Proof of the Collatz Conjecture

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Abstract

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2……The inverted Collatz sequences can be represented as a tree with 1 as its root node. In order to prove the Collatz conjecture, one must demonstrate that the tree covers all natural numbers. In this paper, we construct a Collatz tree with 1 as its root node by connecting infinite number of basic trees. Each basic tree relates to each natural number. We prove that a Collatz tree is a connected binary tree and covers all natural numbers.

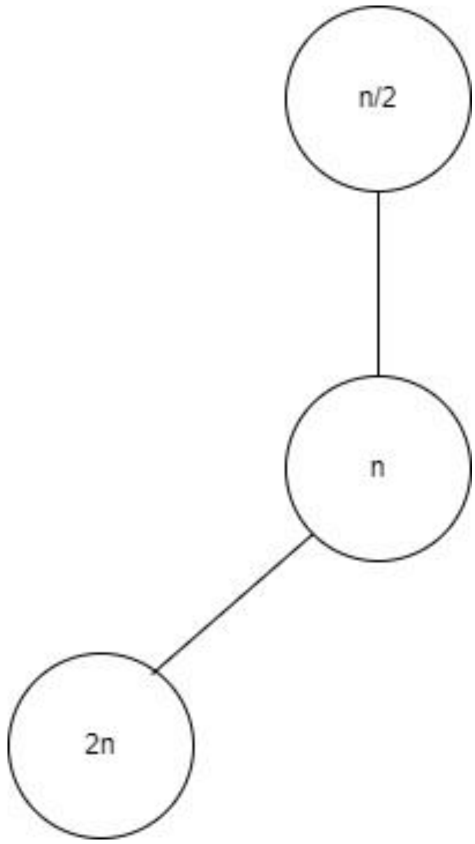
1. Introduction

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2……The inverted Collatz sequences can be represented as a tree with 1 as its root node. In order to prove the Collatz conjecture, one must demonstrate that this tree covers all natural numbers [1].

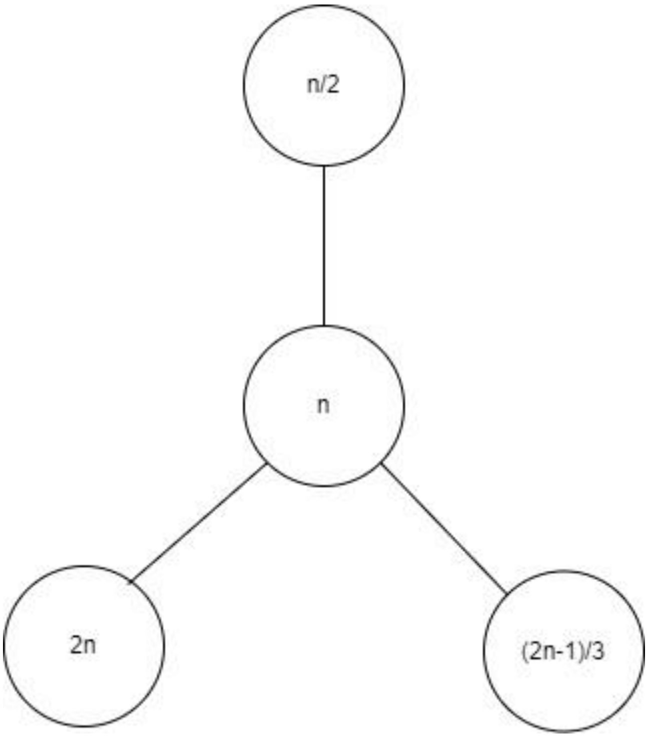
2. **A basic tree**

The basic tree is constructed for each natural number as follows:

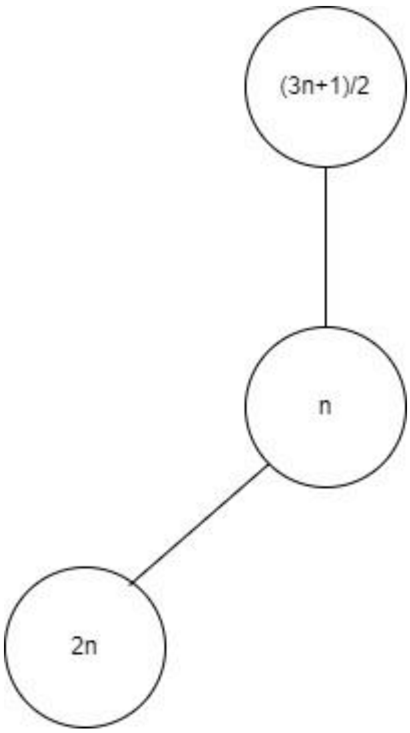
Let n be a positive integer node. Its parent node is $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. Its left child is 2n. Its right child is $\frac{2n-1}{3}$, if n ≡ 2mod3, or no right child , if n ≠ 2mod3. Thus there are four types of basis tree as shown in Figure 1.



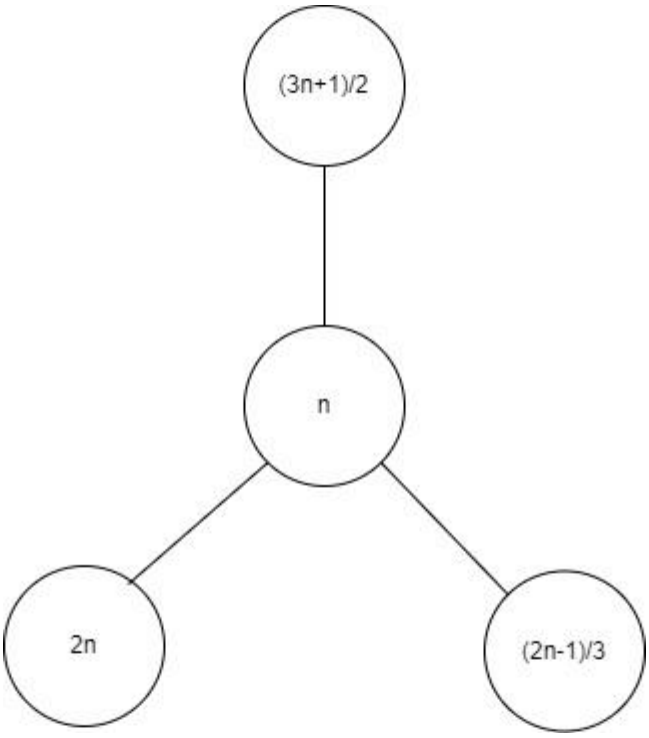
(a) n is even and not equal to 2mod3



(b) n is even and equals to 2mod3



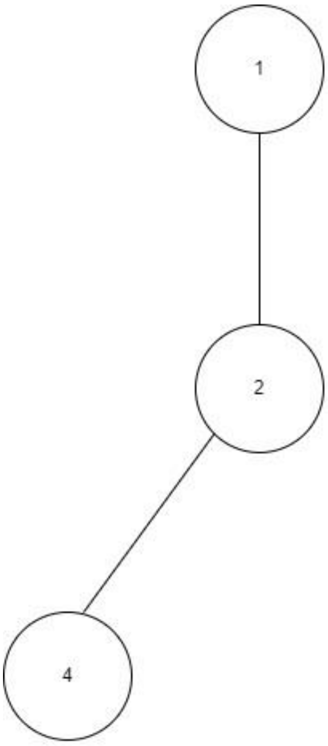
(c) n is odd and not equal to 2mod3

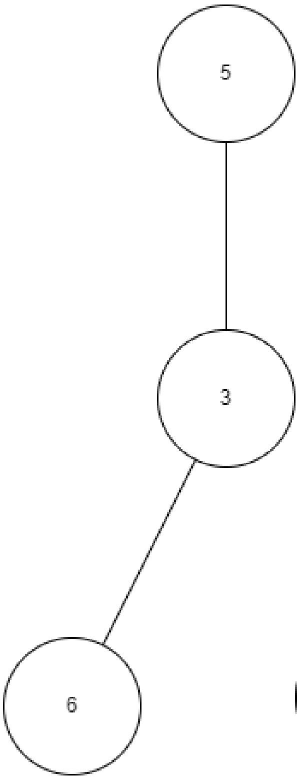


(d) n is odd and equals to 2mod3

Figure 1, four types of basic trees

Examples of basic trees shown in Figure 2.


1
2
4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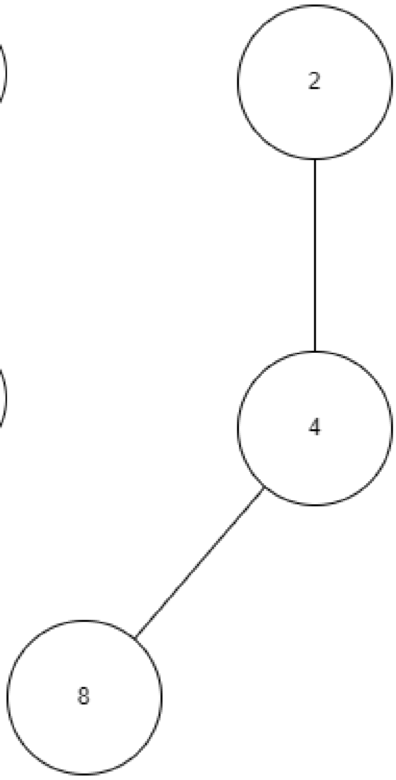
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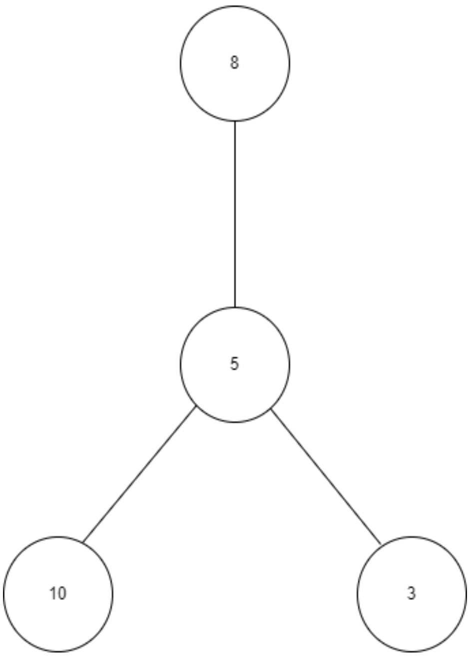
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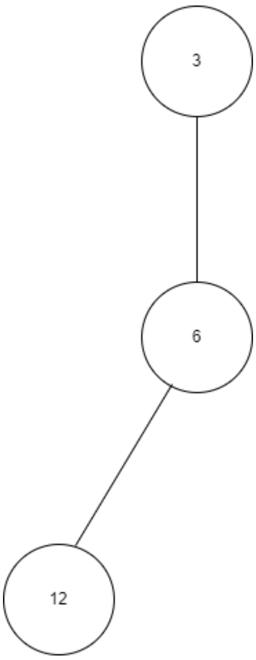
3
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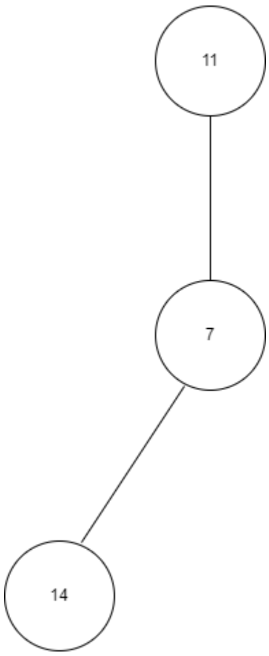
12


11
7
14

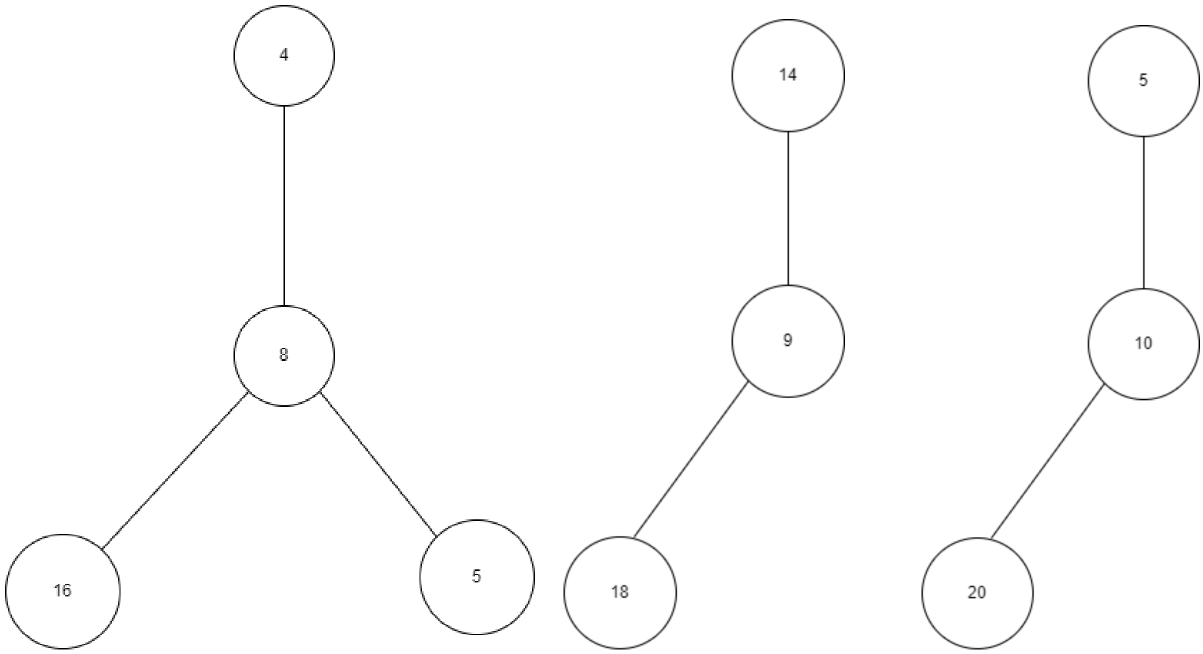
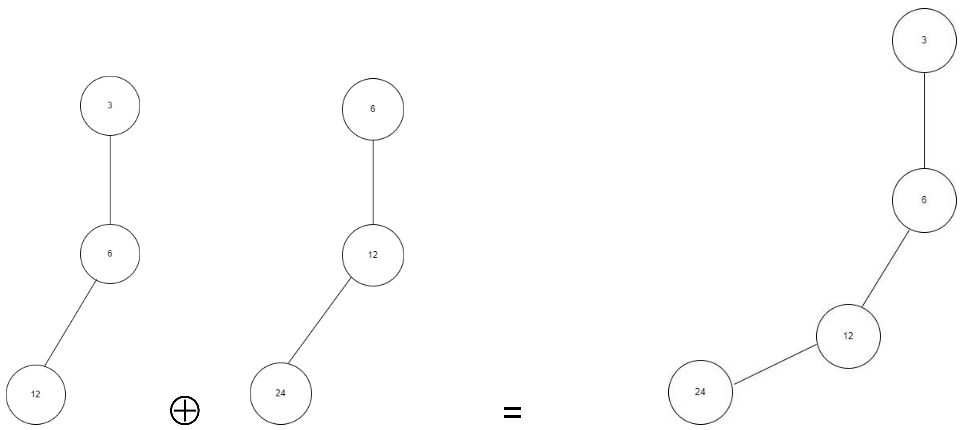


Figure 2. Basic trees of positive integers 2, 3, 4, 5, 6, 7, 8, 9, 10

3. **How to connect two basic trees**

A simple rule to connect two basic trees is that these two basic trees must have a common branch as three cases shown in Figure 3.



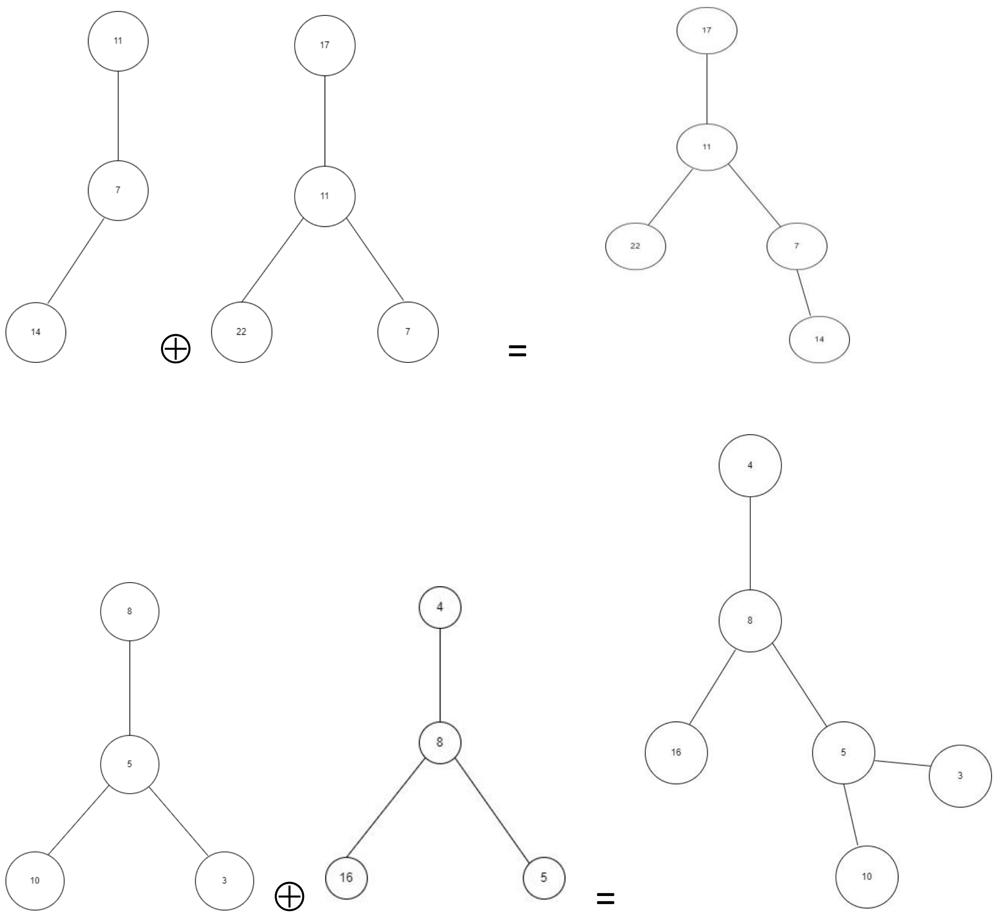


Figure 3. Various cases of basic trees connection

4. **Collatz tree with node 1 as its root node**

By connecting all basic trees , a connected binary tree called a Collatz tree is formed and it covers all natural number as shown in Figure 4.

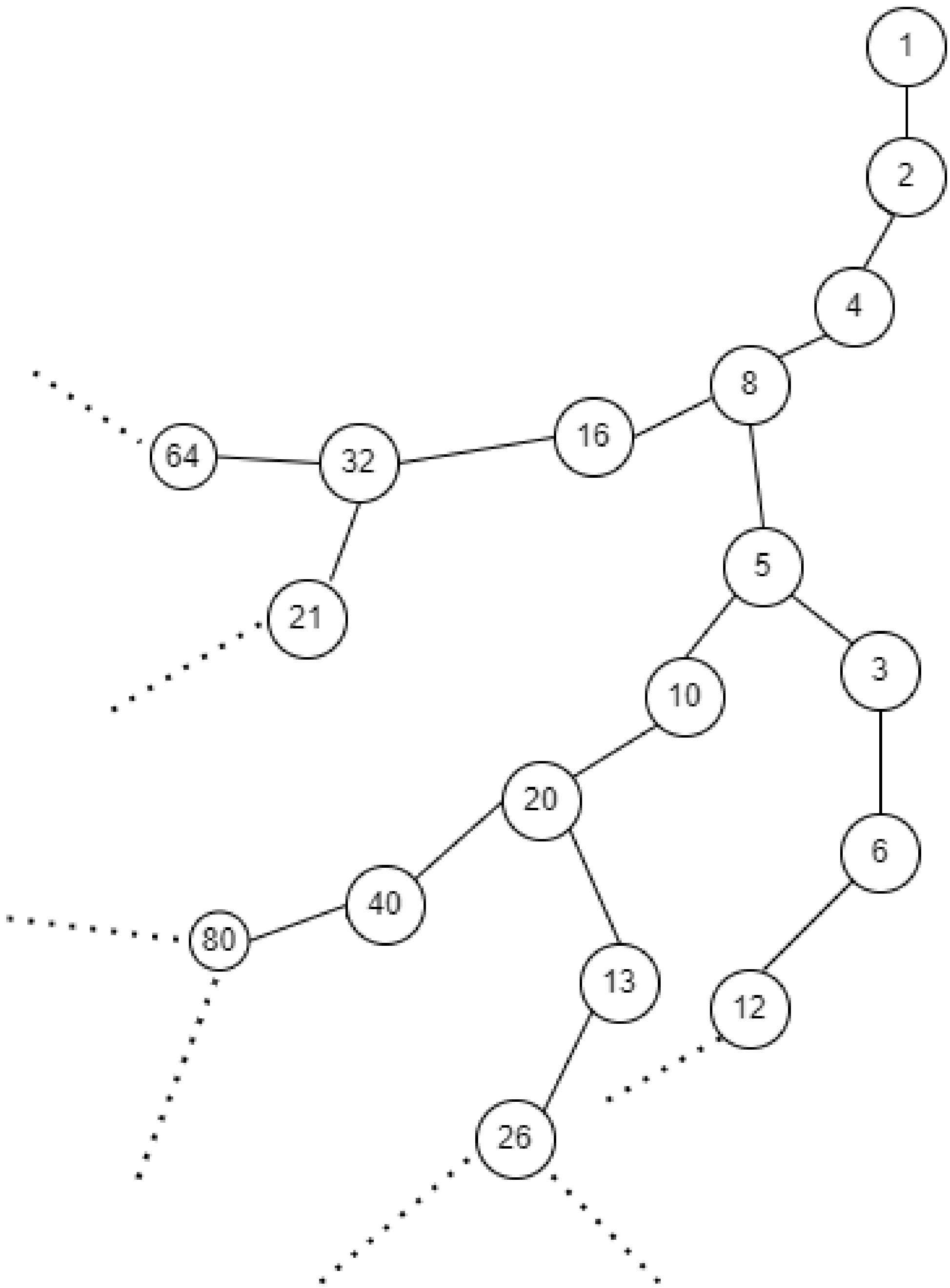


Figure 4. A connected binary tree

5. **Conclusion**

A connected binary tree covers all natural number**.** By starting at any node in a tree, there is a unique path from that node to a node 1.

References

[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.