Homogenization of the first initial-boundary value problem for periodic hyperbolic systems. Principal term of approximation *

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Abstract

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider a matrix elliptic second order differential operator $A_{D,\varepsilon}$ with the Dirichlet boundary condition. Here $\varepsilon > 0$ is a small parameter. The coefficients of the operator $A_{D,\varepsilon}$ are periodic and depend on $\mathbf{x}/\varepsilon$. The principal terms of approximations for the operator cosine and sine functions are given in the $(H^2 \to L_2)$- and $(H^1 \to L_2)$-operator norms, respectively. The error estimates are of the precise order $O(\varepsilon)$ for a fixed time. The results in operator terms are derived from the quantitative homogenization estimate for approximation of the solution of the initial-boundary value problem for the equation $(\partial_t^2 + A_{D,\varepsilon})\mathbf{u}_\varepsilon = \mathbf{F}$.

Key words: periodic differential operators, homogenization, convergence rates, hyperbolic systems.

Introduction

The paper is devoted to homogenization of periodic differential operators (DO's). More precisely, we are interested in the so-called operator error estimates, i. e., in quantitative homogenization results, admitting formulation as estimates in the uniform operator topology.

For elliptic and parabolic problems, estimates of such type are very well studied, see, e. g., books [CDaGr, Chapter 14] and [Sh], the survey [ZhPas2],

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the papers [BSu1, Su2] and references therein. The hyperbolic problems and the non-stationary Schrödinger equation were considered in [BSu4]. See also very resent results [DSu2, Su5].

0.1 The class of operators

Let $\Gamma \subset \mathbb{R}^d$ be a lattice. For a Γ-periodic function ψ in $\mathbb{R}^d$, we denote $\psi^\varepsilon(\mathbf{x}) := \psi(\mathbf{x}/\varepsilon)$, where $\varepsilon > 0$.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$ we study a wide class of matrix strongly elliptic operators $A_{D,\varepsilon}$ given by the differential expression $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, $\varepsilon > 0$, with the Dirichlet boundary condition. Here g is a Hermitian matrix-valued function in $\mathbb{R}^d$ (of size $m \times m$), positive definite and periodic with respect to the lattice Γ. The operator $b(\mathbf{D})$ is an $(m \times n)$-matrix first order DO with constant coefficients. It is assumed that $m \geqslant n$; the symbol of $b(\mathbf{D})$ has maximal rank. This condition ensures strong ellipticity of the operator $A_{D,\varepsilon}$.

The simplest example of the operator under consideration is the acoustics operator $-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$. The operator of elasticity theory also can be written in the required form. These and other examples are considered in [BSu1] in detail.

Let A_D^0 be the effective operator $b(\mathbf{D})^* g^0 b(\mathbf{D})$ defined on $H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n)$. Here g^0 is the constant positive definite effective matrix.

0.2 Known results in a bounded domain

The broad literature is devoted to the operator error estimates in homogenization. In the present subsection, we concentrate on problems in a bounded domain.

Operator error estimates for the Dirichlet problems for second order elliptic equations in a bounded domain with sufficiently smooth boundary were studied by many authors. Apparently, the first result is due to Sh. Moskow and M. Vogelius who proved an estimate

$$\|A_{D,\varepsilon}^{-1} - (A_D^0)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C\varepsilon, \tag{0.1}$$

see [MoV, Corollary 2.2]. Here the operator $A_{D,\varepsilon}$ acts in $L_2(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^2$, and is given by $-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$ with the Dirichlet condition on $\partial\mathcal{O}$. The matrix-valued function g is assumed to be infinitely smooth.

For arbitrary dimension, homogenization problems in a bounded domain with sufficiently smooth boundary were studied in [Zh1, Zh2], and [ZhPas1]. The acoustics and elasticity operators with the Dirichlet or Neumann boundary conditions and without any smoothness assumptions on coefficients were considered. The analog of estimate (0.1), but of order $O(\sqrt{\varepsilon})$, was obtained. (In the case of the Dirichlet problem for the acoustics equation, the $(L_2 \to L_2)$-estimate was improved in [ZhPas1], but the order was not sharp.) Similar results for the operator $-\mathrm{div} g^\varepsilon(\mathbf{x})\nabla$ in a smooth bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were obtained by G. Griso [Gr1, Gr2] with the help of the "unfolding" method. In [Gr2], sharp-order estimate (0.1) (for the same operator) was proven. For elliptic systems similar results were independently obtained in [KeLiSh] and in [PSu, Su1]. Further results and a detailed survey can be found in [Su3, Su4].

The $(L_2 \to L_2)$-approximation for the parabolic semigroup $e^{-tA_{D,\varepsilon}}$ was proven in [MSu].

The first initial-boundary value problem for the hyperbolic systems were studied in [M1]. By using the inverse Laplace transformation and a known result on homogenization of the resolvent in dependence on the spectral parameter, for the operator including the lower order terms it was obtained that

$$\left\| \left(\cos(tA_{D,\varepsilon}^{1/2}) - \cos(t(A_D^0)^{1/2})\right) (A_D^0)^{-2} \right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C\varepsilon(1+|t|^5), \quad (0.2)$$

$$\left\| \left(A_{D,\varepsilon}^{-1/2}\sin(tA_{D,\varepsilon}^{1/2}) - (A_D^0)^{-1/2}\sin(t(A_D^0)^{1/2})\right) (A_D^0)^{-2} \right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C\varepsilon|t|(1+|t|^5). \quad (0.3)$$

According to the known results in the whole space $\mathbb{R}^d$, see [BSu4, M2, DSu1], these estimates do not look optimal with respect to the type of the norm and to the rate of growth with respect to the time t.

0.3 Main results

The main result of the paper is the improvement of estimates (0.2), (0.3) with respect to the time growth and to the type of the operator norm: for $0 < \varepsilon \leqslant 1$ and $t \in \mathbb{R}$,

$$\left\| \cos(tA_{D,\varepsilon}^{1/2}) - \cos(t(A_D^0)^{1/2}) \right\|_{H^2(\mathcal{O})\cap H_0^1(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C\varepsilon(1+|t|), \quad (0.4)$$

$$\left\| A_{D,\varepsilon}^{-1/2}\sin(tA_{D,\varepsilon}^{1/2}) - (A_D^0)^{-1/2}\sin(t(A_D^0)^{1/2}) \right\|_{H_0^1(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C\varepsilon(1+|t|). \quad (0.5)$$

Here the space $H^2(\mathcal{O};\mathbb{C}^n) \cap H_0^1(\mathcal{O};\mathbb{C}^n)$ is equipped with the H^2-norm. For the operators, acting in $\mathbb{R}^d$, in [DSu1] it was shown that estimates of the form (0.4), (0.5) are optimal with respect to time t and to the type of the operator norm in the general case. While in $\mathbb{R}^d$ it is possible to refine such estimates with respect to the type of the operator norm under additional assumptions on the operator (see [DSu1]), for the problems in a bounded domain such a refinement was not obtained in the present paper.

Note that the operators $(A_D^0)^{-1} : L_2(\mathcal{O};\mathbb{C}^n) \to H^2(\mathcal{O};\mathbb{C}^n) \cap H_0^1(\mathcal{O};\mathbb{C}^n)$ and $(A_D^0)^{-1/2} : L_2(\mathcal{O};\mathbb{C}^n) \to H_0^1(\mathcal{O};\mathbb{C}^n)$ are isomorphisms, so estimates (0.4), (0.5) can be reformulated as

$$\left\| \left(\cos(tA_{D,\varepsilon}^{1/2}) - \cos(t(A_D^0)^{1/2}) \right) (A_D^0)^{-1} \right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C\varepsilon(1+|t|),$$

$$\left\| \left(A_{D,\varepsilon}^{-1/2} \sin(tA_{D,\varepsilon}^{1/2}) - (A_D^0)^{-1/2} \sin(t(A_D^0)^{1/2}) \right) (A_D^0)^{-1/2} \right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leqslant C\varepsilon(1+|t|).$$

The L_2-operator error estimate for homogenization of the solution to the first initial-boundary value problem for the hyperbolic system is also obtained.

0.4 Method

The proof is a modification of the method of [PSu, Su1]. Consider solution $\mathbf{u}_\varepsilon$ of the first initial-boundary value problem for the hyperbolic equation $(\partial_t^2 + A_\varepsilon)\mathbf{u}_\varepsilon = \mathbf{F}$ and the solution $\mathbf{u}_0$ of the corresponding effective problem. Introduce the first order approximation $\mathbf{v}_\varepsilon = \mathbf{u}_0 + \varepsilon K(\varepsilon)\mathbf{u}_0$ to the solution, where the term $K(\varepsilon)\mathbf{u}_0$ is the corrector, and $\|\varepsilon K(\varepsilon)\mathbf{u}_0\|_{L_2(\mathcal{O})} = O(\varepsilon)$. The function $K(\varepsilon)\mathbf{u}_0$ does not satisfy the Dirichlet boundary condition, so we consider the corresponding boundary layer discrepancy $\mathbf{w}_\varepsilon$ and estimate the difference $\mathbf{w}_\varepsilon - \varepsilon K(\varepsilon)\mathbf{u}_0$ in L_2. This estimation crucially relies on the L_2-boundedness of the operator $A_{D,\varepsilon}^{-1} A_\varepsilon$. To estimate $\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}$ we use the approximation

$$\|A_\varepsilon(I + \varepsilon K(\varepsilon)) - A^0\|_{H^2(\mathcal{O})\cap H_0^1(\mathcal{O})\to\ H^{-1}(\mathcal{O})} \leqslant C\varepsilon, \quad 0 < \varepsilon \leqslant 1, \qquad (0.6)$$

which is a direct consequence of [PSu, Lemma 7.3].

0.5 Plan of the paper

The paper consists of two sections and Introduction. In Section 1, we define the class of operators $A_{D,\varepsilon}$, introduce the effective operator A_D^0, and formulate the known auxiliary result (0.6). In Section 2, we formulate and prove the main results of the paper.

0.6 Notation

Let $\mathfrak{H}$ and $\mathfrak{H}_\bullet$ be complex separable Hilbert spaces. The symbols $(\cdot,\cdot)_\mathfrak{H}$ and $\|\cdot\|_\mathfrak{H}$ denote the inner product and the norm in $\mathfrak{H}$, respectively; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_\bullet}$ means the norm of the linear continuous operators from $\mathfrak{H}$ to $\mathfrak{H}_\bullet$.

The symbols $\langle\cdot,\cdot\rangle$ and $|\cdot|$ stand for the inner product and the norm in $\mathbb{C}^n$, respectively, $\mathbf{1}_n$ is the identity $(n\times n)$-matrix. If a is an $(m\times n)$-matrix, then the symbol $|a|$ denotes the norm of the matrix a as the operator from $\mathbb{C}^n$ to $\mathbb{C}^m$.

We use the notation $\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1,\dots,d$, $\mathbf{D} = -i\nabla = (D_1,\dots,D_d)$. The classes L_p of vector-valued functions in a domain $\mathcal{O}\subset\mathbb{R}^d$ with values in $\mathbb{C}^n$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1\leqslant p\leqslant\infty$. The Sobolev spaces of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}\subset\mathbb{R}^d$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$. For $n=1$, we simply write $L_p(\mathcal{O})$, $H^s(\mathcal{O})$ and so on, but, sometimes, if this does not lead to confusion, we use such simple notation for the spaces of vector-valued or matrix-valued functions. The symbol $L_p((0,T);\mathfrak{H})$, $1\leqslant p\leqslant\infty$, denotes the L_p-space of $\mathfrak{H}$-valued functions on the interval $(0,T)$.

By C and c (possibly, with indices and marks) we denote various constants in estimates.

1 Class of the operators

1.1 Lattice in $\mathbb{R}^d$

Let $\Gamma\subset\mathbb{R}^d$ be a lattice generated by a basis $\mathbf{a}_1,\dots,\mathbf{a}_d\in\mathbb{R}^d$, i. e.,

$$\Gamma = \left\{ \mathbf{a}\in\mathbb{R}^d : \mathbf{a} = \sum_{j=1}^d \nu_j\mathbf{a}_j,\ \nu_j\in\mathbb{Z}\right\},$$

and let Ω be the elementary cell of Γ:

$$\Omega = \left\{ \mathbf{x}\in\mathbb{R}^d : \mathbf{x} = \sum_{j=1}^d \tau_j\mathbf{a}_j,\ -\frac{1}{2} < \tau_j < \frac{1}{2}\right\}.$$

By $|\Omega|$ we denote the Lebesgue measure of Ω: $|\Omega| = \operatorname{meas}\Omega$.

If $f(\mathbf{x})$ is a Γ-periodic function in $\mathbb{R}^d$, we denote

$$f^\varepsilon(\mathbf{x}) := f(\varepsilon^{-1}\mathbf{x}),\quad \varepsilon > 0.$$

1.2 Operator $A_{D,\varepsilon}$

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider the operator $A_{D,\varepsilon}$ formally given by the differential expression $A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$ with the Dirichlet boundary condition. Here $g(\mathbf{x})$ is a Γ-periodic $(m \times m)$-matrix-valued function (in general, with complex entries). We assume that $g(\mathbf{x}) > 0$ and $g, g^{-1} \in L_\infty(\mathbb{R}^d)$. Next, $b(\mathbf{D})$ is the differential operator given by

$$b(\mathbf{D}) = \sum_{j=1}^{d} b_j D_j, \tag{1.1}$$

where b_j, $j = 1, \dots, d$, are constant $(m \times n)$-matrices (in general, with complex entries). It is assumed that $m \geqslant n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^{d} b_j \xi_j$ of the operator $b(\mathbf{D})$ has maximal rank:

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d.$$

This condition is equivalent to the estimates

$$\alpha_0 \mathbf{1}_n \leqslant b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leqslant \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}, \quad 0 < \alpha_0 \leqslant \alpha_1 < \infty, \tag{1.2}$$

with some positive constants α_0 and α_1. So,

$$|b_l| \leqslant \alpha_1^{1/2}. \tag{1.3}$$

The precise definition of the operator $A_{D,\varepsilon}$ is given in terms of the quadratic form

$$a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] = \int_{\mathcal{O}} \langle g^\varepsilon b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n).$$

This form is closed and positive definite. Indeed, extending $\mathbf{u}$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$, using the Fourier transformation and taking (1.2) into account, it is easy to check that

$$\alpha_0 \|g^{-1}\|_{L_\infty} \int_{\mathcal{O}} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x} \leqslant a_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \leqslant \alpha_1 \|g\|_{L_\infty} \int_{\mathcal{O}} |\mathbf{D}\mathbf{u}|^2 \, d\mathbf{x},$$

$\mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n)$. It remains to note that due to the Friedrichs's inequality the functional $\|\mathbf{D}\mathbf{u}\|_{L_2(\mathcal{O})}$ determines the norm in $H^1(\mathcal{O};\mathbb{C}^n)$ equivalent to the standard one. We have

$$\begin{aligned} \|A_{D,\varepsilon}^{1/2}\mathbf{u}\|_{L_2(\mathcal{O})} &\geqslant 2^{-1/2}(1 + (\operatorname{diam}\mathcal{O})^{-2})^{1/2} \alpha_0^{1/2} \|g^{-1}\|_{L_\infty}^{1/2} \|\mathbf{u}\|_{H^1(\mathcal{O})} \\ &=: c_*^{-1} \|\mathbf{u}\|_{H^1(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n). \end{aligned} \tag{1.4}$$

Hence,

$$\|A_{D,\varepsilon}^{-1/2}\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} = \|A_{D,\varepsilon}^{-1/2}\|_{H^{-1}(\mathcal{O}) \to L_2(\mathcal{O})} \leqslant c_*. \tag{1.5}$$

Lemma 1.1. *The operator* $A_{D,\varepsilon}^{-1} A_\varepsilon$ *is bounded in* $L_2(\mathcal{O};\mathbb{C}^n)$ *and*

$$\|A_{D,\varepsilon}^{-1} A_\varepsilon\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leqslant 1. \tag{1.6}$$

Proof. Let $\mathbf{f},\mathbf{h} \in C_0^\infty(\mathcal{O};\mathbb{C}^n)$. We have

$$\begin{aligned}
&(A_{D,\varepsilon}^{-1} A_\varepsilon \mathbf{f}, \mathbf{h})_{L_2(\mathcal{O})} = (A_\varepsilon \mathbf{f}, A_{D,\varepsilon}^{-1}\mathbf{h})_{L_2(\mathcal{O})} \\
&= \big((g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{f}, (g^\varepsilon)^{1/2} b(\mathbf{D}) A_{D,\varepsilon}^{-1}\mathbf{h}\big)_{L_2(\mathcal{O})} = (A_{D,\varepsilon}^{1/2}\mathbf{f}, A_{D,\varepsilon}^{1/2} A_{D,\varepsilon}^{-1}\mathbf{h})_{L_2(\mathcal{O})} \\
&= (A_{D,\varepsilon}^{1/2}\mathbf{f}, A_{D,\varepsilon}^{-1/2}\mathbf{h})_{L_2(\mathcal{O})} = (\mathbf{f},\mathbf{h})_{L_2(\mathcal{O})}.
\end{aligned}$$

So,

$$|(A_{D,\varepsilon}^{-1} A_\varepsilon \mathbf{f}, \mathbf{h})_{L_2(\mathcal{O})}| \leqslant \|\mathbf{f}\|_{L_2(\mathcal{O})}\|\mathbf{h}\|_{L_2(\mathcal{O})}.$$

Since $\mathbf{h}$ belongs to the set $C_0^\infty(\mathcal{O};\mathbb{C}^n)$ which is dense in $L_2(\mathcal{O};\mathbb{C}^n)$, by continuity,

$$\|A_{D,\varepsilon}^{-1} A_\varepsilon \mathbf{f}\|_{L_2(\mathcal{O})} \leqslant \|\mathbf{f}\|_{L_2(\mathcal{O})}, \quad \mathbf{f} \in C_0^\infty(\mathcal{O};\mathbb{C}^n).$$

By continuity, this inequality is valid for any $\mathbf{f} \in L_2(\mathcal{O};\mathbb{C}^n)$. We arrive at estimate (1.6). □

1.3 The effective operator

Suppose that a Γ-periodic $(n\times m)$-matrix-valued function $\Lambda(\mathbf{x})$ is the (weak) solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \qquad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0.$$

As Λ is the weak solution, its $H^1(\Omega)$-norm is bounded. We will use estimate

$$\|\Lambda\|_{L_2(\Omega)} \leqslant |\Omega|^{1/2} M. \tag{1.7}$$

The constant M can be written explicitly (see [BSu2, Subsection 7.3]) and depends only on m, α_0, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

The effective matrix is given by

$$g^0 = |\Omega|^{-1} \int_\Omega g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)\, d\mathbf{x}.$$

It can be checked that g^0 is positive definite. Due to the Voight-Reuss bracketing (see, e. g., [BSu1, Chapter 3, Theorem 1.5]), the matrix g^0 satisfy estimates

$$|g^0| \leqslant \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \leqslant \|g^{-1}\|_{L_\infty}. \tag{1.8}$$

The effective operator A_D^0 for $A_{D,\varepsilon}$ is given by the differential expression

$$A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) \tag{1.9}$$

with the Dirichlet condition on $\partial\mathcal{O}$. The domain of this operator coincides with $H^2(\mathcal{O};\mathbb{C}^n) \cap H_0^1(\mathcal{O};\mathbb{C}^n)$. Indeed, the operator (1.9) is strongly elliptic and due to the assumption $\partial\mathcal{O} \in C^{1,1}$, we can apply the "additional smoothness" theorems for solutions of strongly elliptic systems (see, e. g., [McL, Chapter 4]). Thus, $(A_D^0)^{-1}$ is a continuous operator from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^2(\mathcal{O};\mathbb{C}^n)$:

$$\|(A_D^0)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \leqslant \widehat{c}. \tag{1.10}$$

The constant $\widehat{c}$ depends only on α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Remark 1.2. *Instead of the condition $\partial\mathcal{O} \in C^{1,1}$ one can impose the following implicit condition: a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with Lipschitz boundary is such that estimate* (1.10) *holds. The results of the paper remain true for such domain. In the case of the scalar elliptic operators, wide sufficient conditions on $\partial\mathcal{O}$ ensuring* (1.10) *can be found in* [KoE] *and* [MaSh, *Chapter* 7] (*in particular, it suffices that $\partial\mathcal{O} \in C^\alpha$ for $\alpha > 3/2$*).

Similarly to (1.4), by (1.2) and (1.8),

$$\|(A_D^0)^{1/2}\mathbf{u}\|_{L_2(\mathcal{O})} \geqslant c_*^{-1}\|\mathbf{u}\|_{H^1(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n).$$

Hence,

$$\|(A_D^0)^{-1/2}\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leqslant c_*. \tag{1.11}$$

Lemma 1.3. *For any $\mathbf{h} \in H^2(\mathcal{O};\mathbb{C}^n) \cap H_0^1(\mathcal{O};\mathbb{C}^n)$ we have*

$$\|A_D^0\mathbf{h}\|_{L_2(\mathcal{O})} \leqslant \alpha_1 d \|g\|_{L_\infty} \|\mathbf{D}^2\mathbf{h}\|_{L_2(\mathcal{O})}.$$

Proof. By (1.1), (1.3), and (1.8),

$$\begin{aligned}\|A_D^0\mathbf{h}\|_{L_2(\mathcal{O})} &\leqslant \sum_{l=1}^d \|b_l^* D_l g^0 b(\mathbf{D})\mathbf{h}\|_{L_2(\mathcal{O})} \leqslant \left(\sum_{l=1}^d |b_l|^2\right)^{1/2} \|\mathbf{D} g^0 b(\mathbf{D})\mathbf{h}\|_{L_2(\mathcal{O})} \\ &\leqslant \alpha_1^{1/2} d^{1/2} \|g\|_{L_\infty} \|b(\mathbf{D})\mathbf{D}\mathbf{h}\|_{L_2(\mathcal{O})} \leqslant \alpha_1 d \|g\|_{L_\infty} \|\mathbf{D}^2\mathbf{h}\|_{L_2(\mathcal{O})}.\end{aligned}$$

$\square$

1.4 Steklov smoothing operator

Let S_ε be the Steklov smoothing (or Steklov averaging) operator [St] in $L_2(\mathbb{R}^d;\mathbb{C}^m)$:

$$(S_\varepsilon \mathbf{u})(\mathbf{x}) = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x} - \varepsilon \mathbf{z})\, d\mathbf{z}.$$

We need the following property of the operator S_ε (see [ZhPas1] or [PSu, Proposition 3.2]).

Proposition 1.4. *Let f be a Γ-periodic function in $\mathbb{R}^d$ such that $f \in L_2(\Omega)$. Let $[f^\varepsilon]$ be the operator of multiplication by the function $f^\varepsilon(\mathbf{x})$. Then*

$$\| [f^\varepsilon] S_\varepsilon \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant |\Omega|^{-1/2} \| f \|_{L_2(\Omega)}, \quad \varepsilon > 0.$$

1.5 Auxiliary result

We fix a linear continuous extension operator

$$P_\mathcal{O} : H^l(\mathcal{O};\mathbb{C}^n) \to H^l(\mathbb{R}^d;\mathbb{C}^n), \quad l = 1, 2. \tag{1.12}$$

Let

$$C_\mathcal{O} := \| P_\mathcal{O} \|_{H^1(\mathcal{O}) \to H^1(\mathbb{R}^d)}. \tag{1.13}$$

By $R_\mathcal{O}$ we denote the operator of restriction of functions in $\mathbb{R}^d$ onto the domain $\mathcal{O}$.

The following result was obtained in [PSu, Lemma 7.3].

Lemma 1.5 ([PSu]). *Let $\mathbf{\Phi} \in L_2(\mathcal{O};\mathbb{C}^n)$. When for $0 < \varepsilon \leqslant 1$ we have*

$$\begin{aligned} &\| A_\varepsilon (I + \varepsilon R_\mathcal{O} [\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D}) P_\mathcal{O}) (A_D^0)^{-1} \mathbf{\Phi} - A^0 (A_D^0)^{-1} \mathbf{\Phi} \|_{H^{-1}(\mathcal{O})} \\ &\leqslant C_1 \varepsilon \| (A_D^0)^{-1} \mathbf{\Phi} \|_{H^2(\mathcal{O})}. \end{aligned} \tag{1.14}$$

The constant here depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Since $(A_D^0)^{-1} : L_2(\mathcal{O};\mathbb{C}^n) \to H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n)$ is an isomorphism, the following analogue of estimate (1.14) holds.

Corollary 1.6. *Let $\mathbf{f} \in H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n)$. When for $0 < \varepsilon \leqslant 1$ we have*

$$\| A_\varepsilon (I + \varepsilon R_\mathcal{O} [\Lambda^\varepsilon] S_\varepsilon b(\mathbf{D}) P_\mathcal{O}) \mathbf{f} - A^0 \mathbf{f} \|_{H^{-1}(\mathcal{O})} \leqslant C_1 \varepsilon \| \mathbf{f} \|_{H^2(\mathcal{O})}.$$

2 Hyperbolic systems. Main result

2.1 Problem setting

Let $\mathbf{u}_\varepsilon$ be the solution of the first initial-boundary value problem for the hyperbolic system:

$$\begin{cases} (\partial_t^2 + A_\varepsilon)\mathbf{u}_\varepsilon(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t), \quad \mathbf{x} \in \mathcal{O},\ t \in (0,T), \\ \mathbf{u}_\varepsilon|_{\partial\mathcal{O}} = 0, \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \quad (\partial_t \mathbf{u}_\varepsilon)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}. \end{cases} \tag{2.1}$$

Here the initial data $\boldsymbol{\phi} \in H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n)$, $\boldsymbol{\psi} \in H^1_0(\mathcal{O};\mathbb{C}^n)$, and the right-hand side $\mathbf{F} \in L_1((0,T);H^1_0(\mathcal{O};\mathbb{C}^n))$ for some $0 < T \leqslant \infty$. When

$$\begin{aligned} \mathbf{u}_\varepsilon(\cdot,t) &= \cos(tA_{D,\varepsilon}^{1/2})\boldsymbol{\phi} + A_{D,\varepsilon}^{-1/2}\sin(tA_{D,\varepsilon}^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t A_{D,\varepsilon}^{-1/2}\sin((t-s)A_{D,\varepsilon}^{1/2})\mathbf{F}(\cdot,s)\,ds. \end{aligned} \tag{2.2}$$

2.2 The effective problem

The effective problem has the form

$$\begin{cases} (\partial_t^2 + A^0)\mathbf{u}_0(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t), \quad \mathbf{x} \in \mathcal{O},\ t \in (0,T), \\ \mathbf{u}_0|_{\partial\mathcal{O}} = 0, \\ \mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \quad (\partial_t \mathbf{u}_0)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O}. \end{cases} \tag{2.3}$$

We have

$$\begin{aligned} \mathbf{u}_0(\cdot,t) &= \cos(t(A_D^0)^{1/2})\boldsymbol{\phi} + (A_D^0)^{-1/2}\sin(t(A_D^0)^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t (A_D^0)^{-1/2}\sin((t-s)(A_D^0)^{1/2})\mathbf{F}(\cdot,s)\,ds. \end{aligned} \tag{2.4}$$

So,

$$\begin{aligned} \|\mathbf{u}_0(\cdot,t)\|_{H^2(\mathcal{O})} &\leqslant \|\boldsymbol{\phi}\|_{H^2(\mathcal{O})} + \|(\mathbf{D}^2+I)(A_D^0)^{-1/2}\boldsymbol{\psi}\|_{L_2(\mathcal{O})} \\ &\quad + \int_0^t \|(\mathbf{D}^2+I)(A_D^0)^{-1/2}\mathbf{F}(\cdot,s)\|_{L_2(\mathcal{O})}\,ds. \end{aligned} \tag{2.5}$$

By (1.11),

$$\begin{aligned} \|(\mathbf{D}^2+I)(A_D^0)^{-1/2}\boldsymbol{\psi}\|_{L_2(\mathcal{O})} &\leqslant \|(\mathbf{D}^2+I)^{1/2}(A_D^0)^{-1/2}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \\ &\quad \times \|(\mathbf{D}^2+I)^{1/2}\boldsymbol{\psi}\|_{L_2(\mathcal{O})} \leqslant c_*\|\boldsymbol{\psi}\|_{H^1(\mathcal{O})}. \end{aligned}$$

The summand with $\mathbf{F}$ in (2.5) can be estimated in the same manner. Combining this with (2.5), we get

$$\|\mathbf{u}_0(\cdot,t)\|_{H^2(\mathcal{O})} \leqslant \|\boldsymbol{\phi}\|_{H^2(\mathcal{O})} + c_*\|\boldsymbol{\psi}\|_{H^1(\mathcal{O})} + c_*\|\mathbf{F}\|_{L_1((0,t);H^1(\mathcal{O}))}. \tag{2.6}$$

2.3 Homogenization of solutions of the first initial-boundary value problem for hyperbolic systems

Our *main result* is the following theorem.

Theorem 2.1. *Under the assumptions of Subsections* 1.1–1.3 *and* 2.1, 2.2, *for* $0 < \varepsilon \leqslant 1$ *and* $t \in (0, T)$, *we have*

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot, t) - \mathbf{u}_0(\cdot, t)\|_{L_2(\mathcal{O})} \\ &\quad\leqslant C_2 \varepsilon (1 + |t|) \left(\|\boldsymbol{\phi}\|_{H^2(\mathcal{O})} + \|\boldsymbol{\psi}\|_{H^1(\mathcal{O})} + \|\mathbf{F}\|_{L_1((0,t);H^1(\mathcal{O}))} \right). \end{aligned} \tag{2.7}$$

The constant C_2 *depends only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

Remark 2.2. *Since the class of operators* $A_{D,\varepsilon}$ *under consideration includes the acoustics operator, the system of elasticity theory, and the model equation of electrodynamics* (*see examples in* [BSu1, *Chapters* 5 *and* 7]), *one can tautologically rewrite Theorem* 2.1 *as a homogenization result for the acoustics equation, for the system of elasticity theory, and for the model equation of electrodynamics.*

2.4 Main results in operator terms

Since functions $\boldsymbol{\phi} \in H^2(\mathcal{O}; \mathbb{C}^n) \cap H^1_0(\mathcal{O}; \mathbb{C}^n)$ and $\boldsymbol{\psi} \in H^1_0(\mathcal{O}; \mathbb{C}^n)$ in (2.2) and (2.4) are arbitrarily and one can take $\mathbf{F} = 0$, Theorem 2.1 admits formulation in operator terms.

Theorem 2.3. *Let* $\mathcal{O}$ *be a bounded domain of class* $C^{1,1}$. *Let the assumptions of Subsections* 1.1–1.3 *be satisfied. For* $0 < \varepsilon \leqslant 1$, $t \in \mathbb{R}$, *we have*

$$\| \cos(t A_{D,\varepsilon}^{1/2}) - \cos(t (A_D^0)^{1/2}) \|_{H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}) \to L_2(\mathcal{O})} \leqslant C_2 \varepsilon (1 + |t|),$$
$$\| A_{D,\varepsilon}^{-1/2} \sin(t A_{D,\varepsilon}^{1/2}) - (A_D^0)^{-1/2} \sin(t (A_D^0)^{1/2}) \|_{H^1_0(\mathcal{O}) \to L_2(\mathcal{O})} \leqslant C_2 \varepsilon (1 + |t|).$$

Here the space $H^2(\mathcal{O}; \mathbb{C}^n) \cap H^1_0(\mathcal{O}; \mathbb{C}^n)$ *is equipped with the* H^2*-norm. The constant* C_2 *depends only on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the parameters of the lattice* Γ, *and the domain* $\mathcal{O}$.

2.5 Start of the proof of Theorem 2.1. Discrepancy

Let $\mathbf{v}_\varepsilon$ be the first order approximation to the solution $\mathbf{u}_\varepsilon$:

$$\mathbf{v}_\varepsilon = \mathbf{u}_0 + \varepsilon R_{\mathcal{O}} \Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\mathbf{u}}_0. \tag{2.8}$$

Here $\widetilde{\mathbf{u}}_0 := P_\mathcal{O}\mathbf{u}_0$, where $P_\mathcal{O}$ is the extension operator (1.12).

The function $\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon$ does not satisfy the Dirichlet boundary condition. It is convenient to introduce the discrepancy $\mathbf{w}_\varepsilon$ as the weak solution of the problem

$$\begin{cases} (\partial_t^2 + A_\varepsilon)\mathbf{w}_\varepsilon = \varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\partial_t^2\widetilde{\mathbf{u}}_0 \quad \text{in}\,\mathcal{O}, \\ \mathbf{w}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0|_{\partial\mathcal{O}}, \\ \mathbf{w}_\varepsilon(\cdot,0) = \varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})P_\mathcal{O}\boldsymbol{\phi}, \quad (\partial_t\mathbf{w}_\varepsilon)(\cdot,0) = \varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})P_\mathcal{O}\boldsymbol{\psi}. \end{cases} \tag{2.9}$$

Let us write (2.9) as

$$\begin{cases} (\partial_t^2 + A_\varepsilon)(\mathbf{w}_\varepsilon - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0) = -A_\varepsilon\varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0 \quad \text{in}\,\mathcal{O}, \\ (\mathbf{w}_\varepsilon - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)|_{\partial\mathcal{O}} = 0, \\ (\mathbf{w}_\varepsilon - \varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\cdot,0) = 0, \quad \partial_t(\mathbf{w}_\varepsilon - \varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\cdot,0) = 0. \end{cases}$$

So,

$$\begin{aligned} &(\mathbf{w}_\varepsilon - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0)(\cdot,t) = \\ &\quad -\int_0^t A_{D,\varepsilon}^{-1/2}\sin((t-s)A_{D,\varepsilon}^{1/2})A_\varepsilon\varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\cdot,s)\,ds =: \mathcal{I}(\varepsilon,t). \end{aligned} \tag{2.10}$$

Lemma 2.4. *Let $\mathbf{u}_0$ be the solution of effective problem* (2.3)*. Let the discrepancy $\mathbf{w}_\varepsilon$ be the solution of problem* (2.9)*. For $0 < \varepsilon \leqslant 1$ and $t \in (0,T)$ we have*

$$\begin{aligned} &\|\mathbf{w}_\varepsilon(\cdot,t) - \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\cdot,t)\|_{L_2(\mathcal{O})} \\ &\quad \leqslant C_3\varepsilon(1+|t|)(\|\boldsymbol{\phi}\|_{H^2(\mathcal{O})} + \|\boldsymbol{\psi}\|_{H^1(\mathcal{O})} + \|\mathbf{F}\|_{L_1((0,t);H^1(\mathcal{O}))}). \end{aligned} \tag{2.11}$$

The constant C_3 depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Proof. Integrating by parts, we have

$$\begin{aligned} \mathcal{I}(\varepsilon,t) &= -\int_0^t \frac{d\cos((t-s)A_{D,\varepsilon}^{1/2})}{ds}A_{D,\varepsilon}^{-1}A_\varepsilon\varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\cdot,s)\,ds \\ &= -A_{D,\varepsilon}^{-1}A_\varepsilon\varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\cdot,t) \\ &+ \cos(tA_{D,\varepsilon}^{1/2})A_{D,\varepsilon}^{-1}A_\varepsilon\varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\cdot,0) \\ &+ \int_0^t \cos((t-s)A_{D,\varepsilon}^{1/2})A_{D,\varepsilon}^{-1}A_\varepsilon\varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\partial_s\widetilde{\mathbf{u}}_0(\cdot,s)\,ds. \end{aligned}$$

By Lemma 1.1 and (2.3),

$$\|\mathcal{I}(\varepsilon,t)\|_{L_2(\mathcal{O})} \leqslant \varepsilon\|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\cdot,t)\|_{L_2(\mathcal{O})} + \varepsilon\|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})P_\mathcal{O}\boldsymbol{\phi}\|_{L_2(\mathcal{O})} + \varepsilon\int_0^t \|\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\partial_s\widetilde{\mathbf{u}}_0(\cdot,s)\|_{L_2(\mathcal{O})}\,ds.$$

Using (1.2), (1.7), (1.13), and Proposition 1.4, we derive that

$$\begin{aligned}\|\mathcal{I}(\varepsilon,t)\|_{L_2(\mathcal{O})} &\leqslant \varepsilon M\alpha_1^{1/2}C_\mathcal{O}\|\widetilde{\mathbf{u}}_0(\cdot,t)\|_{H^1(\mathcal{O})} + \varepsilon M\alpha_1^{1/2}C_\mathcal{O}\|\boldsymbol{\phi}\|_{H^1(\mathcal{O})}\\ &\quad + \varepsilon M\alpha_1^{1/2}\int_0^t \|\mathbf{D}\partial_s\widetilde{\mathbf{u}}_0(\cdot,s)\|_{L_2(\mathcal{O})}\,ds.\end{aligned} \tag{2.12}$$

By (1.3), (1.8), (1.13), and (2.4),

$$\begin{aligned}&\|\mathbf{D}\partial_s\widetilde{\mathbf{u}}_0(\cdot,s)\|_{L_2(\mathbb{R}^d)}\\ &= \Big\|\mathbf{D}P_\mathcal{O}\Big(-(A_D^0)^{1/2}\sin(s(A_D^0)^{1/2})\boldsymbol{\phi} + \cos(s(A_D^0)^{1/2})\boldsymbol{\psi}\\ &+ \int_0^t \cos((t-s)(A_D^0)^{1/2})\mathbf{F}(\cdot,s)\,ds\Big)\Big\|_{L_2(\mathbb{R}^d)}\\ &\leqslant C_\mathcal{O}\left(\alpha_1^{1/2}d^{1/2}\|g\|_{L_\infty}^{1/2}\|\boldsymbol{\phi}\|_{H^2(\mathcal{O})} + \|\boldsymbol{\psi}\|_{H^1(\mathcal{O})} + \|\mathbf{F}\|_{L_1((0,t);H^1(\mathcal{O}))}\right).\end{aligned} \tag{2.13}$$

(We assumed that $\boldsymbol{\phi}\in H^2(\mathcal{O};\mathbb{C}^n)\cap H_0^1(\mathcal{O};\mathbb{C}^n)$, so we can apply the operator $(A_D^0)^{1/2}$ to $\boldsymbol{\phi}$.)

Combining (2.6), (2.12), and (2.13), we get

$$\|\mathcal{I}(\varepsilon,t)\|_{L_2(\mathcal{O})} \leqslant C_3\varepsilon(1+|t|)(\|\boldsymbol{\phi}\|_{H^2(\mathcal{O})}+\|\boldsymbol{\psi}\|_{H^1(\mathcal{O})}+\|\mathbf{F}\|_{L_1((0,t);H^1(\mathcal{O}))}), \tag{2.14}$$

where $C_3 := M\alpha_1^{1/2}C_\mathcal{O}\max\{2;c_*;\alpha_1^{1/2}d^{1/2}\|g\|_{L_\infty}\}$. Taking (2.10) into account, we arrive at estimate (2.11). $\square$

2.6 End of the proof of Theorem 2.1. L_2-estimate for the function $\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon$

We have

$$\begin{cases}(\partial_t^2 + A_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon) = A^0\mathbf{u}_0 - A_\varepsilon\left(I + \varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})P_\mathcal{O}\right)\mathbf{u}_0 \quad \text{in } \mathcal{O},\\ \mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon|_{\partial\mathcal{O}} = 0,\\ (\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)(\cdot,0) = 0, \quad \partial_t(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)(\cdot,0) = 0.\end{cases}$$

So,

$$\begin{aligned}&(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)(\cdot,t)\\ &= \int_0^t A_{D,\varepsilon}^{-1/2}\sin((t-s)A_{D,\varepsilon}^{1/2})\left(A^0\mathbf{u}_0 - A_\varepsilon\left(I + \varepsilon R_\mathcal{O}\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})P_\mathcal{O}\right)\mathbf{u}_0\right)(\cdot,s)\,ds.\end{aligned}$$

By Corollary 1.6,

$$\|(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)(\cdot,t)\|_{L_2(\mathcal{O})} \leqslant \int_0^t \|A_{D,\varepsilon}^{-1/2}\|_{H^{-1}(\mathcal{O})\to L_2(\mathcal{O})} C_1 \varepsilon \|\mathbf{u}_0(\cdot,s)\|_{H^2(\mathcal{O})}\, ds.$$

(Our assumptions on $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, and $\mathbf{F}$ guarantee that $\mathbf{u}_0(\cdot,t) \in H^2(\mathcal{O};\mathbb{C}^n) \cap H^1_0(\mathcal{O};\mathbb{C}^n)$.) Together with (1.5) and (2.6) this implies

$$\begin{aligned}&\|(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)(\cdot,t)\|_{L_2(\mathcal{O})}\\ &\quad \leqslant c_* C_1 \varepsilon |t| (\|\boldsymbol{\phi}\|_{H^2(\mathcal{O})} + c_* \|\boldsymbol{\psi}\|_{H^1(\mathcal{O})} + c_* \|\mathbf{F}\|_{L_1((0,t);H^1(\mathcal{O}))}).\end{aligned}$$

We arrive at the following result.

Lemma 2.5. *Let $\mathbf{u}_\varepsilon$ be the solution of the problem* (2.1)*. Let $\mathbf{v}_\varepsilon$ be the first order approximation* (2.8) *to the solution $\mathbf{u}_\varepsilon$, where $\mathbf{u}_0$ is the solution of the effective problem* (2.3)*. Let the discrepancy $\mathbf{w}_\varepsilon$ be the solution of the problem* (2.9)*. When for $0 < \varepsilon \leqslant 1$ and $t \in (0,T)$ we have*

$$\|(\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon)(\cdot,t)\|_{L_2(\mathcal{O})} \leqslant C_4 \varepsilon |t| (\|\boldsymbol{\phi}\|_{H^2(\mathcal{O})} + \|\boldsymbol{\psi}\|_{H^1(\mathcal{O})} + \|\mathbf{F}\|_{L_1((0,t);H^1(\mathcal{O}))}).$$

The constant $C_4 = c_* C_1 \max\{1; c_*\}$ depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the domain $\mathcal{O}$.

Together with identity (2.8) and inequality (2.11), this result imply estimate (2.7) with the constant $C_2 := C_3 + C_4$. The proof of Theorem 2.1 is complete.

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