New bounds on Mertens function

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Abstract

In this brief paper we study and bound Mertens function, which is defined for all positive integers as $M(n) = \sum_{k=1}^{n} \mu(k)$, where $\mu(x)$ is the Möbius function. The main breakthrough is the obtention of a Möbius-invertible formulation of Mertens function, which with some transformations and the application of the generalization of Möbius inversion formula, allows us to reach the asymptotic equivalence

$$|M(n)| \sim \sqrt{n} \cdot \frac{\log(\sqrt{n})}{\log\log(\sqrt{n})}$$

MSC2020: 11A41

1 Introduction

In 1859, in his paper "*On the Number of Primes Less Than a Given Magnitude*" [3], Bernhard Riemann published the assumption that all non-trivial zero-points of the zeta function extended to the range of complex numbers $\mathbf{C}$ have a real part of $\frac{1}{2}$. Ever since David Hilbert in 1900 added this problem to his list of the 23 most important problems of 20^{th} century, mathematicians have been working on finding evidence for the Riemann hypothesis.

Other hand, for any positive integer n, we define the Möbius function $\mu(n)$ as having the following values depending on the factorization of n into prime factors:

- $\mu(n) = 1$ if n is a square-free positive integer with an even number of prime factors.
- $\mu(n) = -1$ if n is a square-free positive integer with an odd number of prime factors.
- $\mu(n) = 0$ if n has a squared prime factor.

Merten's function $M(n)$ is the summatory function of the Möbius function, so it is defined for all positive integers as

$$M(n) = \sum_{k=1}^{n} \mu(k) \tag{1}$$

The value of Mertens function is closely connected to Riemann hypothesis through the identity

$$\frac{1}{\zeta(s)} = s\int_{1}^{\infty} M(x)x^{-s-1}dx \tag{2}$$

This identity is valid for $\text{Re}(s) > 1$, and $\zeta(s)$ is the Riemann zeta function. If $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$, then the integral converges for $\text{Re}(s) > \frac{1}{2}$, implying that $\frac{1}{\zeta(s)}$ has no poles in this region and that the Riemann hypothesis is true. Concretely, we have that

Theorem (Littlewood):[2] *Riemmann's hypothesis is equivalent to the statement: for every $\epsilon > 0$ the function $M(x)/x^{1/2+\epsilon}$ approaches zero as $x \to \infty$.*

In this brief paper we study and bound Mertens function. Using some transformations and a generalization of Möbius inversion formula, we are able to reach the asymptotic equivalence

$$|M(n)| \sim \sqrt{n} \cdot \frac{\log(\sqrt{n})}{\log\log(\sqrt{n})}$$

2 A new reformulation of Mertens function

From the definition of Möbius function, we have that

$$\sum_{k=1} \frac{\mu\left(k\right)}{k} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \dots \tag{3}$$

Where k runs over the square-free integers.

It is straightforward from the definition of $\mu\left(n\right)$ to note that

$$\sum_{k \leq n} \frac{\mu(k)}{k} = 1 - \sum_{p_i \leq n} \left(\frac{1}{p_i}\right) + \sum_{p_i < p_j \leq \frac{n}{p_i}} \left(\frac{1}{p_i p_j}\right) - \sum_{p_i < p_j < p_k \leq \frac{n}{p_i p_j}} \left(\frac{1}{p_i p_j p_k}\right) + \dots \tag{4}$$

Other hand, as already stated in (1), we have that Merten's function $M\left(n\right)$ is defined for all positive integers as

$$M\left(n\right) = \sum_{k=1}^{n} \mu\left(k\right)$$

Starting from (4), it is pretty straightforward to obtain that

$$M\left(n\right) = 1 - \pi\left(n\right) + \sum_{p_i \leq \frac{n}{p_i}} \left(\pi\left(\frac{n}{p_i}\right) - i\right) - \sum_{p_i < p_j \leq \frac{n}{p_i p_j}} \left(\pi\left(\frac{n}{p_i p_j}\right) - j\right) +$$
$$+ \sum_{p_i < p_j < p_k \leq \frac{n}{p_i p_j p_k}} \left(\pi\left(\frac{n}{p_i p_j p_k}\right) - k\right) - \dots$$

Where $\pi(x)$ is the prime counting function.

It can be noted that the above expansion can be re-expressed more compactly as

$$M(n) = -\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k) \pi\left(\frac{n}{k}\right) + S(n) \tag{5}$$

Where $p_{\pi(\sqrt{n})}$ is the greatest prime number less than $\sqrt{n}$, and where we have that

$$S(n) = -\sum_{p_i \leq \frac{n}{p_i}} i + \sum_{p_i < p_j \leq \frac{n}{p_i p_j}} j - \sum_{p_i < p_j < p_k \leq \frac{n}{p_i p_j p_k}} k + \dots$$

It is worthy to have a look at the expansion of $S(n)$, which looks as follows:

$$S(n) = -(1 + 2 + 3 + \dots + \pi(\sqrt{n}) + ((2 + 3 + ... + \pi\left(\sqrt{\frac{n}{2}}\right)) + (3 + 4 + ... + \pi\left(\sqrt{\frac{n}{3}}\right)) + ...$$
$$+(k + (k+1) + ... + \pi\left(\sqrt{\frac{n}{k}}\right))) - \dots$$

Working on this expansion leads us to a closed form for $S(n)$, which can be expressed as:

$$S(n) = -\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k) \left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right) + \lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right) - \lambda + 1\right)\right)}{2}\right) \tag{6}$$

Where $\lambda = \omega(k) + \pi(gpf(k))$, $\omega(k)$ counts the number of distinct prime factors of k, and $\pi(gpf(k))$ counts the number of prime numbers equal or less than the greatest prime factor of k.

At the end, from (5) and (6), we get that

$$M(n) = -\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu(k) \left(\pi\left(\frac{n}{k}\right) + \left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right) + \lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right) - \lambda + 1\right)\right)}{2}\right)\right) \tag{7}$$

3 Bounding Mertens function

3.1 Transformations and simplifications

Firstly, we need to bound the difference

$$d_{n,k} = \frac{\pi\left(\frac{n}{k}\right)}{\left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right)+\lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right)-\lambda+1\right)\right)}{2}\right)} \tag{8}$$

For the purpose of this paper, it suffices to note that the minimum difference is obtained when $\lambda = 0$, and that this happens only when $k = 1$. For $k = 1$, we have that

$$\left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right)\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right)+1\right)\right)}{2}\right) = \left(\frac{\left(\pi\left(\sqrt{n}\right)\right)\left(\left(\pi\left(\sqrt{n}\right)+1\right)\right)}{2}\right)$$

Applying one of the best currently known explicit bounds for $\pi(x)$[1], we have that for $x > 6$

$$\frac{x}{\log(x)} < \pi(x) \leq \frac{x}{\log(x)}\left(1 + \frac{1}{\log(x)} + \frac{2}{\log^2(x)} + \frac{7.59}{\log^3(x)}\right) \tag{9}$$

And therefore, after substituting and operating, we have that

$$\left(\frac{\left(\pi\left(\sqrt{n}\right)\right)\left(\left(\pi\left(\sqrt{n}\right)+1\right)\right)}{2}\right) \leq \frac{n}{2\log^2(\sqrt{n})}\left(1 + \frac{1}{\log^2(\sqrt{n})} + \frac{4}{\log^4(\sqrt{n})} + \frac{7.59^2}{\log^6(\sqrt{n})}\right) + \frac{\sqrt{n}}{2\log(\sqrt{n})}\left(1 + \frac{1}{\log(\sqrt{n})} + \frac{4}{\log^2(\sqrt{n})} + \frac{7.59^2}{\log^3(\sqrt{n})}\right) \tag{11}$$

For the purpose of this paper, it is sufficient to note that, applying the explicit bounds settled before, we have that, for $x > 6$,

$$d_{n,k} \geq \frac{\log(\sqrt{n})}{2} \tag{10}$$

As a result, we have that, for $n > 6$, for all k,

$$\frac{2 \cdot \pi\left(\frac{n}{k}\right)}{\log(\sqrt{n})} > \left(\frac{\left(\pi\left(\sqrt{\frac{n}{k}}\right)+\lambda\right)\left(\left(\pi\left(\sqrt{\frac{n}{k}}\right)-\lambda+1\right)\right)}{2}\right) \tag{11}$$

Now note that, if we have two functions $f(x)$ and $g(x)$ such that $f\left(\frac{n}{k}\right) \leq g\left(\frac{n}{k}\right)$ for each value of n, k, then

$$\left|\sum_{k=1}^{n} \mu(k) f\left(\frac{n}{k}\right)\right| \leq \left|\sum_{k=1}^{n} \mu(k) g\left(\frac{n}{k}\right)\right|$$

Therefore, substituting in (7) an reordering, we have that

$$\left|\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu\left(k\right)\pi\left(\frac{n}{k}\right)\right| <\mid M(n) \mid< \left|\left(1 + \frac{2}{\log(\sqrt{n})}\right) \cdot \left(\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu\left(k\right)\pi\left(\frac{n}{k}\right)\right)\right| \tag{12}$$

This last expression is clue for the purpose of this paper, as it implies that, as n grows to infinity,

$$\mid M(n) \mid \sim \left|\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu\left(k\right)\pi\left(\frac{n}{k}\right)\right|$$

Getting some asymptotic equivalence of $\left|\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}} \mu\left(k\right)\pi\left(\frac{n}{k}\right)\right|$ would yield an asymptotic equivalence of $\mid M(n) \mid$, which in turn can prove or disprove the Riemann hypothesis.

3.2 Application of Möbius inversion formula

In this section we will apply a generalization of Möbius inversion formula [4] and some transformations. Concretely, if $F(x)$ and $G(x)$ are complex-valued functions such that, as x grows to infinity, $G(x) \sim \sum_{k=1}^{x} F\left(\frac{x}{k}\right)$, we have that $F(x) \sim \sum_{k=1}^{x} \mu(k) G\left(\frac{x}{k}\right)$.

The Prime Number Theorem yields that, when n grows to infinity, $p_{\pi(\sqrt{n})} \sim \pi(\sqrt{n}) \cdot \log(\pi(\sqrt{n}))$ and $\pi(\sqrt{n}) \sim \frac{\sqrt{n}}{\log(\sqrt{n})}$. Thus, we have that

$$p_{\pi(\sqrt{n})} \sim \frac{\sqrt{n}}{\log(\sqrt{n})} \cdot \log\left(\frac{\sqrt{n}}{\log(\sqrt{n})}\right) = \sqrt{n} \cdot \left(1 - \frac{\log\log(\sqrt{n})}{\log(\sqrt{n})}\right)$$

For readability reasons, from now on we establish that

$$\alpha = 1 - \frac{\log\log(\sqrt{n})}{\log(\sqrt{n})}$$

Therefore, we have that, as n grow to infinity, $p_{\pi(\sqrt{n})} \sim \alpha\sqrt{n}$, and therefore $\frac{n}{p_{\pi(\sqrt{n})}} \sim \frac{\sqrt{n}}{\alpha}$.

By the Prime Number Theorem and the explicit bounds for the Prime counting function, we have that $\pi(x) \sim \frac{x}{\log(x)}$. Therefore, if we set $G(x) = \frac{x}{\log(x)}$, we have that

$$\left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}} \mu(k)\pi\left(\frac{n}{k}\right)\right| \sim \left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}} \mu(k) G\left(\frac{n}{k}\right)\right|$$

To apply the generalization of Möbius inversion formula to the right hand side of the above asymptotic, we need to find $F(x)$ such that we have that

$$G(n) \sim \sum_{k=1}^{n} F\left(\frac{n}{k}\right)$$

Or, substituting,

$$\frac{n}{\log(n)} \sim \sum_{k=1}^{n} F\left(\frac{n}{k}\right) \tag{13}$$

Using Stirling's and Riemann sums approximations, when n grows to infinity, we have that

$$\sum_{k=1}^{n} \frac{\log\left(\frac{n}{k}\right)}{\log\log\left(\frac{n}{k}\right)} \sim \frac{n}{\log(n)}$$

Therefore, we have that

$$F\left(n\right) \sim \frac{\log(n)}{\log\log(n)} \tag{14}$$

As a result, applying the generalization of Möbius inversion formula, we get that

$$\left|\sum_{k=1}^{n} \mu(k) G\left(\frac{n}{k}\right)\right| \sim \frac{\log(n)}{\log\log(n)} \tag{15}$$

From this result, we can just substitute n with $\frac{\sqrt{n}}{\alpha}$ to get that

$$\left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}} \mu(k) G\left(\frac{\sqrt{n}}{\alpha \cdot k}\right)\right| \sim \frac{\log\left(\frac{\sqrt{n}}{\alpha}\right)}{\log\log\left(\frac{\sqrt{n}}{\alpha}\right)}$$

Substituting, we have that

$$G\left(\frac{\sqrt{n}}{\alpha \cdot k}\right) \sim \frac{n}{\alpha\sqrt{n} \cdot k \cdot \log\left(\frac{n}{k}\right) - \alpha\sqrt{n} \cdot k \cdot \log\left(\alpha\sqrt{n}\right)}$$

Operating, we have that

$$\frac{1}{G\left(\frac{\sqrt{n}}{\alpha\cdot k}\right)} \sim \frac{\alpha\cdot\sqrt{n}}{G\left(\frac{n}{k}\right)} - \frac{\alpha\cdot k\cdot\log(\alpha\sqrt{n})}{\sqrt{n}}$$

Operating a bit more, we get that

$$G\left(\frac{\sqrt{n}}{\alpha\cdot k}\right) \sim G\left(\frac{n}{k}\right)\cdot\left(\frac{\sqrt{n}}{\alpha\cdot n - \alpha\cdot k\cdot G\left(\frac{n}{k}\right)}\right)$$

$$G\left(\frac{\sqrt{n}}{\alpha\cdot k}\right) \sim G\left(\frac{n}{k}\right)\cdot\frac{\sqrt{n}}{\alpha}\cdot\left(\frac{1}{n - k\cdot G\left(\frac{n}{k}\right)}\right)$$

As we have that

$$k\cdot G\left(\frac{n}{k}\right) = k\cdot\frac{n}{k\log\left(\frac{n}{k}\right)} = \frac{n}{\log\left(\frac{n}{k}\right)}$$

We finally get that

$$G\left(\frac{\sqrt{n}}{\alpha\cdot k}\right) \sim G\left(\frac{n}{k}\right)\cdot\frac{\sqrt{n}}{\alpha}\cdot\left(\frac{1}{n\cdot\left(1-\frac{1}{\log\left(\frac{n}{k}\right)}\right)}\right)$$

$$G\left(\frac{\sqrt{n}}{\alpha\cdot k}\right) \sim G\left(\frac{n}{k}\right)\cdot\frac{1}{\alpha\sqrt{n}}\cdot\left(\frac{1}{1-\frac{1}{\log\left(\frac{n}{k}\right)}}\right)$$

$$G\left(\frac{\sqrt{n}}{\alpha\cdot k}\right) \sim G\left(\frac{n}{k}\right)\cdot\frac{1}{\alpha\sqrt{n}}\cdot\frac{\log\left(\frac{n}{k}\right)}{\log\left(\frac{n}{k}\right)-1} \sim G\left(\frac{n}{k}\right)\cdot\frac{1}{\alpha\sqrt{n}} \tag{16}$$

Therefore, we have that

$$\left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}}\mu(k)G\left(\frac{n}{k}\right)\right| \sim\ \alpha\sqrt{n}\cdot\left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}}\mu(k)G\left(\frac{\sqrt{n}}{\alpha\cdot k}\right)\right|$$

$$\left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}}\mu(k)G\left(\frac{n}{k}\right)\right| \sim\ \alpha\sqrt{n}\cdot\frac{\log\left(\frac{\sqrt{n}}{\alpha}\right)}{\log\log\left(\frac{\sqrt{n}}{\alpha}\right)}$$

And therefore, we have that

$$\left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}}\mu(k)G\left(\frac{n}{k}\right)\right| \sim\ \sqrt{n}\cdot\frac{\alpha\cdot\log\left(\frac{\sqrt{n}}{\alpha}\right)}{\log\log\left(\frac{\sqrt{n}}{\alpha}\right)}$$

Let us recall that $\alpha = 1-\frac{\log\log(\sqrt{n})}{\log(\sqrt{n})}$. As n grows to infinity, we have that $\alpha\sim 1$. Therefore, we finally have that

$$\left|\sum_{k=1}^{\frac{\sqrt{n}}{\alpha}}\mu(k)G\left(\frac{n}{k}\right)\right| \sim\ \sqrt{n}\cdot\frac{\log\left(\sqrt{n}\right)}{\log\log\left(\sqrt{n}\right)}$$

And thus, as n grows to infinity, we have that

$$\mid M(n)\mid \sim \left|\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}}\mu\left(k\right)\pi\left(\frac{n}{k}\right)\right| \sim \left|\sum_{k=1}^{\frac{n}{p_{\pi(\sqrt{n})}}}\mu(k)G\left(\frac{n}{k}\right)\right| \sim\ \sqrt{n}\cdot\frac{\log\left(\sqrt{n}\right)}{\log\log\left(\sqrt{n}\right)} \tag{17}$$

4 Final Remarks

The asymptotic obtained for the absolute value of Mertens function is sufficient to prove the Riemann Hypothesis, as from the result obtained we have that $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$. Moreover, we are sure that the reformulation obtained in Section 2 and expressed compactly in (7), as well as the explicit bounds set in (12), can be (and will be) improved to get better explicit bounds for Mertens function for sufficiently large x.

I want to specially thank my caring wife Elena for supporting me throughout this marvellous journey of free-time researching and learning during this last eight years. And "*I praise you, Father, Lord of Heaven and Earth, because you have hidden these things from the wise and learned, and revealed them to little children*" (Matthew 11, 25).

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