{p^2}+\left(\frac{1}{2p}-\left(\frac{1}{2p}-\frac{1}{2}\right)\frac{1}{p-1}\right)n^{p-1}+\sum_{k=2}^{p-1}n^{p-k}\left(\frac{\binom{p-1}{k-1}}{p}\frac{B_k}{k}+\frac{1}{p-k}\left(\frac{1}{p(k+1)}-\frac{1}{2k}+\sum_{i=2}^{k}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k-i+1}\right)\right.$$

$$\left.+\frac{1}{2}\left(\frac{1}{pk}-\frac{1}{2(k-1)}+\sum_{i=2}^{k-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k-i}\right)+\sum_{j=2}^{k-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\frac{B_{k-j+1}}{k-j+1}\binom{p-j}{k-j}\right).$$

Thus, we have

$$Y_p(n)=\frac{n^p}{p^2}+\sum_{k=2}^{p-1}n^{p-k}\left(\frac{\binom{p-1}{k-1}}{p}\frac{B_k}{k}+\frac{1}{p-k}\left(\frac{1}{p(k+1)}-\frac{1}{2k}+\sum_{i=2}^{k}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k-i+1}\right)\right.$$

$$\left.+\frac{1}{2}\left(\frac{1}{pk}-\frac{1}{2(k-1)}+\sum_{i=2}^{k-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{k-i}\right)+\sum_{j=2}^{k-1}\left(\frac{1}{pj}-\frac{1}{2(j-1)}+\sum_{i=2}^{j-1}\frac{B_i}{i}\frac{\binom{p-1}{i-1}}{j-i}\right)\frac{B_{k-j+1}}{k-j+1}\binom{p-j}{k-j}\right).$$

and this is (3.3).

4. Remarks

We use Mathematica software to calculate a few first terms.

$$\ln L_1=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\ln k-\left(n+\frac{1}{2}\right)\ln n+n\right]$$
$$=0.91893853320467274178\ldots\left(=\ln\sqrt{2\pi}\right),$$
$$\ln L_2=\lim_{n\to\infty}\left[\sum_{k=1}^{n}k\ln k-\left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n+\frac{n^2}{4}\right]$$
$$=0.24875447703378426254\ldots(=\ln A),$$
$$\ln L_3=\lim_{n\to\infty}\left[\sum_{k=1}^{n}k^2\ln k-\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\ln n+\left(\frac{n^3}{9}-\frac{n}{12}\right)\right]$$
$$=0.03044845705839327078\ldots,(=\ln B)$$
$$\ln L_4=\lim_{n\to\infty}\left[\sum_{k=1}^{n}k^3\ln k-\left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}\right)\ln n+\left(\frac{n^4}{16}-\frac{n^2}{12}\right)\right]$$
$$=-0.02065635413555207892\ldots,(=\ln C)$$

$$\ln L_5=\lim_{n\to\infty}\left[\sum_{k=1}^{n}k^4\ln k-\left(\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}\right)\ln n+\left(\frac{n^5}{25}-\frac{n^3}{12}+\frac{13n}{360}\right)\right] \quad (4.1)$$
$$=-0.00798381145026862428070\ldots$$

$$\ln L_6=\lim_{n\to\infty}\left[\sum_{k=1}^{n}k^5\ln k-\left(\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}+\frac{1}{252}\right)\ln n+\left(\frac{n^6}{36}-\frac{n^4}{12}+\frac{47n^2}{720}\right)\right] \quad (4.2)$$
$$=0.00963383254104519605155\ldots$$

$$\ln L_7 = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^6 \ln k - \left(\frac{n^7}{7}+\frac{n^6}{2}+\frac{n^5}{2}-\frac{n^3}{6}+\frac{n}{42}\right)\ln n + \left(\frac{n^7}{49}-\frac{n^5}{12}+\frac{37n^3}{360}-\frac{29n}{840}\right)\right] \quad (4.3)$$

$$= 0.0058997591435159374506 3...$$

We used (4.1) in calculation of limit of (2.6).

Conclusions

In this paper, a generalization of Glaisher-Kinkelin's constant has been made. This study has been based on the our new method. In a forthcoming work, we will take properties of our new constant .

New constants will play important role in calculating of the sum of multi-series which has the multi-power in general term in mathematical analysis.

References

[1] C. Mortici, Approximating the constants of Glaisher-Kinkelin type, J. Number Theory 133(2013)2465-2469

[2] CholHo Ji, HyonChol Kim, KwangIl Ri: The interesting problems and solutions in Mathematical Analysis, Scholars'press, 2021.

[3] Dawei Lu, Cristinel Mortici: Some new quicker approximations of Glaisher-Kinkelin's and Bendersky-Adamchik's constants, J. Number Theory 144(2014) 340-352, Elsevier 2014

[4] Dence, T.P., Dence, J.B., A survey of Euler's constant. Math. Mag. 82 (4), 255-265 (2009)

[5] Finch, S.R., Mathematical Constants,Encyclopedia of Mathematics and Its Applications 94. Cambridge University Press, New York (2003)

[6] Furdui, O. : Limits, series, and Fractional Part Integrals problems in Mathematical analysis. Springer 2013

[7] Furdui, O., A convergence criteria for multiple harmonic series. Creat. Math. Inform.18(1), 22-25 (2009)

[8] HyonChol Kim ,SongIl Ri , Advanced topics in Mathematical Analysis (Approximants of the Euler-Mascheroni constant and the Gamma function), Scholar'Press,2020.

[9] J.-X.Cheng, C.-P. Chen : Asymptotic expansions of the Glaisher-Kinkelin and Choi-Srivastava constants, J. Number Theory 144(2014) 105-110 ,Elsevier 2014

[10] J. Choi, H.M.Srivastava, Certain classes of series associated with the zeta function and multiple Gamma functions,J. Comput. Appl. Math. 118(2000)87-109

[11] J. Choi, Some mathematical constants, Appl. Math. Comput 187(2007) 122-140

[12] Songhao Liu and Dawei Lu, Improved Convergence Towards Glaisher–Kinkelin's and Bendersky–Adamchik's Constants, Results Math 71 (2017), 731–747

[13] Xu You, Some Approximations of Glaisher–Kinkelin and Bendersky–Adamchik Constants, Results Math 72 (2017), 585–594

.