A Simple Proof That $e^{p/q}$ is Irrational

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Abstract

Using a simple application of the mean value theorem, we show that rational powers of e are irrational.

Introduction

Hermite proved that e is transcendental in 1873 [3]. His proof has been improved over the years by several mathematicians. A similar evolution has not taken place for proofs that show the irrationality of rational powers of e. In this note, we use relatively recent transcendence techniques [4, 6] to prove that the powers of e are irrational.

This approach may have pedagogical advantages in that it allows for the understanding of recent transcendental techniques, for both e and π, in the simpler context of an irrationality proof. It also gives a nice use of the mean value theorem that in suitable for first-year calculus students.

e^p is irrational

Assume, to the contrary, that $e^p = a/b$ with a, b, and p positive integers.

Since factorial growth exceeds polynomial, we can choose a positive integer n large enough that

$$be^p p^{2n+1} < n!. \tag{1}$$

Choose a value of n satisfying (1) and define $f(x) = x^n(p-x)^n$. Define $P(x)$ as the sum of $f(x)$ and its derivatives; that is,

$$F(x) = f(x) + f'(x) + \cdots + f^{(2n)}(x).$$

Next, let $G(x) = -e^{-x}F(x)$. Then $G'(x) = e^{-x}f(x)$. Using the mean value theorem on the interval $[0, p]$, we know there exists $\zeta \in (0, p)$ such that

$$\frac{G(p) - G(0)}{p} = G'(\zeta),$$

or

$$\frac{-e^p F(p) + F(0)}{p} = e^{-\zeta} f(\zeta) \tag{2}$$

Now, multiplying both sides of (2) by pe^p gives

$$-F(p) + e^p F(0) = pe^{p-\zeta} f(\zeta),$$

and then substituting $e^p = a/b$ and multiplying by b gives

$$-bF(p) + aF(0) = bpe^{p-\zeta} f(\zeta). \tag{3}$$

We claim that the left side of (3) is an integer multiple of $n!$. When we repeatedly differentiate $f(x)$, we find that every term of every derivative includes either a factor of x or a factor of $n!$. Similarly, each term includes either a factor of $(p - x)$ or a factor of $n!$. It follows that both $F(0)$ and $F(p)$ are integer multiples of $n!$, and so the left side of (3) is also an integer multiple of $n!$. A Leibniz table, developed in [7], shows these properties succinctly.

Meanwhile, the right-hand side of (3) is strictly positive, and it is at most $bp^{2n+1}e^p$. This follows as the maximum values of x^n and $(p - x)^n$ on $(0, p)$ are both p^n, so that $f(\zeta)$ is bounded above by p^{2n}. The additional p factor in $pbe^{p-\zeta} f(\zeta)$ gives the $2n + 1$ exponent. Therefore, by (1), the right side of (3) is strictly less than $n!$.

We have, then, a contradiction. An integer multiple of $n!$ is positive, but less than $n!$.

$e^{p/q}$ is irrational

To show that rational powers of e are irrational, suppose to the contrary that $e^{p/q} = a/b$, where p, q, a, and b are positive integers. Then

$$(e^{p/q})^q = e^p = (a/b)^q,$$

and, as $(a/b)^q$ is rational, this contradicts the irrationality of e^p.

Further reading

To see how the techniques used in this article can be applied, with some modifications, to show the irrationality of π, see [7]. Readers interested in a transcendence proof for e should give Herstein's proof a try [4]. After mastering the transcendence of e, we are ready to approach the big brother and big sister of all these irrationality and transcendence proofs: the transcendence of π, which shows that you can't square the circle. Hobson gives the history of attempts to square the circle from antiquity up to the proof of its impossibility [5]. Niven's 1939 transcendence of π proof [8] adds some further historical perspectives while giving a simplification of Lindemann's original 1882 proof. Original proofs of e and π can be found in [1].

References

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