Sum of Three Cubes Explored
Proof

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Abstract

Using already known techniques along with some not so obvious innovations on my part, I was able to show (prove) that there are solutions for all K (except those of the form 9m+/-4 and 9m+/-5 which are impossible) for +/-K = +/- (x^3) +/- (y^3) +/- (z^3). A further stipulation is that x, y and z must be whole numbers that can be a combination of positives and negatives. This is achieved through simple subtraction.

Setting up a table showing that all K can be represented using a multiple of 27 plus a mask lends validity to a portion of the proof. These representations may and often do contain many more than the required number of cubes summed up. I side step that problem by showing that no matter the K picked and how ever many cubes are required to create it in my represenions, they can all be reduced to a maximum of cubes summed. Exactly what we require for the proof. Having done that we are complete. The three new cubes we have just reduced to are already included in table. They are items I have already represented in the above format.

I hope you enjoy this 'proof'.

Introduction

This is a rather simple concept to grasp. For all +/-K there is at least one corresponding combination of 3 cubes summed. That is, there exists K= +/- (x^3) +/- (y^3) +/- (z^3); -K= +/-(x^3) +/- (y^3) +/- (z^3). This is true for all K except for those of the form 9m+4; 9m+5; 9m-4 and 9m-5. No solutions are possible for those, so they are excluded from this proof.

I will include some of Euclid's research adding in the negative components so that the proof is more inclusive. I will introduce a method of representing the remaining K's as a sum of cubes (not limited to 3 max) created using multiples of 27 plus unique masks. This conforms to a very neat structure that can be used to formulate the proof.

From those representations, I will introduce how to reduce that many (more than 3 cubes) down to a 3 cube maximum. This is required for the proof. I show that all +/-K are reducable in this way.

Since we can relatively easily reduce to 3 cubes; we know that those cubes are already available. When we were building the structure with the multiples of 27 and the masks, it included those as well. There are already implied representations of them if we extent out our tables accordingly.

I look forward to any feedback that may help improve upon this 'proof '.

Relationship of Perfect Cubes

Here is a partial list of the positive and negative perfect cubes:

Base	+ve Base Cubed	-ve Base Cubed
0	0	0
1	1	-1
2	8	-8
3	27	-27
4	64	-64
5	125	-125
6	216	-216
7	343	-343
8	512	-512
9	729	-729
10	1000	-1000
11	1331	-1331
12	1728	-1728
13	2197	-2197
14	2744	-2744
15	3375	-3375
16	4096	-4096
17	4913	-4913
18	5832	-5832
19	6859	-6859
20	8000	-8000

If you extend the base to include ALL whole numbers you will get the following partial chart:

Base	Base Cubed	Absolute Difference	Separation by	Absolute Difference	Divisible by 6
			Multiples of 6		
-20	-8000		in Both Directions		
-19	-6859	1141			
-18	-5832	1027		114	19
-17	-4913	919		108	18
-16	-4096	817		102	17
-15	-3375	721		96	16
-14	-2744	631		90	15
-13	-2197	547		84	14
-12	-1728	469		78	13
-11	-1331	397		72	12
-10	-1000	331		66	11
-9	-729	271		60	10
-8	-512	217		54	9
-7	-343	169		48	8
-6	-216	127		42	7
-5	-125	91		36	6
-4	-64	61	61=37+24	30	5
-3	-27	37	37=19+18	24	4
-2	-8	19	19=7+12	18	3
-1	-1	7	7=1+6	12	2
0	0	1	1=1+0	6	1
1	1	1	1=1+0	0	0
2	8	7	7=1+6	6	1
3	27	19	19=7+12	12	2
4	64	37	37=19+18	18	3
5	125	61	61=37+24	24	4
6	216	91		30	5
7	343	127		36	6
8	512	169		42	7
9	729	217		48	8
10	1000	271		54	9
11	1331	331		60	10
12	1728	397		66	11
13	2197	469		72	12
14	2744	547		78	13
15	3375	631		84	14
16	4096	721		90	15
17	4913	817		96	16
18	5832	919		102	17
19	6859	1027		108	18
20	8000	1141		114	19

I may not be able to use this fact directly in any proof, but it does show that there is a pattern to the growth from one cube to the next in line, which is related to a multiple of '6' and an addition of a multiple of 6. Now that is interesting because X1 = X0+(6*y) in the two possible directions, right? That set of Xn's can be further defined as XX1 = XX0+(6*y) in both directions. That's a neat and consistant pattern that predicts what the next cube will be.

This is a good spot to point out that the third column shows the abosolute difference between the adjacent cubes...1, 7, 19, 37, ... These will aid in the search of 3 cubes that add up to exactly K. The cubes themselves get us started. For example K=0 can be found by adding 3 cubes of '0' or 0^3 + x^3 + (-x)^3. K=1 can be found by adding a single cude of 1 to any pair of +/-cubes...1= 1^3 + 5^3 + (-5)^3. K=2 can be 2 cubes of '1' plus 0^3. K=3... 3 cubes of 1. Note that we will be skipping over any 9m+4 and 9m+5 since they are impossible...these include 4, 5, 13, 14, 22, 23, 31, 32, ... So our next candidate is K=6 which we can easily get with a difference from column 3, like '7' and '-1'. '7' is a combination of '8' and '-1'... so ultimately '8', '-1' and '-

1'. K=7 like above can be '7' and '0'...'8', '-1' and '0' or simply '8', '-1' and '0' since we don't need the intermediate '7'. K=8 is simply '8', and two '0'.

Euclid's Division Lemma (Extended to Include Negative Whole Numbers)

I'm not going to re-invent the wheel so I will briefly describe this lemma and how it becomes useful to our search for a proof.

In summary Euclid was able to prove that any cubed positive whole number can be represented in one of three forms: 9m+0; 9m+1 or 9m+8. I totally agree with this lemma. Now that lemma can be extended to include all whole numbers whether they be positive of negative. We need only make two simple changes in the 9m subsets: 9m-1 or 9m-8. These are directional changes only! See the following chart:

a	a cubed	Euclid's	-a cubed	Euclid's Extended
0	0	9m	0	9m
1	1	9m+1	-1	9m-1
2	8	9m+8	-8	9m-8
3	27	9m	-27	9m
4	64	9m+1	-64	9m-1
5	125	9m+8	-125	9m-8
6	216	9m	-216	9m
7	343	9m+1	-343	9m-1
8	512	9m+8	-512	9m-8
9	729	9m	-729	9m
10	1000	9m+1	-1000	9m-1
11	1331	9m+8	-1331	9m-8
12	1728	9m	-1728	9m
13	2197	9m+1	-2197	9m-1
14	2744	9m+8	-2744	9m-8
15	3375	9m	-3375	9m
16	4096	9m+1	-4096	9m-1
17	4913	9m+8	-4913	9m-8
18	5832	9m	-5832	9m
19	6859	9m+1	-6859	9m-1
20	8000	9m+8	-8000	9m-8
21	9261	9m	-9261	9m
22	10648	9m+1	-10648	9m-1
23	12167	9m+8	-12167	9m-8
24	13824	9m	-13824	9m
25	15625	9m+1	-15625	9m-1
26	17576	9m+8	-17576	9m-8
27	19683	9m	-19683	9m
28	21952	9m+1	-21952	9m-1
29	24389	9m+8	-24389	9m-8

As can be readily seen, if a cubed number results in a multiple of 3, there is no need to add a directional component. So 9m will suffice; the direction is built in. Needless to say, the other two which have directional components of +1 and +8 must be changed to -1 and -8 (opposite direction) to match the cubed number exactly. When looking at the negative cubes remember that 9m-x only works if you make m negative; (9)(-7)-1. Right? I'm making every effort to keep this easily understandable. So expanding on Euclid's work I have 17 9m's to work with to cover the entire number set K. They include 9m, 9m+1, 9m-1, 9m+2, 9m-2, 9m+3, 9m-3, 9m+4, 9m-4, 9m+5, 9m-5, 9m+6, 9m-6, 9m+7, 9m-7, 9m+8 and 9m-8. Note that 9m works for both positive and negative...which leads me to quickly conclude that there should be twice as many 9m as any of the others. This will play into some 'density' considerations further on in our discussion.

I have not done any serious investigation into Euclid's work but he likely talks about my next topic which is this subset of 9m's. It becomes obvious after taking a quick look at all positive whole numbers that they can be sub-divided into 9 groups of 9m's: 9m; 9m+1; 9m+2; 9m+3; 9m+4; 9m+5; 9m+6; 9m+7; and 9m+8. See the following chart:

9m	9m+1	9m+2	9m+3	9m+4	9m+5	9m+6	9m+7	9m+8
0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	60	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98
99	100	101	102	103	104	105	106	107
108	109	110	111	112	113	114	115	116
117	118	119	120	121	122	123	124	125
126	127	128	129	130	131	132	133	134
135	136	137	138	139	140	141	142	143
144	145	146	147	148	149	150	151	152
153	154	155	156	157	158	159	160	161
162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179
180	181	182	183	184	185	186	187	188
189	190	191	192	193	194	195	196	197
198	199	200	201	202	203	204	205	206
207	208	209	210	211	212	213	214	215
216	217	218	219	220	221	222	223	224
225	226	227	228	229	230	231	232	233
234	235	236	237	238	239	240	241	242
243	244	245	246	247	248	249	250	251
252	253	254	255	256	257	258	259	260
261	262	263	264	265	266	267	268	269
270	271	272	273	274	275	276	277	278
279	280	281	282	283	284	285	286	287
288	289	290	291	292	293	294	295	296
297	298	299	300	301	302	303	304	305

There is a corresponding chart for the negative whole numbers. I kept that chart much smaller since it is for all intents and purpose a mirror directional image of the above chart.

9m	9m-1	9m-2	9m-3	9m-4	9m-5	9m-6	9m-7	9m-8
0	-1	-2	-3	-4	-5	-6	-7	-8
-9	-10	-11	-12	-13	-14	-15	-16	-17
-18	-19	-20	-21	-22	-23	-24	-25	-26
-27	-28	-29	-30	-31	-32	-33	-34	-35
-36	-37	-38	-39	-40	-41	-42	-43	-44
-45	-46	-47	-48	-49	-50	-51	-52	-53
-54	-55	-56	-57	-58	-59	-60	-61	-62
-63	-64	-65	-66	-67	-68	-69	-70	-71
-72	-73	-74	-75	-76	-77	-78	-79	-80
-81	-82	-83	-84	-85	-86	-87	-88	-89
-90	-91	-92	-93	-94	-95	-96	-97	-98
-99	-100	-101	-102	-103	-104	-105	-106	-107

If one were to place the actual cubes of the whole numbers (positive or negative), one would find that they fall in the 9m; 9m+1; 9m+8; 9m-1; 9m-8 columns. Right? I've highlighted them in yellow in both the above charts. Euclid has already proven this much, I just added the negative whole number components.

'Modulo 9' (9m+/-4; 9m+/-5 are Impossible to Create with Sum of 3 Cubes)

As I've eluded to previously, the columns represented by 9m+4, 9m+5, 9m-4 and 9m-5 are impossible to create using the 'sum of 3 cubes'. Why you might ask? As I have discovered in the limited literature on this subject, the use of modulo 9 is the key. See the following chart:

9m's	1^{st} 18	Mod 9
9m	0	0
9m+1	1	1
9m+2	2	2
9m+3	3	3
9m+4	4	4
9m+5	5	5
9m+6	6	6
9m+7	7	7
9m+8	8	8
9m	0	0
9m-1	-1	8
9m-2	-2	7
9m-3	-3	6
9m-4	-4	5
9m-5	-5	4
9m-6	-6	3
9m-7	-7	2
9m-8	-8	1

As seen, Mod 9 on the positive whole numbers gives expected remainders. The negative whole numbers seems backwards but it is correct; for example -8 Mod 9 is the same as saying 9-8/9 which is 1. Right? But do note that this does not affect what we are trying to show here. Note that above, the cubes only show up in 9m, 9m+1, 9m+8, 9m-1 and 9m-8. For all intents and purpose 9m-1 is the same modulo result as 9m+8 and 9m-8 is the same as 9m+1. Those two are reversed. The three Mod 9 answers remain the same for both postive and negative whole numbers with respect to cubes.

Base	cubes	Mod 9
0	0	0
1	1	1
2	8	8
3	27	0
4	64	1
5	125	8
6	216	0
7	343	1
8	512	8
0	0	0
-1	-1	8
-2	-8	1
-3	-27	0
-4	-64	8
-5	-125	1
-6	-216	0
-7	-343	8
-8	-512	1

Now my understanding is that since the only possible Mod 9 entries for any cube is 0, 1 or 8...we can create 0, 1, 2, 3, 6, 7, and 8 from these base cubes mod 9...but we can not for 4 and 5. There is a second chart for the negative K's which complements (opposite direction) of the first. See the following two charts:

K	K = a^3 + b^3 + c^3
	0, 1, -1, 8 or -8
0	0 + 0 + 0
1	0 + 0 + 1
2	0 + 1 + 1
3	1 + 1 + 1
4	Impossible
5	Impossible
6	8 + (-1) + (-1)
7	8 + 0 + (-1)
8	8 + 0 + 0

K	K = a^3 + b^3 + c^3
	0, 1, -1, 8 or -8
0	0 + 0 + 0
-1	0 + 0 + (–1)
-2	0 + (-1) + (-1)
-3	(-1) + (-1) + (-1)
-4	Impossible
-5	Impossible
-6	(-8) + 1 + 1
-7	(-8) + 0 + 1
-8	(-8) + 0 + 0

Note that the second column in the above charts can be formulated in different ways. So we are not limited to the easiest ones I have shown. For example 3 can be formed by adding (-5)^3; 4^3 and 4^3; -125 + 64 + 64 = 3. Note that -125 is a 9m-8 and 64 is a 9m+1. Of course a -3 would be opposite +125 – 64 – 64 = -3. Why do I point this out? To show the number play occuring here: -125 = 9*(-13)-8 and 64 = 9*(7)+1.

Let me expand upon this using K= 0. '0' can also be formed using 0 + (-1) + 1 or 0 + (-8) + 8 or of course 0 + (-0) + 0. Do you see a pattern here? You can form 0 from addiing any K to it's inverse. So in the case of 0 there are infinitely many of them... any cube plus it's inverse cube plus 0 cubed. The same idea for +/- 1...there are infinitely many because you can plug in another plus it's inverse. Right.

Another easy example is K=2; 343 – 125 – 216 = 2; that 7^3; 6^3 and 5^5; 9m; 9m-1 and 9m-8. So this is the same as saying a 9m+1 minus a 9m and a 9m-8. 10-0-8 which is 9(1)+1 plus -9(0) plus -(9(0)+8).

Forming K with multiples of 27 plus patterns (masking)!

After some serious investigation I realized that I could represent all K (except those of the form 9m+/-4 and 9m+/-5) as a multiple of 27 plus some combination of 0, 1, -1, 8 and -8. There are some serious patterns that have emerged as I compiled the following chart that will become very useful in my proof concept.

	9m		9m+1		9m+2		9m+3		9m+6		9m+7		9m+8
0	0(27)	1	0(27)+1	2	0(27)+1+1	3	0(27)+1+1+1	6	0(27)+8-1-1	7	0(27)+8-1	8	0(27)+8
9	0(27)+8+1	10	0(27)+8+1+1	11	1(27)-8-8	12	1(27)-8-8+1	15	0(27)+8+8-1	16	0(27)+8+8	17	0(27)+8+8+1
18	1(27)-8-1	19	1(27)-8	20	1(27)-8+1	21	1(27)-8+1+1	24	1(27)-1-1-1	25	1(27)-1-1	26	1(27)-1
27	1(27)	28	1(27)+1	29	1(27)+1+1	30	1(27)+1+1+1	33	1(27)+8-1-1	34	1(27)+8-1	35	1(27)+8
36	1(27)+8+1	37	1(27)+8+1+1	38	2(27)-8-8	39	2(27)-8-8+1	42	1(27)+8+8-1	43	1(27)+8+8	44	1(27)+8+8+1
45	2(27)-8-1	46	2(27)-8	47	2(27)-8+1	48	2(27)-8+1+1	51	2(27)-1-1-1	52	2(27)-1-1	53	2(27)-1
54	2(27)	55	2(27)+1	56	2(27)+1+1	57	2(27)+1+1+1	60	2(27)+8-1-1	61	2(27)+8-1	62	2(27)+8
63	2(27)+8+1	64	2(27)+8+1+1	65	3(27)-8-8	66	3(27)-8-8+1	69	2(27)+8+8-1	70	2(27)+8+8	71	2(27)+8+8+1
72	3(27)-8-1	73	3(27)-8	74	3(27)-8+1	75	3(27)-8+1+1	78	3(27)-1-1-1	79	3(27)-1-1	80	3(27)-1
81	3(27)	82	3(27)+1	83	3(27)+1+1	84	3(27)+1+1+1	87	3(27)+8-1-1	88	3(27)+8-1	89	3(27)+8
90	3(27)+8+1	91	3(27)+8+1+1	92	4(27)-8-8	93	4(27)-8-8+1	96	3(27)+8+8-1	97	3(27)+8+8	98	3(27)+8+8+1
99	4(27)-8-1	100	4(27)-8	101	4(27)-8+1	102	4(27)-8+1+1	105	4(27)-1-1-1	106	4(27)-1-1	107	4(27)-1
108	4(27)	109	4(27)+1	110	4(27)+1+1	111	4(27)+1+1+1	114	4(27)+8-1-1	115	4(27)+8-1	116	4(27)+8
117	4(27)+8+1	118	4(27)+8+1+1	119	5(27)-8-8	120	5(27)-8-8+1	123	4(27)+8+8-1	124	4(27)+8+8	125	4(27)+8+8+1
126	5(27)-8-1	127	5(27)-8	128	5(27)-8+1	129	5(27)-8+1+1	132	5(27)-1-1-1	133	5(27)-1-1	134	5(27)-1
135	5(27)	136	5(27)+1	137	5(27)+1+1	138	5(27)+1+1+1	141	5(27)+8-1-1	142	5(27)+8-1	143	5(27)+8
144	5(27)+8+1	145	5(27)+8+1+1	146	6(27)-8-8	147	6(27)-8-8+1	150	5(27)+8+8-1	151	5(27)+8+8	152	5(27)+8+8+1
153	6(27)-8-1	154	6(27)-8	155	6(27)-8+1	156	6(27)-8+1+1	159	6(27)-1-1-1	160	6(27)-1-1	161	6(27)-1
162	6(27)	163	6(27)+1	164	6(27)+1+1	165	6(27)+1+1+1	168	6(27)+8-1-1	169	6(27)+8-1	170	6(27)+8
171	6(27)+8+1	172	6(27)+8+1+1	173	7(27)-8-8	174	7(27)-8-8+1	177	6(27)+8+8-1	178	6(27)+8+8	179	6(27)+8+8+1
180	7(27)-8-1	181	7(27)-8	182	7(27)-8+1	183	7(27)-8+1+1	186	7(27)-1-1-1	187	7(27)-1-1	188	7(27)-1
189	7(27)	190	7(27)+1	191	7(27)+1+1	192	7(27)+1+1+1	195	7(27)+8-1-1	196	7(27)+8-1	197	7(27)+8
198	7(27)+8+1	199	7(27)+8+1+1	200	8(27)-8-8	201	8(27)-8-8+1	204	7(27)+8+8-1	205	7(27)+8+8	206	7(27)+8+8+1
207	8(27)-8-1	208	8(27)-8	209	8(27)-8+1	210	8(27)-8+1+1	213	8(27)-1-1-1	214	8(27)-1-1	215	8(27)-1
216	8(27)	217	8(27)+1	218	8(27)+1+1	219	8(27)+1+1+1	222	8(27)+8-1-1	223	8(27)+8-1	224	8(27)+8
225	8(27)+8+1	226	8(27)+8+1+1	227	9(27)-8-8	228	9(27)-8-8+1	231	8(27)+8+8-1	232	8(27)+8+8	233	8(27)+8+8+1
234	9(27)-8-1	235	9(27)-8	236	9(27)-8+1	237	9(27)-8+1+1	240	9(27)-1-1-1	241	9(27)-1-1	242	9(27)-1
243	9(27)	244	9(27)+1	245	9(27)+1+1	246	9(27)+1+1+1	249	9(27)+8-1-1	250	9(27)+8-1	251	9(27)+8
252	9(27)+8+1	253	9(27)+8+1+1	254	10(27)-8-8	255	10(27)-8-8+1	258	9(27)+8+8-1	259	9(27)+8+8	260	9(27)+8+8+1
261	10(27)-8-1	262	10(27)-8	263	10(27)-8+1	264	10(27)-8+1+1	267	10(27)-1-1-1	268	10(27)-1-1	269	10(27)-1
270	10(27)	271	10(27)+1	272	10(27)+1+1	273	10(27)+1+1+1	276	10(27)+8-1-1	277	10(27)+8-1	278	10(27)+8
279	10(27)+8+1	280	10(27)+8+1+1	281	11(27)-8-8	282	11(27)-8-8+1	285	10(27)+8+8-1	286	10(27)+8+8	287	10(27)+8+8+1
288	11(27)-8-1	289	11(27)-8	290	11(27)-8+1	291	11(27)-8+1+1	294	11(27)-1-1-1	295	11(27)-1-1	296	11(27)-1
297	11(27)	298	11(27)+1	299	11(27)+1+1	300	11(27)+1+1+1	303	11(27)+8-1-1	304	11(27)+8-1	305	11(27)+8
306	11(27)+8+1	307	11(27)+8+1+1	308	12(27)-8-8	309	12(27)-8-8+1	312	11(27)+8+8-1	313	11(27)+8+8	314	11(27)+8+8+1
315	12(27)-8-1	316	12(27)-8	317	12(27)-8+1	318	12(27)-8+1+1	321	12(27)-1-1-1	322	12(27)-1-1	323	12(27)-1
324	12(27)	325	12(27)+1	326	12(27)+1+1	327	12(27)+1+1+1	330	12(27)+8-1-1	331	12(27)+8-1	332	12(27)+8

My audience will appreciate the importance of this as they read on. Let's point out that each column above, 9m, 9m+1, 9m+2, 9m+3, 9m+6, 9m+7 and 9m+8 have three repeating patterns to infinity. I haven't shown it in the above table but this holds true to negative infinity as well. It's not important to explicitly include those. Negatives are just the opposite of positives; they go in the opposite direction on the number line. What do I mean by a repeating pattern? Lets look at 9m. 0 is created by 0*27(+0); 9 is created by 0*27(+8+1); 18 is created by 1*27(-8-1); 27 is created by 1*27(+0); 36 is created by 1*27(+8+1); and 45 is created by 2*27(-8-1). As you can plainly see there are three n*27 each applying one of the upper patterns/masks (+0; +8+1 and -8-1). This is consistent through the entire chart. So for every K (except 9m+4, 9m+5) there is a pattern for it's creation that we now have access to. You'll see why this is important in later sections.

You are likely asking yourself why I am considering this since it is clearly obvious that I am in many cases adding more than 3 cubes, many, many more. It is to show that any legal K's can be created using 1 to infinitely many cubes summed. I have a method to take these excess cubes and shrink them down to a maximum of 3...which is our goal. Your heart will likely skip a beat when you see this connection. I'll lead you into that once I have covered off the preliminaries. It is shockingly easy to understand the concept.

I've decided to go one step further by creating the following chart that lays out the multiples of 27 and the applied patterns that lead to potential cubes to see if there are any observable patterns. Indeed there are...we find there are cubes available in the patterns 0, 1, -1, 8, -8, 10 and 17 and they appear to display their own density patterns.

Included is another chart that shows there are no similar cubes in any of the other patterns. See the 'headings' in that chart for those patterns. There are clearly 21 patterns... 7 of them result in possible cubes; 14 do not. Without actually showing it in this report you can appreciate that there is something eerily similar happening with the negative K's. But we don't actually need to include that side since it is easily reproducable using subtraction in the positive realm. Right?

Note that I do all my own research in a vacuum and do not consult other's research. This way I am not biased or exclude something important. It also allows me to progress through my own discoveries and branch out accordingly. In the process I am reinventing the wheel for myself. Unfortunately this approach leaves me in the dark as to whether or not others have already made these connections. Prior research into where the problem sits give me a good idea where to push my own research. To date no one seems to have tried to come up with a proof and only seem to be interested in finding a solution for all K from 1 to 1000. They are using smarter and smarter alogorithms to search them down leaving only a handful.

I believe that my approach will make it easer to come up with extremely fast algorithms that zero in on much more precise locations to start searching, allowing skip-overs of impossible areas. In other words, precision searching; like looking searching through an unsorted list versus a sorted list.

This idea will become crystal clear as I introduce more of my research findings.

27*?		(+)0	(+)1	(-)1	(+)8	(-)8	(+)10	(+)17
0		0	1	-1	8	-8	10	17
27		27	28	26	35	19	37	44
54		54	55	53	62	46	64	71
81		81	82	80	89	73	91	98
108		108	109	107	116	100	118	125
135		135	136	134	143	127	145	152
162		162	163	161	170	154	172	179
189		189	190	188	197	181	199	206
216		216	217	215	224	208	226	233
243		243	244	242	251	235	253	260
270		270	271	269	278	262	280	287
297		297	298	296	305	289	307	314
324		324	325	323	332	316	334	341
351		351	352	350	359	343	361	368
378		378	379	377	386	370	388	395
405		405	406	404	413	397	415	422
432		432	433	431	440	424	442	449
459		459	460	458	467	451	469	476
486		486	487	485	494	478	496	503
513		513	514	512	521	505	523	530
540		540	541	539	548	532	550	557
567		567	568	566	575	559	577	584
594		594	595	593	602	586	604	611
621		621	622	620	629	613	631	638
648		648	649	647	656	640	658	665
675		675	676	674	683	667	685	692
702		702	703	701	710	694	712	719
729		729	730	728	737	721	739	746
756		756	757	755	764	748	766	773
783		783	784	782	791	775	793	800
810		810	811	809	818	802	820	827
837		837	838	836	845	829	847	854
864		864	865	863	872	856	874	881
891		891	892	890	899	883	901	908
918		918	919	917	926	910	928	935
945		945	946	944	953	937	955	962
972		972	973	971	980	964	982	989
999		999	1000	998	1007	991	1009	1016
1026		1026	1027	1025	1034	1018	1036	1043
1053		1053	1054	1052	1061	1045	1063	1070
1080		1080	1081	1079	1088	1072	1090	1097
1107		1107	1108	1106	1115	1099	1117	1124
1134		1134	1135	1133	1142	1126	1144	1151
1161		1161	1162	1160	1169	1153	1171	1178
1188		1188	1189	1187	1196	1180	1198	1205
1215		1215	1216	1214	1223	1207	1225	1232
1242		1242	1243	1241	1250	1234	1252	1259

The following chart shows those patterns that do not result in possible cubes...

(+2)	(-)2	(+)3	(-)3	(+)6	(-)6	(+)7	(-)7	(+)9	(-)9	(+)15	(-)15	(+)16	(-1)16
2	-2	3	-3	6	-6	7	-7	9	-9	15	-15	16	-16
29	25	30	24	33	21	34	20	36	18	42	12	43	11
56	52	57	51	60	48	61	47	63	45	69	39	70	38
83	79	84	78	87	75	88	74	90	72	96	66	97	65
110	106	111	105	114	102	115	101	117	99	123	93	124	92
137	133	138	132	141	129	142	128	144	126	150	120	151	119
164	160	165	159	168	156	169	155	171	153	177	147	178	146
191	187	192	186	195	183	196	182	198	180	204	174	205	173
218	214	219	213	222	210	223	209	225	207	231	201	232	200
245	241	246	240	249	237	250	236	252	234	258	228	259	227
272	268	273	267	276	264	277	263	279	261	285	255	286	254
299	295	300	294	303	291	304	290	306	288	312	282	313	281
326	322	327	321	330	318	331	317	333	315	339	309	340	308
353	349	354	348	357	345	358	344	360	342	366	336	367	335
380	376	381	375	384	372	385	371	387	369	393	363	394	362
407	403	408	402	411	399	412	398	414	396	420	390	421	389
434	430	435	429	438	426	439	425	441	423	447	417	448	416
461	457	462	456	465	453	466	452	468	450	474	444	475	443
488	484	489	483	492	480	493	479	495	477	501	471	502	470
515	511	516	510	519	507	520	506	522	504	528	498	529	497
542	538	543	537	546	534	547	533	549	531	555	525	556	524
569	565	570	564	573	561	574	560	576	558	582	552	583	551
596	592	597	591	600	588	601	587	603	585	609	579	610	578
623	619	624	618	627	615	628	614	630	612	636	606	637	605
650	646	651	645	654	642	655	641	657	639	663	633	664	632
677	673	678	672	681	669	682	668	684	666	690	660	691	659
704	700	705	699	708	696	709	695	711	693	717	687	718	686
731	727	732	726	735	723	736	722	738	720	744	714	745	713
758	754	759	753	762	750	763	749	765	747	771	741	772	740
785	781	786	780	789	777	790	776	792	774	798	768	799	767
812	808	813	807	816	804	817	803	819	801	825	795	826	794
839	835	840	834	843	831	844	830	846	828	852	822	853	821
866	862	867	861	870	858	871	857	873	855	879	849	880	848
893	889	894	888	897	885	898	884	900	882	906	876	907	875
920	916	921	915	924	912	925	911	927	909	933	903	934	902
947	943	948	942	951	939	952	938	954	936	960	930	961	929
974	970	975	969	978	966	979	965	981	963	987	957	988	956
1001	997	1002	996	1005	993	1006	992	1008	990	1014	984	1015	983
1028	1024	1029	1023	1032	1020	1033	1019	1035	1017	1041	1011	1042	1010
1055	1051	1056	1050	1059	1047	1060	1046	1062	1044	1068	1038	1069	1037
1082	1078	1083	1077	1086	1074	1087	1073	1089	1071	1095	1065	1096	1064
1109	1105	1110	1104	1113	1101	1114	1100	1116	1098	1122	1092	1123	1091
1136	1132	1137	1131	1140	1128	1141	1127	1143	1125	1149	1119	1150	1118
1163	1159	1164	1158	1167	1155	1168	1154	1170	1152	1176	1146	1177	1145
1190	1186	1191	1185	1194	1182	1195	1181	1197	1179	1203	1173	1204	1172
1217	1213	1218	1212	1221	1209	1222	1208	1224	1206	1230	1200	1231	1199
1244	1240	1245	1239	1248	1236	1249	1235	1251	1233	1257	1227	1258	1226
1271	1267	1272	1266	1275	1263	1276	1262	1278	1260	1284	1254	1285	1253
1298	1294	1299	1293	1302	1290	1303	1289	1305	1287	1311	1281	1312	1280
1325	1321	1326	1320	1329	1317	1330	1316	1332	1314	1338	1308	1339	1307
1352	1348	1353	1347	1356	1344	1357	1343	1359	1341	1365	1335	1366	1334
1379	1375	1380	1374	1383	1371	1384	1370	1386	1368	1392	1362	1393	1361

The availability of cubes in those 7 patterns is clearly defined in the following chart that looks at only cubes to determine the density of them in those patterns.

(+)0	(+)1	(-)1	(+)8	(-)8	(+)10	(+)17
-8	-8	-8	-8	**-8**	-8	-8
-1	-1	**-1**	-1	-1	-1	-1
0	0	0	0	0	0	0
1	**1**	1	1	1	1	1
8	8	8	**8**	8	8	8
27	27	27	27	27	27	27
64	64	64	64	64	**64**	64
125	125	125	125	125	125	**125**
216	216	216	216	216	216	216
343	343	343	343	**343**	343	343
512	512	**512**	512	512	512	512
729	729	729	729	729	729	729
1000	**1000**	1000	1000	1000	1000	1000
1331	1331	1331	**1331**	1331	1331	1331
1728	1728	1728	1728	1728	1728	1728
2197	2197	2197	2197	2197	**2197**	2197
2744	2744	2744	2744	2744	2744	**2744**
3375	3375	3375	3375	3375	3375	3375
4096	4096	4096	4096	**4096**	4096	4096
4913	4913	**4913**	4913	4913	4913	4913
5832	5832	5832	5832	5832	5832	5832
6859	**6859**	6859	6859	6859	6859	6859
8000	8000	8000	**8000**	8000	8000	8000
9261	9261	9261	9261	9261	9261	9261
10648	10648	10648	10648	10648	**10648**	10648
12167	12167	12167	12167	12167	12167	**12167**
13824	13824	13824	13824	13824	13824	13824
15625	15625	15625	15625	**15625**	15625	15625
17576	17576	**17576**	17576	17576	17576	17576
19683	19683	19683	19683	19683	19683	19683
21952	**21952**	21952	21952	21952	21952	21952
24389	24389	24389	**24389**	24389	24389	24389
27000	27000	27000	27000	27000	27000	27000
29791	29791	29791	29791	29791	**29791**	29791
32768	32768	32768	32768	32768	32768	**32768**
35937	35937	35937	35937	35937	35937	35937
39304	39304	39304	39304	**39304**	39304	39304
42875	42875	**42875**	42875	42875	42875	42875
46656	46656	46656	46656	46656	46656	46656
50653	**50653**	50653	50653	50653	50653	50653
54872	54872	54872	**54872**	54872	54872	54872
59319	59319	59319	59319	59319	59319	59319
64000	64000	64000	64000	64000	**64000**	64000
68921	68921	68921	68921	68921	68921	**68921**
74088	74088	74088	74088	74088	74088	74088
79507	79507	79507	79507	**79507**	79507	79507
85184	85184	**85184**	85184	85184	85184	85184
91125	91125	91125	91125	91125	91125	91125
97336	**97336**	97336	97336	97336	97336	97336
103823	103823	103823	**103823**	103823	103823	103823
110592	110592	110592	110592	110592	110592	110592
117649	117649	117649	117649	117649	**117649**	117649

In the previous chart we can see that the seven patterns combined touch every single cubed number. And this is not by coincidence. Column 9m has three hits in every 9 cubes; the rest of the columns have only 1 hit in every 9. Since there are 7 columns total you can clearly see that there are a combined total of 9 hits for every 9 cubes. This means they are all touched.

This brings us back to a topic we touched on above; and that is the density of 9m's. I indicated that there are twice as many as the other patterns/masks. Only now, when looking at only the 'cubes' we see that there are 3 times as many. Interesting. Looking more closely at the chart shows us why this is the case. It appears that is simply the way it works. Some 'Power' has designed it thus! I couldn't resist working that in here...

Collapsing Infinitely Many Cubes Summed to Just Three Summed

After all this preamble you have likely deduced that you will have to find a combination of up to 3 cubes that are formed by the 7 patterns outlined above. None of the other 14 patterns afford us that option. You at least know that you can exclude 2/3rds as dead ends. This points us to likely stacks on where to begin our search...but which stacks to consider is our problem. Can we isolate specific combinations that will lead us there? And ignore the remainder? Well, yes we can. Read on.

But that still does not help if we have a large number (more than 3 cubes) summed up. By doing this we were able to show that any K can be formed with a combination of infinately many summed 'cubes'. A requirement for this proof is to take those that are more than 3 and somehow reduce them down to that magic number 3...a maximum of 3 cubes only. The idea of having this entire table revolve around multiples of 27 is the key and why I spent the time to set it up that way. Having repeating masks really helps too. The multiples of 27 step up consistantly through each of the columns. We'll have sub-groups with three x(27)...example three 1(27) each one with a different mask (remember there are three of them repeating) before jumping up to the next 2(27) where there are three again, and so on. Nice and neat! 0(27) is the starting point for each column and the only situation where the sub-group may not contain all 3 masks. But 0(27) sets the stage.

I'll begin with a simpler example 11(27)+0 = 297 = K. That's 27+27+27+27+27+27+27+27+27+27+27+0. Since we are dealing with the easiest of patterns (+0) we are confining/limiting our search to the 9m column. This lead to the next leap when I asked myself if there was a way to combine the multiples of 27 into three distinct groups that will yield a perfect cube for each...that not only means limiting ourselves to 11 total but any number where the total of +27s and -27s is exacly 11. How about 27(27)-8(27)-8(27). This is 27-8-8=11. 27(27)=729=9*9*9=9^3. 8(27)=216=6*6*6=6^3. And guess what; 729-216-216= 297...so (9^3)-(6^3)-(6^3). There was no mask to worry about so that eliminated all but the 9m column (+0).

Let's pick another that is a little more involved...say 98 = 3(27)+8+8+1. I quickly see that 125-27=98 from the previous chart. This means that we can use 4(27)+8+8+1 and 1(27)+0. You can see that 4-1 gives us our magic number of 3(27s). We also notice that I can take the entire +8+8+1 pattern and apply it to the 9m+8 column entity because 9m has no such pattern. We can't break this pattern into two distinct patterns since there are none in the seven available masks! We could break it into 3 if we choose to look for x(27)+/-y(27)+/-z(27) to give say x(27)+8; y(27)+8 and z(27)+1...but my initial search seems to indicate that path will yield no results. That path is impossible. You can see how this method really simplifies the search. Continuing on we see that 4(27)+8+8+1=125=5*5*5=5^3 and 1(27)=27=3*3*3=3^3. That is (5^3)-(3^3)=98. Right?

How about another one to hammer this point home. This one is going to set loose on the 1, -1, 8 and -8 masks and their interchangablity. Let's look at 47 which is 2(27)-8+1. If we consult the chart we can see that splitting this into just two parts -8 and +1 will yield no results so we will proceed to three parts and start our search there. This is the same as saying -8+1+0; three parts. So we will be looking for something like x(27)-8 +/- y(27)+1 +/- z(27)+0. Make note here that we have something else to consider when doing our search and that is with respect to +/-. If you want to include a (-) of a +1 then you are actually going to be searching the -1 column entities. The same with 8 and -8. I mention that here because you will become confused as to why I picked a number from the -1 instead of the +1 column to solve this one. So I can see a possibility jump out: 343+216-512=47. That is 13(27)-8 + 8(27) – 19(27)-1. See how that (-) on the +1 part made it a -1 part? That only happens when you are subtracting the opposite sign. Subtracting a +8 gives -8; subtracting a -8 gives +8 and likewise -1 gives +1. This gives us more variability. 13+8-19=2...so we have the 2(27s)! That variability allows us to search both +1, -1 when searching any '1' whether or not it is negative...the same with 8 and -8. BUT, depending on how you intend to apply the sign will dictate which of the two masks to search. This is a little difficult to understand suffice to say it is not a free for all and continues to severely limit our searches. I've only included this to explain how I can manipulate a +1 search to a -1 search by subtracting that cube. The opposite is also true going from -1 to +1. Of course we can do the same with +8 and -8. I guess what I'm confusing is that we can initially search both signs for potential candidates then isolate the exact column with sign manipulation.

The above three examples should be sufficient to show the concept of how we can reduce many summed cubes down to our requirement of no more than 3. We can do this for all legal K's whether or not they are positive or negative. The only problem remains in that we may have to search out to extremely large numbers before coming accross a solution. Because of the repeating nature of those charts there will eventually be some combination following the above approach that will yield results.

If we apply my approach to the recently discovered k=30...which is not too far into monstrous numbers we can see my approach at work. Is it a valid approach? Others have found that K=30=(2220422932)^3-(2218888517)^3-(283059965)^3.

K=30=1(27)+1+1+1. So applying my approach we will be looking for something like x(27)+1 plus y(27)+1 plus z(27)+1. But remember that it is legal for us to subtract (-1)s to give us access to +1. In this case they found a solution that resembled x(27)+1 subtract y(27)-1 subtract z(27)-1? But I quickly consulted my chart and it appears that there are no possible solutions using just entries from +/-1. So something else must be occuring! Let's see if I can easily figure out what masks were used to arrive at these numbers.

(2220422932)^3 = 10,947,302,325,566,084,787,191,541,568 = 405,455,641,687,632,769,895,983,021(27)+1
(2218888517)^3 = 10,924,622,727,902,378,924,946,084,413 = 404,615,656,588,976,997,220,225,349(27)-10
(283059965)^3 = 22,679,597,663,705,862,245,457,125 = 839,985,098,655,772,675,757,671(27)+8

This is interesting because it did not do as I expected. Instead it added another angle of complexity. This new twist does conform to my approach and simply adds to the validity of the 'proof' I'm trying to present. So untimately the 'smallest' solution to 30 was found to be of the form x(27)+1 subtract y(27)-10 subtract z(27)+8. Do you see how this relates to my approach? The x-y-z still works out to 1. So we continue to have 1(27)! But the masks are strange at first glance, right? Not really, when you realize that the ultimate mask +1+1+1 must result from the summing of the individual masks. So we have +1 subtract -10 subtract 8... or (+1) - (-8-1-1) - (+8) which is actually +1+8+1+1-8 which reduces to 1+1+1. Cool, eh?

I hadn't considered that we could do that to the mask and am mighty pleased I looked at K=30 for verification. That openned up many new doors to potential solutions.

With all this research I will have to question whether or not a solution for K=30 couldn't be found using the +1+1+1 split into three +1's, but as I noted above a quick glance at my own charts would indicate it is impossible. What I do want to point out is that +8+1+1 and -8 are both masks along with +1 that appear as the three repeating masks for the +1 column. Right? So in effect we are actually doing three +1s. What happens when we subtract a mask allows us to consider other stacks...in the case of -8, it let's us consider entries in 9m+8, right? This is becoming a tad bit confusing yet manageable.

The above example for K=30 makes it easier to see how subtracting a mask can turn it into another mask. This idea of subtracting allows us to introduce all the negative K's into the proof. Without doing any further work specifically on -K, we can easily show that we can create that -K by inverting what we did to get

+K. Right? A quick example could be K=27=1(27)+0; -K=-27=(Subtract){1(27)+0}. K=0+K; -K=0-K.

Conclusion

This approach should be sufficient to prove that for those K that are legal (not 9m+4, 9m+5, 9m-4 and 9m-5) we have repeating patterns consisting of repeating masks that afford us the opportunity to create any of them using a multiple of 27 plus a mask... x(27)+mask. This proves that for all K that we seek there is a specific form. These specific forms are simply a structured way to form any of them using nothing but cubes. The actual number of cubes at this point is not important. We simply want to prove to ourselves that there are solutions that can be formed using nothing but cubes.

Once that was established it was important to show that these could actually be reduced to 3 or fewer. I believe I have successfully proved that with my above multiple of 27 plus mask. There is always a way to rewrite a complicated large number of cubes as three or less. Exactly what we want to prove.

The meat and bones of this proof is that each of the reduced cube forms (max three of them, now!) can already be found somewhere. My writing them in the original starting form x(27)+mask allows us to reduce to at most three that take the form x(27)+mask as well. Logic dictates that when creating the charts all these forms were already accounted for. So those three new forms are already out there. Given enough time searching one could locate a solution. With the approach I've taken, I have shown without a doubt, that they are out there. They exist. They may be difficult to find and/or become very large numbers, but they are there.

Once the concept of masking and multiples of 27 were injected into the proof it naturally unfolded into something that was easily manageable.

This 'proof' could be used to find faster ways to identify combinations of cubes for whatever K you are searching. I believe you can also use this proof to predict the 'density' of that K as well. I wasn't interested with researching that component simply because one could deduce infinite solutions by looking at the 'masks' and the number of cubes required. For example if K=27=3^3+0^3+0^3...really one cube...makes infinite solutions possible because the other two can hold a K and it's -K cancelling each other out. 27+64-64; 27+125-125; 27+8-8;... You get the idea.

Ultimately, I have shown that all legal +/-K can be reduced to a maximum of 3 cubes which are already defined and redily available. So this is the Proof!

I hope you enjoyed my presentation of this proof.