Extension Formulas and Norm Inequalities in Sobolev Hilbert Spaces

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Abstract: In this note, we shall consider extension formulas and norm inequalities for some typical Sobolev Hilbert spaces. We see many related open problems.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

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1 Introduction

In order to consider the restriction and extension of reproducing kernel Hilbert spaces, we first recall the fundamental general property based on [5].

We consider a positive definite quadratic form function $K : E \times E \to \mathbb{C}$ and its restriction of K to $E_0 \times E_0$, where E_0 is a subset of E. Of course, the restriction is again a positive definite quadratic form function on the subset $E_0 \times E_0$. We shall consider the relation between two reproducing kernel Hilbert spaces derived from the positive definite quadratic form functions.

Theorem A. *Let E_0 be a subset of E. Then the reproducing kernel Hilbert space that $K|E_0 \times E_0 : E_0 \times E_0 \to \mathbb{C}$ defines is given by:*

$$H_{K|E_0\times E_0}(E_0) = \{f \in \mathcal{F}(E_0) \,:\, f = \tilde{f}|E_0 \textit{ for some } \tilde{f} \in H_K(E)\}. \tag{1.1}$$

Furthermore, the norm is expressed in terms of the one of $H_K(E)$:

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} = \min\{\|\tilde{f}\|_{H_K(E)} \,:\, \tilde{f} \in H_K(E),\, f = \tilde{f}|E_0\}. \tag{1.2}$$

In Theorem A, note that the inequality, for any function $f \in H_K(E)$

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} \leq \|f\|_{H_K(E)} \tag{1.3}$$

holds, that is, the restriction map is a bounded linear operator.

At first, we shall consider the simplest Sobolev Hilbert space.

The space $H_S(\mathbb{R})$ is comprising of absolutely continuous functions f on $\mathbb{R}$ with the norm

$$\|f\|_{H_S(\mathbb{R})} \equiv \sqrt{\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2)dx}. \tag{1.4}$$

The Hilbert space $H_S(\mathbb{R})$ admits the reproducing kernel

$$K(x,y) \equiv \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}\exp(i(x-y)\xi)d\xi = \frac{1}{2}e^{-|x-y|} \quad (x, y \in \mathbb{R}). \tag{1.5}$$

Its restriction to the closed interval $[a, b]$ is the reproducing kernel Hilbert space $H_S[a, b] = W^{1,2}[a, b]$ as a set of functions, and the norm is given by

$$\|f\|_{H_S[a,b]} \equiv \sqrt{\left(\int_a^b (|f(x)|^2 + |f'(x)|^2)\,dx\right) + |f(a)|^2 + |f(b)|^2} \tag{1.6}$$

([5], pages 10-16).

The representation (1.5) means that the functions $f(x)$ of $H_S(\mathbb{R})$ are represented in the form

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}\exp(ix\xi)F(\xi)d\xi$$

with the functions $F(\xi)$ satisfying

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}|F(\xi)|^2 d\xi < \infty$$

and the norm is represented by

$$\|f\|_{H_S(\mathbb{R})} = \sqrt{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+\xi^2}|F(\xi)|^2 d\xi}.$$

The restriction mapping L from the space $H_S(\mathbb{R})$ to the space $H_S[a,b]$ is, of course, not injective and so, in particular, we obtain the norm inequality

$$\|f\|_{H_S(\mathbb{R})} \geq \|f\|_{H_S[a,b]};$$

that is,

$$\int_{\mathbb{R}} (|f(x)|^2 + |f'(x)|^2)dx \geq \left(\int_a^b (|f(x)|^2 + |f'(x)|^2)\, dx\right) + |f(a)|^2 + |f(b)|^2. \tag{1.7}$$

By our general theory, we can give the precise correspondence of the two spaces; that is,

$$f|_{[a,b]}(x) = (f(\xi), K(\xi,x))_{H_S(\mathbb{R})} \tag{1.8}$$

and

$$f(x) = (f_{[a,b]}(\xi), K(\xi,x))_{H_S[a,b])}, \tag{1.9}$$

with the minimum extension f of $f_{[a,b]}$ in $H_S[a,b])$ to $H_S(\mathbb{R})$. Indeed, we can derive directly the identity (1.9) for the minimum extension f of $f_{[a,b]}$ in $H_S[a,b])$ to $H_S(\mathbb{R})$. See the following proof of Theorem 2.1 for the space $W^{2,2}(\mathbb{R})$.

However, for the minimum extension formula we have the general formula in Theorem A,

$$f(p) = (f|E_0(\cdot), K(\cdot, p))_{H_{K|E_0 \times E_0}(E_0)},$$

for the minimum extension f of $f|E_0$. See the proof of Proposition 2.5 in [5] (pages 79-80), in particular, (2.4).

We obtained several realizations of restricted reproducing kernel Hilbert spaces as in (1.6), however, they are, in general, involved. See [4], [5]. The formula (1.6) is a simple result, however, the realization of the restricted reproducing kernel spaces is, in general, complicated in this sense.

Open Problem : *Let $m > \frac{n}{2}$ be an integer. Denote by ${}_NC_K$ the binomial coefficient and by $W^{m,2}(\mathbb{R}^n)$ the Sobolev space whose norm is given by*

$$\|F\|_{W^{m,2}(\mathbb{R}^n)} = \sqrt{\sum_{\nu=0}^{m} {}_mC_\nu \left(\sum_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \le \nu} \frac{\nu!}{\alpha!} \int_{\mathbb{R}^n} \left| \frac{\partial^\nu F(x)}{\partial x^\nu} \right|^2 dx \right)}. \tag{1.10}$$

Then, the reproducing kernel K is given by

$$K(x,y) \equiv \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\exp(i(x-y)\cdot\xi)}{(1+|\xi|^2)^m}\, d\xi \quad (x, y \in \mathbb{R}^n) \tag{1.11}$$

([5], page 22). How will be the realization of the norm for the restricted reproducing kernel Hilbert space to some nontrivial subset (the typical case is a sphere $\{r < a\}$) of $\mathbb{R}^n$ as in the case (1.6) of one dimensional way?

2 The typical case for the space $W^{2,2}(\mathbb{R})$

For the Sobolev Hilbert space $W^{2,2}(\mathbb{R})$ defined to be the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm:

$$\|f\|_{W^{2,2}(\mathbb{R})} = \sqrt{\|f''\|_{L^2(\mathbb{R})}{}^2 + 2\|f'\|_{L^2(\mathbb{R})}{}^2 + \|f\|_{L^2(\mathbb{R})}{}^2},$$

we have the reproducing kernel

$$G(s,t) \equiv \frac{1}{4} e^{-|s-t|}(1 + |s-t|) \quad (s, t \in \mathbb{R})$$

([5], page 21-22).

For simplicity, in this section, we shall consider functions in real valued functions.

In order to look for the reproducing kernel Hilbert space $W_S([a,b]), (a < b)$ admitting the restricted reproducing kernel $G(s,t)$ to the interval $[a,b]$, we calculate the integral, for any function $f \in W^{2,2}(\mathbb{R})$

$$(f(s), G(s,t))_{W^{2,2}([a,b])}.$$

By setting $G_t(s) = G(s,t)$, we note that

$$\begin{aligned}
G_t(s) &= \frac{t-s+1}{4}\exp(s-t)\chi_{(a,t)}(s) + \frac{s-t+1}{4}\exp(t-s)\chi_{[t,b)}(s),\\
\frac{dG_t}{ds}(s) &= \frac{t-s}{4}\exp(s-t)\chi_{(a,t)}(s) - \frac{s-t}{4}\exp(t-s)\chi_{[t,b)}(s),\\
\frac{d^2G_t}{ds^2}(s) &= \frac{t-s-1}{4}\exp(s-t)\chi_{(a,t)}(s) + \frac{s-t-1}{4}\exp(t-s)\chi_{[t,b)}(s). \quad (2.1)
\end{aligned}$$

Then, by integration by parts repeatedly, we have

$$\begin{aligned}
&(f(s), G(s,t))_{W^{2,2}([a,b])}\\
&= f(t)\\
&+f(a)\frac{-t+a-2}{4}\exp(a-t) - f'(a)\frac{t-a-1}{4}\exp(a-t)\\
&-f(b)\frac{b-t+2}{4}\exp(t-b) + f'(b)\frac{b-t-1}{4}\exp(t-b). \quad (2.2)
\end{aligned}$$

That is

$$\begin{aligned}
&(f(s), G(s,t))_{W_S([a,b])}\\
&= f(t)\\
&+f(a)G(a,t)\left(-1+\frac{-1}{1+t-a}\right)\\
&+f'(a)G'(a,t)\left(\frac{1}{t-a}-1\right)\\
&+f(b)G(b,t)\left(-1+\frac{-1}{1+b-t}\right)
\end{aligned}$$

$$+f'(b)G'(b,t)\left(\frac{1}{b-t}-1\right). \tag{2.3}$$

We thus have the desired identity admitting the restricted reproducing kernel of $G(s,t)$ to the interval $[a,b]$

$$(f, G(\cdot,t))_{W_S([a,b])} = (f, G(\cdot,t))_{W^{2,2}(\mathbf{R})} \tag{2.4}$$
$$-f(a)\frac{-t+a-2}{4}\exp{(a-t)} + f'(a)\frac{t-a-1}{4}\exp{(a-t)}$$
$$+f(b)\frac{b-t+2}{4}\exp{(t-b)} - f'(b)\frac{b-t-1}{4}\exp(t-b).$$

We can see that this identity is right, indeed, we shall give another natural method in order to see it.

In order to look for the norm admitting the restricted reproducing kernel of $G(s,t)$ to the interval $[a,b]$, note that the integral

$$\int_{-\infty}^{a} \left(f''(x)^2 + 2f'(x)^2 + f(x)^2\right) dx \tag{2.5}$$

is identical with its integral of the function

$$f(x) = 4f(a)G_a(x) - 4f'(a)G'_a x) \tag{2.6}$$

that is the minimum integral over $(-\infty, a)$ of the functions $W^{2,2}(\mathbf{R})$ taking the values $f(a)$ and $f'(a)$.

The function is given by

$$f(x) = [(A+B)(a-x)+A]\exp(x-a)$$

with

$$A = f(a), \quad B = -f'(a).$$

Then, by direct calculations, we have

$$\int_{-\infty}^{a} \left(f''(x)^2 + 2f'(x)^2 + f(x)^2\right) dx = 2(A^2 + AB + B^2) \tag{2.7}$$

$$= 2(f(a)^2 - f(a)f'(a) + f'(a)^2).$$

From this result, we see that the corresponding inner product over $(-\infty, a)$ is represented by

$$(f_1, f_2)_{W^{2,2}(-\infty,a)} = 2(f_1(a)f_2(a) + f_1'(a)f_2'(a)) \tag{2.8}$$

$$+\frac{1}{2}(f_1(a) - f_1'(a))(f_2(a) - f_2'(a)) - \frac{1}{2}(f_1(a) + f_1'(a))(f_2(a) + f_2'(a)).$$

The situation for the integrals over $(b, +\infty)$ is similar and so we obtain the desired isometric identity

$$\|f\|^2_{W^{2,2}(\mathbf{R})} = \|f\|^2_{W_S([a,b])} \tag{2.9}$$

$$+2(f(a)^2 - f(a)f'(a) + f'(a)^2)$$

$$+2(f(b)^2 - f(b)f'(b) + f'(b)^2).$$

Therefore, the inner product relation is given by

$$(f_1, f_2)_{W^{2,2}(\mathbf{R})} = (f_1, f_2)_{W_S([a,b])}$$

$$+2(f_1(a)f_2(a) + f_1'(a)f_2'(a)) + \frac{1}{2}(f_1(a) - f_1'(a))(f_2(a) - f_2'(a))$$

$$-\frac{1}{2}(f_1(a) + f_1'(a))(f_2(a) + f_2'(a))$$

$$+2(f_1(b)f_2(b) + f_1'(b)f_2'(b)) + \frac{1}{2}(f_1(b) - f_1'(b))(f_2(b) - f_2'(b))$$

$$-\frac{1}{2}(f_1(b) + f_1'(b))(f_2(b) + f_2'(b)).$$

We can confirm that (2.4) and (2.9) are consistent, directly.
Indeed,

$$2(f(a)G_t(a) + f'(a)G_t'(a)) + \frac{1}{2}(f(a) - f'(a))(G_t(a) - G_t'(a))$$

$$-\frac{1}{2}(f(a) + f'(a))(G_t(a) + G_t'(a))$$

is identical with

$$+f(a)\frac{-t + a - 2}{4}\exp(a - t) + f'(a)\frac{t - a - 1}{4}\exp(a - t).$$

For the point b, the result is similar.

In particular, we have

Theorem 2.1: *The extension of the functions f in $W_S([a,b])$ to $W^{2,2}(\mathbf{R})$ with the minimum norm is given by*

$$f(t) = (f, G(\cdot,t))_{W^{2,2}([a,b])}$$

$$+2(f(a)G_t(a) + f'(a)G_t'(a)) + \frac{1}{2}(f(a) - f'(a))(G_t(a) - G_t'(a))$$

$$-\frac{1}{2}(f(a) + f'(a))(G_t(a) + G_t'(a))$$

$$+2(f(b)G_t(b) + f'(b)G_t'(b)) + \frac{1}{2}(f(b) - f'(b))(G_t(b) - G_t'(b))$$

$$-\frac{1}{2}(f(b) + f'(b))(G_t(b) + G_t'(b)).$$

Corollary 2.1: *We obtain the inequality for real valued functions f of $W^{2,2}(\mathbb{R})$*

$$\|f\|^2_{W^{2,2}(\mathbb{R})} = \|f''\|^2_{L^2(\mathbb{R})} + 2\|f'\|^2_{L^2(\mathbb{R})} + \|f\|^2_{L^2(\mathbb{R})}$$

$$\geq \|f''\|^2_{L^2([a,b])} + 2\|f'\|^2_{L^2([a,b])} + \|f\|^2_{L^2([a,b])}$$

$$+2(f(a)^2 - f(a)f'(a) + f'(a)^2) + 2(f(b)^2 - f(b)f'(b) + f'(b)^2).$$

Equality holds for the minimum extension stated in Theorem 2.1.

Related versions

By the similar method or directly we have the following results.

Let

$$K(s,t) \equiv \int_0^\infty \frac{\cos(s\,u)\cos(t\,u)}{u^2+1}\,du = \frac{\pi}{4}\left(\exp(-|s-t|) + \exp(-s-t)\right) \tag{2.10}$$

for $s, t > 0$. Then $H_K(0,\infty) = W^{1,2}(0,\infty)$ as a set of functions and the norm is given by:

$$\|f\|_{H_K(0,\infty)} = \sqrt{\frac{2}{\pi}\int_0^\infty (|f'(u)|^2 + |f(u)|^2)\,du} \tag{2.11}$$

([5], page12-13). From the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the norm inequality

$$\|f\|^2_{H_K(0,\infty)} \geq \frac{2}{\pi}\frac{1-\exp(-2a)}{1+\exp(-2a)}|f(a)|^2 \tag{2.12}$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du + \frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t) \equiv \int_0^\infty \frac{\sin(s\,u)\sin(t\,u)}{u^2+1}\,du = \frac{\pi}{4}\left(\exp(-|s-t|)-\exp(-s-t)\right) \tag{2.13}$$

for $s,t>0$. Then we have

$$H_K(0,\infty) = \{f \in \mathrm{AC}(0,\infty)\,:\, f(0)=0\} \tag{2.14}$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)} = \sqrt{\frac{2}{\pi}\int_0^\infty (|f'(u)|^2+|f(u)|^2)\,du} \tag{2.15}$$

([5], 13-14). From the restriction of the kernel $K(s,t)$ to $[a,b], a>0$, we have the norm inequality

$$\|f\|^2_{H_K(0,\infty)} \geq \frac{2}{\pi}\frac{1+\exp(-2a)}{1-\exp(-2a)}|f(a)|^2 \tag{2.16}$$

$$+\frac{2}{\pi}\int_a^b (|f'(u)|^2+|f(u)|^2)\,du + \frac{2}{\pi}|f(b)|^2.$$

Let

$$K(s,t) \equiv \min(s,t) \quad (s,t>0). \tag{2.17}$$

Then we have

$$H_K(0,\infty) = \left\{f \in W^{1,2}(0,\infty)\,:\, \lim_{\varepsilon\downarrow 0} f(\varepsilon)=0\right\} \tag{2.18}$$

as a set of functions and the norm is given by

$$\|f\|_{H_K(0,\infty)} = \sqrt{\int_0^\infty |f'(u)|^2\,du} \tag{2.19}$$

([5], 14-15). From the restriction of the kernel $K(s,t)$ to $[a,b], a > 0$, we have the norm inequality

$$\|f\|^2_{H_K(0,\infty)} \geq \frac{1}{a}|f(a)|^2 + \int_a^b |f'(u)|^2\, du. \tag{2.20}$$

We have many type Sobolev Hilbert spaces. For example,
for $\omega^2 = \gamma^2 - \alpha^2 > 0$, the kernel

$$K(s,t) = \frac{\exp(-\alpha|s-t|)}{4\alpha\gamma^2}\cos(\omega|s-t|) + \frac{\alpha}{\omega}\sin(\omega|s-t|)$$

is the reproducing kernel for the Sobolev Hilbert space admitting the norm

$$||u||^2 = 4\alpha\gamma^2 u(a)^2 + 4\alpha u'(a)^2$$

$$+ \int_a^b \left(u''(t) + 2\alpha^2 u'(t) + \gamma^2 u(t)\right)^2 dt$$

(E. Parzen, [2]). For the case $\alpha = 0, \omega = \gamma$, by the division by zero calculus ([6]), we have the corresponding reproducing kernel

$$K(s,t) = \frac{-1}{4\gamma^2}|s-t|\cos(\omega|s-t|).$$

See also [1] and the recent paper A. Yamada ([7]).

Basic applications of the realization of the restricted reproducing kernel Hilbert space

Theorem 2.1 and other derived identities show that the extension of the function with the minimum norm to the whole space (the half space) from a closed interval $[a,b]$ is given simply. This means that in the related Fourier transform, the inversion that corresponds to the function with the minimum norm may be calculated in terms of the values on the interval $[a,b]$.

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