Questions of Quantum Field Theory 1

prof. Angelo Bassi

Fall 2015

Alessandro Giacchetto & Danny Fainozzi

Contents

Klein-Gordon and Dirac equations

1 Klein-Gordon and Dirac equations

a) Write down the Klein-Gordon equation for a free relativistic particle (will all units explicitly shown). Show that plane waves are solutions of the equation and derive the energy spectrum. What is the problem with it?

b) Show that the Klein-Gordon equation satisfies a continuity equation and write down the explicit expression of the 4-current $j^\mu(x)$. Be careful in making sure that it has the correct dimensions. What is the problem here? Show it with a specific example.

c) Write down the free-particle Dirac equation (will all units explicitly shown) in the form

$$i\hbar\frac{\partial}{\partial t}\psi(x) = H_D\,\psi(x),$$

where H_D is the Dirac Hamiltonian, and starting from the requirement that the relativistic relation $E^2 = p^2c^2 + m^2c^4$ be satisfied, derive all relevant constraints on α_i and β. Write an explicit representation of α_i and β.

d) Show that the Dirac equation satisfies a continuity equation and write down the explicit expression of the 4-current $j^\mu(x)$. Be careful in making sure that it has the correct dimensions. Show that this current does not have the problem, which shows up for the current associated to the Klein-Gordon equation.

e) The free-particle Dirac equation still suffers from the problem of negative energy solutions. How did Dirac propose to cure this problem? Why does it work for the Dirac equation, but not for the Klein-Gordon equation? What does Dirac's idea imply?

---

a) The Klein-Gordon equation (KGE) for a free relativistic particle of mass m is

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0,$$

where $x = (ct, \boldsymbol{x})$ and $\Box = \partial^\mu\partial_\mu$. Let us consider a relativistic plane wave

$$\phi(x) = Ne^{-\frac{i}{\hbar}p^\mu x_\mu},$$

where N is a normalization factor. To show that it satisfies the KGE, we plug it in:

$$\begin{aligned}\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi(x) &= \left[\left(-\frac{i}{\hbar}\right)^2\left(\frac{E^2}{c^2} - p^2\right) + \frac{m^2c^2}{\hbar^2}\right]\phi(x) \\ &= -\frac{1}{\hbar^2c^2}(E^2 - p^2c^2 - m^2c^4)\phi(x) = 0,\end{aligned}$$

since we want the relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$ to hold. Thus, the energy spectrum is given by $E = \pm\sqrt{p^2c^2 + m^2c^4}$. The problem is that a free particle admits negative energies, in contrast with the classical case. It is not clear what it means that a free particle has a negative energy.

b) Multiplying the KGE by ϕ^* and the conjugate KGE by ϕ, we obtain

$$\phi^*\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi = 0, \qquad \phi\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi^* = 0.$$

Subtracting the second from the first, we have

$$\begin{aligned}0 = \phi^*\,\partial_\mu\partial^\mu\phi - \phi\,\partial_\mu\partial^\mu\phi^* &= [\partial_\mu(\phi^*\partial^\mu\phi) - (\partial_\mu\phi^*)(\partial^\mu\phi)] - [\partial_\mu(\phi\,\partial^\mu\phi^*) - (\partial_\mu\phi)(\partial^\mu\phi^*)] \\ &= \partial_\mu(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*).\end{aligned}$$

Defining $j^\mu = \frac{i\hbar}{2m}(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*)$, we obtain the continuity equation $\partial_\mu j^\mu = 0$.

In analogy with the Schrödinger equation, we would like to interpret j^μ as a probability current and $\rho = j^0/c$ as a probability density. However we face the problem of negative probability. For example, let ϕ be a plane wave. Then

$$\begin{aligned} \rho(x) &= \frac{i\hbar}{2mc}\left[\phi^*(x)\left(-\frac{iE}{\hbar c}\right)\phi(x) - \phi(x)\left(\frac{iE}{\hbar c}\right)\phi^*(x)\right] \\ &= \frac{E}{mc^2}|N|^2, \end{aligned}$$

which is negative for negative-energy waves.

c) The free-particle Dirac equation (DE) is a linear first-order partial differential equation

$$i\hbar \frac{\partial}{\partial t}\psi(x) = \underbrace{\left(-i\hbar c\, \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta\right)}_{=H_D}\psi(x).$$

The relativistic dispersion relation is satisfied if ψ is a solution of the KGE. We first notice that α_i, β cannot be numbers, otherwise the DE would not be invariant under space rotations. Thus, they must be $N \times N$ matrices. Now, the second time-derivative of ψ is

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi(x) = \left(-\hbar^2 c^2 \sum_{j,k=1}^{3} \frac{\alpha_j\alpha_k + \alpha_k\alpha_j}{2}\frac{\partial^2}{\partial x^j \partial x^k} - i\hbar m c^3 \sum_{j=1}^{3}(\alpha_j\beta + \beta\alpha_j)\frac{\partial}{\partial x^j} + m^2c^4\beta^2\right)\psi(x),$$

that is

$$\left(\frac{\partial^2}{\partial (ct)^2} - \sum_{j,k=1}^{3}\frac{\{\alpha_j,\alpha_k\}}{2}\frac{\partial^2}{\partial x^j \partial x^k} - i\frac{mc}{\hbar}\sum_{j=1}^{3}\{\alpha_j,\beta\}\frac{\partial}{\partial x^j} + \frac{m^2c^2}{\hbar^2}\beta^2\right)\psi(x) = 0.$$

Comparing it with the KGE, we obtain the relations

$$\begin{aligned} &\{\alpha_j, \alpha_k\} = 2\delta_{jk}\mathbb{1} \\ &\{\alpha_j, \beta\} = 0 \\ &\alpha_j^2 = \beta^2 = \mathbb{1}. \end{aligned}$$

This is the so-called Clifford algebra. Since we want H_D to be self-adjoint, then $\alpha_j^\dagger = \alpha_j$, $\beta^\dagger = \beta$. Then the matrices are diagonalizable. When squared, they give the identity matrix. Then their eigenvalues are ± 1. Further, from the anticommutation relation, we have $\alpha_j = -\beta\alpha_j\beta$. Hence,

$$\begin{aligned} \operatorname{tr}(\alpha_j) &= \operatorname{tr}(-\beta\alpha_j\beta) \\ &= -\operatorname{tr}(\beta^2\alpha_j) \\ &= -\operatorname{tr}(\alpha_j), \end{aligned}$$

that is, $\operatorname{tr}(\alpha_j) = 0$. Traceless matrices with eigenvalues ± 1 must be of even dimension. The smallest possible value is $N = 2$, but there exist only three anticommuting matrices of dimension 2×2 (namely, the Pauli matrices σ_j), and they do not do the job. Thus, the smallest admittable dimension is $N = 4$. A representation of the matrices is

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

d) Multiplying the DE by $\psi^\dagger$ on the left and the conjugate DE by ψ on the right, we obtain

$$\begin{aligned} i\hbar\psi^\dagger \frac{\partial \psi}{\partial t} &= (-i\hbar c\, \psi^\dagger \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + mc^2\psi^\dagger\beta\psi) \\ -i\hbar\frac{\partial \psi^\dagger}{\partial t}\psi &= (i\hbar c\, \boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\alpha}\psi + mc^2\psi^\dagger\beta\psi). \end{aligned}$$

Dividing both expressions by $i\hbar c$ and subtracting the second from the first, we obtain

$$\psi^\dagger \frac{\partial \psi}{\partial (ct)} + \frac{\partial \psi^\dagger}{\partial (ct)}\psi = -\psi^\dagger \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\alpha}\psi.$$

By using Einstein's summation convention, we can rewrite the above expression as follows

$$\underbrace{\psi^\dagger(\partial_0\psi) + (\partial_0\psi^\dagger)\psi}_{=\partial_0(\psi^\dagger\psi)} + \underbrace{\psi^\dagger\alpha_j(\partial_j\psi) + (\partial_i\psi^\dagger)\alpha_i\psi}_{\partial_j(\psi^\dagger\alpha_j\psi)} = 0.$$

Let us define the 4-current $j = (c\rho, \boldsymbol{j}) = (c\,\psi^\dagger\psi, c\,\psi^\dagger\boldsymbol{\alpha}\psi)$. The we have the continuity equation

$$\partial_\mu j^\mu = 0.$$

The first component ρ of the 4-current is

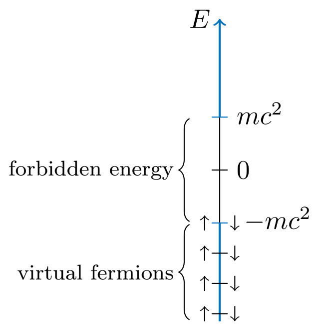
$$\rho(x) = \psi^\dagger(x)\psi(x) = \sum_{r=1}^{4}|\psi_r(x)|^2 \geq 0.$$

It is positive by construction, and can be safely interpreted as a probability density.

e) The free-particle DE still suffers the problem of negative-energy solutions, as the relativistic dispersion relation holds true. Dirac proposed the following solution: all negative-energy states are occupied by *negative-energy* particles with opposite spin and, thanks to Pauli's exclusion principle, these states cannot be occupied by any other particle. This is what we call the *vacuum* or *ground state*. All other particles must then have positive energy, and form matter as we know it.

This solution does not apply to the KGE, since it refers to bosons and the Pauli principle does not apply to them. The negative energy states cannot be filled in a stable way.

The vacuum corresponds to a *many-body state* in which all negative-energy particle states are occupied, forming the so-called Dirac sea. Then the Dirac equation does not make sense as a single particle equation. It makes sense as a many-particle equation. This is the first step towards Quantum Field Theory.



There is an important consequence of assuming the Dirac sea. A negative-energy particle can be promoted to a positive-energy particle by giving it energy through some potential. We are left with a positive energy particle coming out of the sea, and a hole in the sea. This hole can be seen as a new particle with positive energy (since energy was required to create it) and opposite charge. Thus, the DE and the Dirac see predict the existence of what we now call antiparticles.

2 Free-paricle solutions of the Dirac equation

a) Write down the free-particle Dirac equation (will all units explicitly shown) in the form

$$i\hbar\frac{\partial}{\partial t}\psi(x) = H_D\psi(x),$$

where H_D is the Dirac Hamiltonian. Consider a plane wave solution

$$\psi_{\boldsymbol{p}}(x) = N_p w(\boldsymbol{p}) e^{-\frac{i}{\hbar}(Et-\boldsymbol{p}\cdot\boldsymbol{x})},$$

where N_p is a normalization factor and $w(\boldsymbol{p})$ a spinor, and show that it implies the relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$.

b) Derive the explicit expression for the four Dirac spinors $w(\boldsymbol{p})$, so that the general solution of the Dirac equation can be written as:

$$\psi_{\boldsymbol{p}}^{(r)}(x) = N_p w_r(\boldsymbol{p}) e^{-\frac{i}{\hbar}\epsilon_r(E_p t - \boldsymbol{p}\cdot\boldsymbol{x})} \qquad r = 1,2,3,4,$$

with $E_p = \sqrt{p^2c^2 + m^2c^4}$, and $\epsilon_r = +1$ for $r = 1,2$ and $\epsilon_r = -1$ for $r = 3,4$.

c) Prove that the Dirac spinors (with the appropriate factor in front) satisfy the orthogonality relations

$$\begin{aligned} w_r^\dagger(\epsilon_r\boldsymbol{p}) w_{r'}(\epsilon_{r'}\boldsymbol{p}) &= \frac{E_p}{mc^2}\,\delta_{r,r'} \\ \bar{w}_r(\boldsymbol{p}) w_{r'}(\boldsymbol{p}) &= \epsilon_r\,\delta_{r,r'} \end{aligned}$$

d) Prove that the Dirac spinors satisfy the completeness relation

$$\begin{aligned} \sum_{r=1}^{4} [w_r(\epsilon_r\boldsymbol{p})]_\alpha [w_r^\dagger(\epsilon_r\boldsymbol{p})]_\beta &= \frac{E_p}{mc^2}\,\delta_{\alpha,\beta} \\ \sum_{r=1}^{4} \epsilon_r [w_r(\boldsymbol{p})]_\alpha [\bar{w}_r(\boldsymbol{p})]_\beta &= \delta_{\alpha,\beta} \end{aligned}$$

e) Using the orthogonality relations, derive the explicit expression for the normalization factor N_p so that:

$$\int d^3x\; \psi_{\boldsymbol{p}}^{\dagger(r)}(x)\psi_{\boldsymbol{p}'}^{(r')}(x) = \delta_{r,r'}\,\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}').$$

---

a) The free-particle Dirac equation (DE) is linear first-order partial differential equation

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \underbrace{\left(-i\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta\right)}_{=H_D}\psi(x),$$

where the matrices α_j, β are self-adjoint matrices satisfying the Clifford algebra

$$\begin{aligned} &\{\alpha_j, \alpha_k\} = 2\delta_{jk}\mathbb{1} \\ &\{\alpha_j, \beta\} = 0 \\ &\alpha_j^2 = \beta^2 = \mathbb{1}. \end{aligned}$$

Let us take the representation

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

and consider a plane wave solution $\psi_{\boldsymbol{p}}(x) = N_p\, w(\boldsymbol{p}) e^{-\frac{i}{\hbar}p^\mu x_\mu}$. Inserting it into the DE, we obtain

$$i\hbar\left(-\frac{iE}{\hbar}\right)\psi_{\boldsymbol{p}}(x) = \left[-i\hbar c\,\boldsymbol{\alpha}\cdot\left(\frac{i\boldsymbol{p}}{\hbar}\right) + mc^2\beta\right]\psi_{\boldsymbol{p}}(x).$$

Writing $w = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ (omitting the dependence on $\boldsymbol{p}$) we obtain

$$E\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \left[c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} + mc^2\beta\right]\begin{pmatrix} \varphi \\ \chi \end{pmatrix}.$$

The product $\boldsymbol{\alpha}\cdot\boldsymbol{p}$ can be written as

$$\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix}$$

and the equation becomes

$$\begin{pmatrix} (E-mc^2)\mathbb{1} & -c\,\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -c\,\boldsymbol{\sigma}\cdot\boldsymbol{p} & (E+mc^2)\mathbb{1} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0.$$

In order to have non-trivial solution, it must be

$$0 = \begin{vmatrix} (E-mc^2)\mathbb{1} & -c\,\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -c\,\boldsymbol{\sigma}\cdot\boldsymbol{p} & (E+mc^2)\mathbb{1} \end{vmatrix} = (E^2-m^2c^4) - c^2(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2.$$

Now, from $(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \sum_{j,k=1}^{3} \frac{\{\sigma_j,\sigma_k\}}{2} p_j p_k = p^2\,\mathbb{1}$, we obtain the relativistic dispersion relation

$$E^2 = p^2c^2 + m^2c^4.$$

b) Let us write the plane wave solution as

$$\psi^{(r)}_{\boldsymbol{p}}(x) = N_p\, w_r(\boldsymbol{p}) e^{-\frac{i}{\hbar}\epsilon_r(E_p t - \boldsymbol{p}\cdot\boldsymbol{x})},$$

where E_p is the positive energy solution. The system above becomes now

$$\begin{pmatrix} (\epsilon_r E_p - mc^2)\mathbb{1} & -c\epsilon_r\,\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -c\epsilon_r\,\boldsymbol{\sigma}\cdot\boldsymbol{p} & (\epsilon_r E_p + mc^2)\mathbb{1} \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0.$$

Let us consider the positive energy ($r=1,2$). From the second row, we obtain

$$\chi = \frac{c\,\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p + mc^2}\varphi.$$

The two-row complex vector φ is left undetermined. This means that we have two independent degrees of freedom (φ has four degrees of freedom, but one has to subtract the normalization factor and an unimportant global phase). Thus, we have two independent solutions: one for $\varphi = M_{\boldsymbol{p}}\begin{pmatrix}1\\0\end{pmatrix}$ and one for $\varphi = M_{\boldsymbol{p}}\begin{pmatrix}0\\1\end{pmatrix}$, where $M_{\boldsymbol{p}}$ is an appropriate normalization factor. Since

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix},$$

setting $p_\pm = p_1 \pm ip_2$ we have the two independent spinors

$$w_1(\boldsymbol{p}) = M_{\boldsymbol{p}} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \end{pmatrix}, \qquad w_2(\boldsymbol{p}) = M_{\boldsymbol{p}} \begin{pmatrix} 0 \\ 1 \\ \frac{cp_-}{E_p+mc^2} \\ \frac{-cp_3}{E_p+mc^2} \end{pmatrix}.$$

Let us consider now the negative energy ($r=3,4$). From the first row, we obtain

$$\varphi = \frac{c\,\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p + mc^2}\chi.$$

Again, we have two independent solutions: the one for $\chi = M_{\boldsymbol{p}}\begin{pmatrix}1\\0\end{pmatrix}$ and the one for $\chi = M_{\boldsymbol{p}}\begin{pmatrix}0\\1\end{pmatrix}$. The two corresponding independent spinors are

$$w_3(\boldsymbol{p}) = M_{\boldsymbol{p}} \begin{pmatrix} \frac{cp_3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \\ 1 \\ 0 \end{pmatrix}, \qquad w_4(\boldsymbol{p}) = M_{\boldsymbol{p}} \begin{pmatrix} \frac{cp_-}{E_p+mc^2} \\ \frac{-cp_3}{E_p+mc^2} \\ 0 \\ 1 \end{pmatrix}.$$

We have chosen the spinors this way because they turn out to be orthogonal and complete.

c) Let us check the orthogonality relations. We do it for some meaningful cases. For $r = r' = 1$, we have (note that $p_\pm^* = p_\mp$)

$$\begin{aligned} w_1^\dagger(\boldsymbol{p}) w_1(\boldsymbol{p}) &= |M_{\boldsymbol{p}}|^2 \begin{pmatrix} 1 & 0 & \frac{cp_3}{E_p+mc^2} & \frac{cp_-}{E_p+mc^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \end{pmatrix} \\ &= |M_{\boldsymbol{p}}|^2 \left(1 + \frac{p^2c^2}{(E_p+mc^2)^2} \right) \\ &= |M_{\boldsymbol{p}}|^2 \, \frac{E_p^2 + 2E_p mc^2 + m^2c^4 + p^2c^2}{(E_p+mc^2)^2} \\ &= |M_{\boldsymbol{p}}|^2 \, \frac{2E_p}{E_p+mc^2}. \end{aligned}$$

For $r = 2, r' = 3$, we have

$$\begin{aligned} w_2^\dagger(\boldsymbol{p}) w_3(-\boldsymbol{p}) &= |M_{\boldsymbol{p}}|^2 \begin{pmatrix} 0 & 1 & \frac{cp_+}{E_p+mc^2} & \frac{-cp_3}{E_p+mc^2} \end{pmatrix} \begin{pmatrix} \frac{-cp_3}{E_p+mc^2} \\ \frac{-cp_+}{E_p+mc^2} \\ 1 \\ 0 \end{pmatrix} \\ &= |M_{\boldsymbol{p}}|^2 \left(-\frac{cp_+}{E_p+mc^2} + \frac{cp_+}{E_p+mc^2} \right) = 0. \end{aligned}$$

Finally, for $r = r' = 4$, we have

$$\begin{aligned} w_4^\dagger(-\boldsymbol{p}) w_4(-\boldsymbol{p}) &= |M_{\boldsymbol{p}}|^2 \begin{pmatrix} \frac{-cp_+}{E_p+mc^2} & \frac{cp_3}{E_p+mc^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-cp_-}{E_p+mc^2} \\ \frac{cp_3}{E_p+mc^2} \\ 0 \\ 1 \end{pmatrix} \\ &= |M_{\boldsymbol{p}}|^2 \left(\frac{p^2c^2}{(E_p+mc^2)^2} + 1 \right) \\ &= |M_{\boldsymbol{p}}|^2 \, \frac{2E_p}{E_p+mc^2}. \end{aligned}$$

The other cases are similar. Then, we can choose $M_{\boldsymbol{p}} = \sqrt{\frac{E_p+mc^2}{2mc^2}}$ to have

$$w_r^\dagger(\epsilon_r \boldsymbol{p}) w_{r'}(\epsilon_{r'} \boldsymbol{p}) = \frac{E_p}{mc^2} \delta_{rr'}.$$

The other orthogonal relation can be obtained in a similar way. For $r = r' = 1$,

$$\begin{aligned} \bar{w}_1(\boldsymbol{p}) w_1(\boldsymbol{p}) &= |M_{\boldsymbol{p}}|^2 \begin{pmatrix} 1 & 0 & \frac{cp_3}{E_p+mc^2} & \frac{cp_-}{E_p+mc^2} \end{pmatrix} \beta \begin{pmatrix} 1 \\ 0 \\ \frac{cp_3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \end{pmatrix} \\ &= |M_{\boldsymbol{p}}|^2 \left(1 - \frac{p^2c^2}{(E_p+mc^2)^2} \right) \\ &= |M_{\boldsymbol{p}}|^2 \, \frac{E_p^2 + 2E_p mc^2 + m^2c^4 - p^2c^2}{(E_p+mc^2)^2} \\ &= |M_{\boldsymbol{p}}|^2 \, \frac{2mc^2}{E_p+mc^2} = 1. \end{aligned}$$

For $r=2, r'=3$, we have

$$\begin{aligned}\bar{w}_2(\boldsymbol{p})w_3(\boldsymbol{p}) &= |M_{\boldsymbol{p}}|^2 \begin{pmatrix}0 & 1 & \frac{cp_+}{E_p+mc^2} & \frac{-cp_3}{E_p+mc^2}\end{pmatrix} \beta \begin{pmatrix}\frac{cp_3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \\ 1 \\ 0\end{pmatrix} \\ &= |M_{\boldsymbol{p}}|^2 \left(\frac{cp_+}{E_p+mc^2} - \frac{cp_+}{E_p+mc^2}\right) = 0.\end{aligned}$$

Finally, for $r=r'=4$, we have

$$\begin{aligned}\bar{w}_4(\boldsymbol{p})w_4(\boldsymbol{p}) &= |M_{\boldsymbol{p}}|^2 \begin{pmatrix}\frac{cp_+}{E_p+mc^2} & \frac{-cp_3}{E_p+mc^2} & 0 & 1\end{pmatrix} \beta \begin{pmatrix}\frac{cp_-}{E_p+mc^2} \\ \frac{-cp_3}{E_p+mc^2} \\ 0 \\ 1\end{pmatrix} \\ &= |M_{\boldsymbol{p}}|^2 \left(\frac{p^2c^2}{(E_p+mc^2)^2} - 1\right) \\ &= |M_{\boldsymbol{p}}|^2 \frac{-2mc^2}{E_p+mc^2} = -1.\end{aligned}$$

The other cases are similar and in general

$$\bar{w}_r(\epsilon_r\boldsymbol{p})w_{r'}(\epsilon_{r'}\boldsymbol{p}) = \epsilon_r\delta_{rr'}.$$

d) Let us check the completeness relation. For $\alpha=\beta=1$

$$\sum_{r=1}^{4}[w_r(\epsilon_r\boldsymbol{p})]_1[w_r^\dagger(\epsilon_r\boldsymbol{p})]_1 = |M_{\boldsymbol{p}}|^2\left(1+0+\frac{c^2p_3^2}{(E_p+mc^2)^2}+\frac{c^2(p_1^2+p_2^2)}{(E_p+mc^2)^2}\right) = \frac{E_p}{mc^2}.$$

For $\alpha=2, \beta=3$

$$\sum_{r=1}^{4}[w_r(\epsilon_r\boldsymbol{p})]_2[w_r^\dagger(\epsilon_r\boldsymbol{p})]_3 = |M_{\boldsymbol{p}}|^2\left(0+\frac{-cp_+}{E_p+mc^2}+\frac{cp_+}{E_p+mc^2}+0\right) = 0.$$

The other cases are similar. In general,

$$\sum_{r=1}^{4}[w_r(\epsilon_r\boldsymbol{p})]_\alpha[w_r^\dagger(\epsilon_r\boldsymbol{p})]_\beta = \frac{E_p}{mc^2}\delta_{\alpha\beta}.$$

The second completeness relation for $\alpha=\beta=1$ is

$$\sum_{r=1}^{4}\epsilon_r[w_r(\boldsymbol{p})]_1[\bar{w}_r(\boldsymbol{p})]_1 = |M_{\boldsymbol{p}}|^2\left(1+0+\frac{-c^2p_3^2}{(E_p+mc^2)^2}+\frac{-c^2(p_1^2+p_2^2)}{(E_p+mc^2)^2}\right) = 1.$$

For $\alpha=2, \beta=3$

$$\sum_{r=1}^{4}\epsilon_r[w_r(\boldsymbol{p})]_2[\bar{w}_r(\boldsymbol{p})]_3 = |M_{\boldsymbol{p}}|^2\left(0+\frac{-cp_+}{E_p+mc^2}-\frac{-cp_+}{E_p+mc^2}+0\right) = 0.$$

The other cases are similar. In general,

$$\sum_{r=1}^{4}\epsilon_r[w_r(\boldsymbol{p})]_\alpha[\bar{w}_r(\boldsymbol{p})]_\beta = \delta_{\alpha\beta}.$$

e) Let us evaluate the scalar product at equal-time t.

$$\langle\psi_{\boldsymbol{p}}^{(r)}|\psi_{\boldsymbol{p}'}^{(r')}\rangle = \int d^3x\,\psi_{\boldsymbol{p}}^{(r)\dagger}(x)\psi_{\boldsymbol{p}'}^{(r')}(x)$$
$$= N_p^* N_{p'}\, w_r^\dagger(\boldsymbol{p})w_{r'}(\boldsymbol{p}')e^{-\frac{i}{\hbar}(\epsilon_{r'}E_{p'}-\epsilon_r E_p)t}\int d^3x\, e^{+\frac{i}{\hbar}(\epsilon_{r'}\boldsymbol{p}'\cdot\boldsymbol{x}-\epsilon_r\boldsymbol{p}\cdot\boldsymbol{x})}.$$

We know that

$$\int d^3x\, e^{-\frac{i}{\hbar}(\epsilon_{r'}\boldsymbol{p}'\cdot\boldsymbol{x}-\epsilon_r\boldsymbol{p}\cdot\boldsymbol{x})} = (2\pi\hbar)^3\,\delta^{(3)}(\epsilon_r\boldsymbol{p}-\epsilon_{r'}\boldsymbol{p}').$$

If $|\boldsymbol{p}|\neq|\boldsymbol{p}'|$, then everything is zero. If $\boldsymbol{p}=-\boldsymbol{p}'$, then ϵ_r and $\epsilon_{r'}$ must have opposite sign ($r\neq r'$) in order for the Dirac delta not to vanish. But in this case $w_r^\dagger(\boldsymbol{p})w_{r'}(-\boldsymbol{p})=0$. Thus, the only case that survives is $\boldsymbol{p}=\boldsymbol{p}'$, therefore one can write:

$$\langle\psi_{\boldsymbol{p}}^{(r)}|\psi_{\boldsymbol{p}'}^{(r')}\rangle = |N_p|^2\frac{E_p}{mc^2}\,\delta_{rr'}(2\pi\hbar)^3\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'),$$

that is

$$N_p = \frac{1}{\sqrt{(2\pi\hbar)^3}}\sqrt{\frac{mc^2}{E_p}}.$$

3 Dirac equation and relativity

a) Write down the free-particle Dirac equation (will all units be explicitly shown) in the form

$$i\hbar\frac{\partial}{\partial t}\psi(x) = H_D\psi(x),$$

where H_D is the Dirac Hamiltonian. Using the minimal-coupling prescription, write the corresponding equation for a Dirac particle interacting with an electromagnetic field. Show that at the non-relativistic limit, it gives the Pauli equation.

b) Using the γ matrices, write the free-particle Dirac equation in a covariant form. Given the relation

$$\psi'(x') = S(\Lambda)\psi(x)$$

for the Dirac spinor expressed in two different inertial frames connected by a Lorentz transformation Λ, derive the conditions S must satisfy, in order for the Dirac equation to be Lorentz invariant.

c) Consider an infinitesimal Lorentz transformation and derive the explicit (infinitesimal) expression of S in terms of the infinitesimal parameters defining Λ and of the γ matrices.

d) Derive the (finite) expression for S for a rotation of an angle ϑ along the z axis. Generalize it to the case of a rotation along an arbitrary direction $\hat{\mathbf{n}}$ (without proof). Derive also the (finite) expression for S for a boost along the x axis. Generalize it to the case of a boost along an arbitrary direction $\hat{\mathbf{n}}$ (without proof).

e) Consider the free-particle solution of the Dirac equation with $p=0$:

$$\psi_{\boldsymbol{p}=\mathbf{0}}^{(r)}(x) = N_0 w_r(\mathbf{0})e^{-\frac{i}{\hbar}\epsilon_r E_{p=0}t} \qquad r=1,2,3,4,$$

with

$$w_1(\mathbf{0})=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\qquad w_2(\mathbf{0})=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\qquad w_3(\mathbf{0})=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\qquad w_4(\mathbf{0})=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Using the previous results, derive the (correctly normalized) free-particle solution for an arbitrary momentum $\boldsymbol{p}$.

a) The free-particle Dirac equation (DE) is a linear first-order partial differential equation

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \underbrace{\left(-i\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta\right)}_{=H_D}\psi(x),$$

where the matrices α_j, β are self-adjoint matrices satisfying the Clifford algebra

$$\begin{aligned}&\{\alpha_j, \alpha_k\} = 2\delta_{jk}\mathbb{1}\\ &\{\alpha_j, \beta\} = 0\\ &\alpha_j^2 = \beta^2 = \mathbb{1}.\end{aligned}$$

The minimal coupling prescription

$$p^\mu \longrightarrow \Pi^\mu = p^\mu - \frac{e}{c}A^\mu$$

can be written in operatorial form in time as

$$\left(i\hbar\frac{\partial}{\partial t}, -i\hbar c\boldsymbol{\nabla}\right) \longrightarrow \left(i\hbar\frac{\partial}{\partial t} - eA^0, c\boldsymbol{\Pi}\right).$$

Thus, the DE for a charged particle interacting with an electromagnetic field is

$$i\hbar\frac{\partial}{\partial t}\psi(x) = \left(c\,\boldsymbol{\alpha}\cdot\boldsymbol{\Pi} + mc^2\beta + eA^0\right)\psi(x).$$

Let us separate the rest energy:

$$\psi(x) = \begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t}.$$

Then

$$i\hbar\frac{\partial}{\partial t}\psi(x) = i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t} + mc^2\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t},$$

$$\left(c\,\boldsymbol{\alpha}\cdot\boldsymbol{\Pi} + mc^2\beta + eA^0\right)\psi(x) = \left[c\,\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} + mc^2\begin{pmatrix}\varphi(x)\\ -\chi(x)\end{pmatrix} + eA^0\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix}\right] e^{-\frac{i}{\hbar}mc^2 t}.$$

Hence, the equation becomes

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} = \left[c\,\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \chi(x)\end{pmatrix} + eA^0\begin{pmatrix}\varphi(x)\\ \chi(x)\end{pmatrix}\right].$$

From the second equation, using the non-relativistic assumptions $|i\hbar\frac{\partial}{\partial t}\chi(x)| \ll |mc^2\chi(x)|$ and $|eA^0\chi(x)| \ll |mc^2\chi(x)|$ (*i.e*, kinetic and potential energy respectively much smaller than the rest energy), and using the fact that

$$\boldsymbol{\alpha}\cdot\boldsymbol{\Pi} = \begin{pmatrix}0 & \boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\\ \boldsymbol{\sigma}\cdot\boldsymbol{\Pi} & 0\end{pmatrix},$$

we obtain

$$\chi(x) = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}}{2mc}\varphi(x).$$

This means that χ represents a small component of the wave function ($\chi \sim (v/2c)\phi$), and can be neglected. Inserting it in the first equation, we get

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\varphi(x) &= c\,(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}}{2mc}\varphi(x) + eA^0\varphi(x)\\ &= \left[\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})^2}{2m} + eA^0\right]\varphi(x).\end{aligned}$$

Finally, from

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})^2 &= \Pi^2 + i\boldsymbol{\sigma}\cdot\left[\left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}\right)\times\left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}\right)\right]\\ &= \Pi^2 - \frac{\hbar e}{c}\left[\,\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) + \boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{\nabla})\right],\end{aligned}$$

and by applying the operators to $\varphi(x)$

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})^2\varphi(x) &= \Pi^2\varphi(x) - \frac{\hbar e}{c}\big[\,\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) + \boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{\nabla})\,\big]\varphi(x) \\
&= \Pi^2\varphi(x) - \frac{\hbar e}{c}\big[\,\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}\varphi(x)) + \boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{\nabla}\varphi(x))\,\big] \\
&= \Pi^2\varphi(x) - \frac{\hbar e}{c}\big[\,\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\varphi(x) + \underbrace{\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\varphi(x)\times\boldsymbol{A}) + \boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{\nabla}\varphi(x))}_{=\,0}\,\big],
\end{aligned}
$$

we get the Pauli equation

$$
i\hbar\frac{\partial}{\partial t}\varphi(x) = \left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}\right)^2 - \frac{\hbar e}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B} + eA^0\right]\varphi(x).
$$

b) Let us define $\gamma^0 = \beta$, $\gamma^j = \beta\alpha_j$. We can write the DE in terms of the γ matrices by multiplying on the left by β/c as

$$
i\hbar\gamma^0\frac{\partial}{\partial(ct)}\psi(x) = (-i\hbar\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + mc)\psi(x).
$$

The γ-matrices satisfies the algebra

$$
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}.
$$

Since $\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} = \gamma^i\partial_i$, we can write the DE in the covariant notation as

$$
(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0
$$

or, setting $\partial\!\!\!/ = \gamma^\mu\partial_\mu$,

$$
(i\hbar\partial\!\!\!/ - mc)\psi(x) = 0.
$$

Let us consider now a Lorentz transformation

$$
x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu.
$$

Let us suppose that the spinor $\psi(x)$ transforms according to a linear transformation:

$$
\psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x),
$$

where $S(\Lambda)$ is a 4×4 matrix. Since also the opposite transformation is possible according to special relativity, then $S(\Lambda)$ must be invertible, with $S(\Lambda)^{-1} = S(\Lambda^{-1})$; in fact:

$$
\psi(x) = \psi(\Lambda^{-1}x') = S(\Lambda^{-1})\psi'(x').
$$

Imposing the covariance of the DE, we have

$$
\begin{aligned}
(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) &= 0 \\
(i\hbar\gamma'^\mu\partial'_\mu - mc)\psi'(x') &= 0.
\end{aligned}
$$

Since different representations of the Clifford algebra (with self-adjoint matrices) are related by unitary transformations which do not change the physics, we can suppose that $\gamma^\mu = \gamma'^\mu$. Now, let us take the first equation multiplied on the left by $S(\Lambda)$. Expressing $\psi(x)$ as $S(\Lambda)^{-1}\psi'(x')$, we have

$$
(i\hbar S(\Lambda)\gamma^\mu S(\Lambda)^{-1}\partial_\mu - mc)\psi'(x') = 0.
$$

On the other hand, for the chain rule,

$$
\partial_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu = \Lambda^\nu{}_\mu\partial'_\nu.
$$

Thus, the equation becomes

$$
(i\hbar S(\Lambda)\gamma^\mu S(\Lambda)^{-1}\Lambda^\nu{}_\mu\partial'_\nu - mc)\psi'(x') = 0.
$$

Comparing it with the DE for $\psi'(x')$, we obtain the fundamental relation

$$S(\Lambda)\gamma^\mu S(\Lambda)^{-1}\Lambda^\nu{}_\mu = \gamma^\nu,$$

or equivalently

$$S(\Lambda)\gamma^\mu S(\Lambda)^{-1} = \Lambda_\nu{}^\mu\gamma^\nu.$$

The covariance of the DE is demonstrated once we have found a solution $S(\Lambda)$ of the equation above.

c) Let us consider an infinitesimal Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu.$$

By the fact that $\Lambda^\sigma{}_\nu\Lambda_\sigma{}^\mu = \delta^\mu_\nu$, we derive that

$$\delta^\mu_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + \omega_\nu{}^\mu + \text{h.o.t.}$$

(where h.o.t. means "higher order terms"), that is $\omega^\mu{}_\nu = -\omega_\nu{}^\mu$ or equivalently $\omega^{\mu\nu} = -\omega^{\nu\mu}$. Let us suppose now that $S(\Lambda)$ is of the form

$$S(\Lambda) = \mathbb{1} - \frac{i}{4}\sigma_{\mu\nu}\,\omega^{\mu\nu},$$

with $\sigma_{\mu\nu}$ 4×4 matrices such that $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$. We note that, with this assumption, we can obtain a finite transformation by summing infinitesimal ones, in the following way:

$$S(\Lambda) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{4}\sigma_{\mu\nu}\frac{\omega^{\mu\nu}}{N}\right)^N = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}},$$

where $\omega^{\mu\nu}/N$ represents the infinitesimal term of the transformation.

Since $(\Lambda^{-1})^\mu{}_\nu = \delta^\mu_\nu - \omega^\mu{}_\nu$, the fundamental relation becomes

$$\left(\mathbb{1} - \frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\right)\gamma^\mu\left(\mathbb{1} + \frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\right) = \left(\delta^\mu_\nu + \omega_\nu{}^\mu\right)\gamma^\nu.$$

This can be reduced to

$$-\frac{i}{4}\omega^{\alpha\beta}[\sigma_{\alpha\beta},\gamma^\mu] + \text{h.o.t.} = \omega_\nu{}^\mu\gamma^\nu,$$

that is

$$\begin{aligned}\omega^{\alpha\beta}[\sigma_{\alpha\beta},\gamma^\mu] &= 4i\,\eta^\mu{}_\lambda\omega_\nu{}^\lambda\gamma^\nu\\ &= 4i\,\eta^\mu{}_\lambda\omega^{\nu\lambda}\gamma_\nu.\end{aligned}$$

Thanks to the antisymmetry $2\omega^{\nu\lambda} = -(\omega^{\lambda\nu} - \omega^{\nu\lambda})$, we have

$$\begin{aligned}\omega^{\alpha\beta}[\sigma_{\alpha\beta},\gamma^\mu] &= -2i(\eta^\mu{}_\lambda\omega^{\lambda\nu}\gamma_\nu - \eta^\mu{}_\lambda\omega^{\nu\lambda}\gamma_\nu)\\ &= -2i\,\omega^{\alpha\beta}(\eta^\mu{}_\alpha\gamma_\beta - \eta^\mu{}_\beta\gamma_\alpha),\end{aligned}$$

where in the last step we have changed the indices $(\lambda,\nu)\to(\alpha,\beta)$ in the first addendum and $(\nu,\lambda)\to(\alpha,\beta)$ in the last one. Since $\omega^{\alpha\beta}$ is arbitrary and multiplies an antisymmetric expression on both sides (and we sum over α and β), they must be the same. Thus, we have the commutation relation

$$[\sigma_{\alpha\beta},\gamma^\mu] = -2i(\eta^\mu{}_\alpha\gamma_\beta - \eta^\mu{}_\beta\gamma_\alpha).$$

This relation follows from the fundamental one for the $S(\Lambda)$ matrices, in the case of infinitesimal transformation. The problem is now reduced to that of determining the six matrices $\sigma_{\alpha\beta}$. The natural candidate is:

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha,\gamma_\beta].$$

As a proof of it:

$$\begin{aligned}
[\sigma_{\alpha\beta}, \gamma^\mu] &= \frac{i}{2}\left[[\gamma_\alpha, \gamma_\beta], \gamma^\mu\right] \\
&= \frac{i}{2}\big([\gamma_\alpha\gamma_\beta, \gamma^\mu] - [\gamma_\beta\gamma_\alpha, \gamma^\mu]\big) \\
&= \frac{i}{2}\big([\gamma_\alpha\gamma_\beta, \gamma^\mu] - [-\gamma_\alpha\gamma_\beta + 2\eta_{\alpha\beta}\mathbb{1}, \gamma^\mu]\big) \\
&= i\,[\gamma_\alpha\gamma_\beta, \gamma^\mu].
\end{aligned}$$

where we have used the anticommutation relations $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}\mathbb{1}$. On the other hand, we have

$$\begin{aligned}
[\gamma_\alpha\gamma_\beta, \gamma^\mu] &= \gamma_\alpha\gamma_\beta\gamma^\mu - \gamma^\mu\gamma_\alpha\gamma_\beta \\
&= \gamma_\alpha\gamma_\beta\gamma^\mu + \gamma_\alpha\gamma^\mu\gamma_\beta - 2\eta^\mu{}_\alpha\gamma_\beta \\
&= -2(\eta^\mu{}_\alpha\gamma_\beta - \eta^\mu{}_\beta\gamma_\alpha).
\end{aligned}$$

Therefore, we have obtained the commutation relation above

$$[\sigma_{\alpha\beta}, \gamma^\mu] = -2i(\eta^\mu{}_\alpha\gamma_\beta - \eta^\mu{}_\beta\gamma_\alpha).$$

d) As follows from the last paragraph, for rotation and boost transformations, the finite transformation can be recovered by infinitesimal ones.

A rotation of an angle ϑ around the z-axis can be written as

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta & 0 \\ 0 & \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\vartheta & 0 \\ 0 & \vartheta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \text{h.o.t.}$$

Since $\vartheta^{12} = -\vartheta^{21} = \vartheta$ are the only non-zero element of $\vartheta^{\mu\nu}$, we only need to calculate

$$\begin{aligned}
\sigma_{12} &= \frac{i}{2}\left[\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\right] \\
&= \frac{i}{2}\left[\begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} - \begin{pmatrix} -\sigma_2\sigma_1 & 0 \\ 0 & -\sigma_2\sigma_1 \end{pmatrix}\right] \\
&= -\frac{i}{2}\begin{pmatrix} [\sigma_1, \sigma_2] & 0 \\ 0 & [\sigma_1, \sigma_2] \end{pmatrix} \\
&= -\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = -\Sigma_3.
\end{aligned}$$

Then, thanks to the fact that $\Sigma_3^{2n} = \mathbb{1}$ and $\Sigma_3^{2n+1} = \Sigma_3$, we have

$$\begin{aligned}
S(\Lambda) &= e^{\frac{i}{2}\vartheta\Sigma_3} \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}\left(\frac{\vartheta}{2}\right)^n \Sigma_3^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\vartheta}{2}\right)^{2n} \mathbb{1} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\vartheta}{2}\right)^{2n+1} \Sigma_3 \\
&= \cos\frac{\vartheta}{2}\,\mathbb{1} + i\sin\frac{\vartheta}{2}\,\Sigma_3.
\end{aligned}$$

In general, for a rotation of an angle ϑ around the direction $\hat{\boldsymbol{n}}$, we have $\Sigma_3 \to \hat{\boldsymbol{n}} \cdot \boldsymbol{\Sigma}$, where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

that is

$$S(\Lambda) = \cos\frac{\vartheta}{2}\,\mathbb{1} + i\sin\frac{\vartheta}{2}\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma}.$$

A boost of rapidity ω along the x-axis can be written as

$$\Lambda = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} + \begin{pmatrix} 0 & -\omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \text{h.o.t.}$$

Since $\omega^{01} = -\omega^{10} = \omega$ are the only non-zero element of $\omega^{\mu\nu}$, we only need to calculate

$$\begin{aligned}
\sigma_{01} &= \frac{i}{2}[\gamma_0,\gamma_1] = \frac{i}{2}\eta_{0\alpha}\eta_{1\beta}[\gamma^\alpha,\gamma^\beta] = -\frac{i}{2}[\gamma^0,\gamma^1] \\
&= -\frac{i}{2}\left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\right] \\
&= -\frac{i}{2}\left[\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\right] \\
&= -i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = -i\alpha_1.
\end{aligned}$$

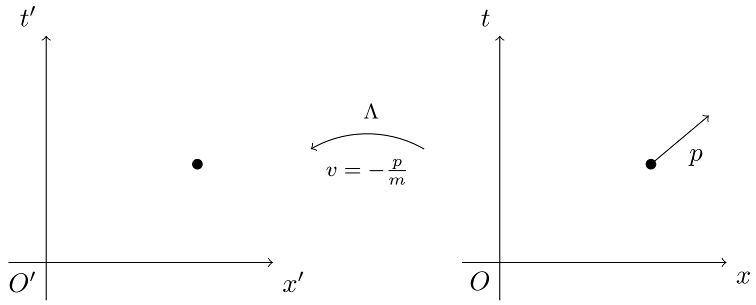
Then, thanks to the fact that $\alpha_1^{2n} = \mathbb{1}$ and $\alpha_1^{2n+1} = \alpha_1$, we have

$$\begin{aligned}
S(\Lambda) &= e^{-\frac{1}{2}\omega\alpha_1} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\omega}{2}\right)^n\alpha_1^n \\
&= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(-\frac{\omega}{2}\right)^{2n}\mathbb{1} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\frac{\omega}{2}\right)^{2n+1}\alpha_1 \\
&= \cosh\frac{\omega}{2}\,\mathbb{1} - \sinh\frac{\omega}{2}\,\alpha_1.
\end{aligned}$$

In general, for a boost of rapidity ω along the direction $\hat{\boldsymbol{n}}$, we have $\alpha_1 \to \hat{\boldsymbol{n}}\cdot\boldsymbol{\alpha}$, that is

$$S(\Lambda) = \cosh\frac{\omega}{2}\,\mathbb{1} - \sinh\frac{\omega}{2}\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\alpha}.$$

e)



Let Λ be the boost which transforms the 4-momentum $(mc, \mathbf{0})$ to the 4-momentum $(\frac{E_p}{c}, \boldsymbol{p})$. By using the relations $E_p^2 = p^2c^2 + m^2c^4$ and $cosh^2\ \omega$ - $sinh^2\ \omega = 1$ we can define the rapidity ω such that

$$\cosh\omega = \gamma = \frac{E_p}{mc^2} \qquad \sinh\omega = \beta\gamma = -\frac{p}{mc},$$

since the reference frame moves with velocity opposite to the particle. We will also need the following relations

$$\cosh\frac{\omega}{2} = \sqrt{\frac{\cosh\omega+1}{2}} = \sqrt{\frac{\frac{E_p}{mc^2}+1}{2}} = \sqrt{\frac{E_p+mc^2}{2mc^2}},$$
$$\tanh\frac{\omega}{2} = \frac{\sinh\omega}{\cosh\omega+1} = \frac{-\frac{p}{mc}}{\frac{E_p}{mc^2}+1} = -\frac{pc}{E_p+mc^2}.$$

By setting $p_\pm = p_1 \pm ip_2$,

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\alpha} = \frac{1}{p}\begin{pmatrix} 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} = \frac{1}{p}\begin{pmatrix} 0 & 0 & p_3 & p_- \\ 0 & 0 & p_+ & -p_3 \\ p_3 & p_- & 0 & 0 \\ p_+ & -p_3 & 0 & 0 \end{pmatrix}.$$

Hence,

$$S(\Lambda) = \cosh\frac{\omega}{2}\begin{pmatrix} 1 & 0 & -\frac{p_3}{p}\tanh\frac{\omega}{2} & -\frac{p_-}{p}\tanh\frac{\omega}{2} \\ 0 & 1 & -\frac{p_+}{p}\tanh\frac{\omega}{2} & \frac{p_3}{p}\tanh\frac{\omega}{2} \\ -\frac{p_3}{p}\tanh\frac{\omega}{2} & -\frac{p_-}{p}\tanh\frac{\omega}{2} & 1 & 0 \\ -\frac{p_+}{p}\tanh\frac{\omega}{2} & \frac{p_3}{p}\tanh\frac{\omega}{2} & 0 & 1 \end{pmatrix}$$

$$= \sqrt{\frac{E_p+mc^2}{2mc^2}}\begin{pmatrix} 1 & 0 & \frac{cp_3}{E_p+mc^2} & \frac{cp_-}{E_p+mc^2} \\ 0 & 1 & \frac{cp_+}{E_p+mc^2} & \frac{-cp_3}{E_p+mc^2} \\ \frac{cp_3}{E_p+mc^2} & \frac{cp_-}{E_p+mc^2} & 1 & 0 \\ \frac{cp_+}{E_p+mc^2} & \frac{-cp_3}{E_p+mc^2} & 0 & 1 \end{pmatrix}.$$

The transformed wave function will be

$$\psi_{\boldsymbol{p}}^{(r)}(x) = S(\Lambda)\psi_{\boldsymbol{0}}^{(r)}(\Lambda^{-1}x).$$

Since $\omega_r(\boldsymbol{0})$ are the standard basis, we have that $\omega_r(\boldsymbol{p})$ is the r-th column of $S(\Lambda)$:

$$\omega_1(\boldsymbol{p}) = \sqrt{\frac{E_p+mc^2}{2mc^2}}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \end{pmatrix}, \quad \omega_2(\boldsymbol{p}) = \sqrt{\frac{E_p+mc^2}{2mc^2}}\begin{pmatrix} 0 \\ 1 \\ \frac{cp_-}{E_p+mc^2} \\ \frac{-cp_3}{E_p+mc^2} \end{pmatrix},$$
$$\omega_3(\boldsymbol{p}) = \sqrt{\frac{E_p+mc^2}{2mc^2}}\begin{pmatrix} \frac{cp_3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad \omega_4(\boldsymbol{p}) = \sqrt{\frac{E_p+mc^2}{2mc^2}}\begin{pmatrix} \frac{cp_-}{E_p+mc^2} \\ \frac{-cp_3}{E_p+mc^2} \\ 0 \\ 1 \end{pmatrix}.$$

On the other hand,

$$e^{-\frac{i}{\hbar}\epsilon_r mc^2 t'} = e^{-\frac{i}{\hbar}\epsilon_r p'^\mu x'_\mu} = e^{-\frac{i}{\hbar}\epsilon_r p^\mu x_\mu},$$

so that the final expression for the wave function is

$$\psi_{\boldsymbol{p}}^{(r)}(x) = N_p\,\omega_r(\boldsymbol{p})e^{-\frac{i}{\hbar}\epsilon_r p^\mu x_\mu}.$$

Classical Field Theory and Second Quantization

4 Classical Field Theory and Noether's theorem

a) Present a concise review of classical field theory: Lagrangian formalism, Euler-Lagrange equations, Hamiltonian formalism, Hamilton equations, Poisson brackets. In particular, discuss the Poisson brackets between the fields and their conjugate momenta.

b) In the previous review, you have used functional derivatives. Explain what they are (in the simplified formulation, which is sufficient for this course), and describe their main properties.

c) Consider the classical Klein-Gordon (KG) equation as an example of how classical field theory works (repeat the main steps of (a)). Write down the prescription to quantize the KG field, according to the canonical quantization scheme.

d) State and prove Noether's theorem.

e) Apply Noether's theorem to the case of an action invariant under translations and define the energy-momentum tensor. What are the conserved quantities in this case? Repeat the same analysis in the case of an action invariant under phase rotation (internal symmetry).

---

b) Let $F[\psi]$ be a functional of the variable $\psi = \psi(x)\colon \mathbb{R}^4 \to \mathbb{R}$. Two remarkable examples are

$$F[\psi] = \psi(x)$$
$$F[\psi] = \int dx\, g(\psi(x)).$$

In the second case $F[\psi]$ can be interpreted as the action, g as the Lagrangian density and ψ as the field.

We define $\frac{\delta F[\psi]}{\delta\psi(x)}$ as the functional derivative of $F[\psi]$ with respect to $\psi(x)$. We want the functional derivative to satisfy the following properties.

- The functional derivative is a derivation, *i.e.* it is linear and satisfies the Leibniz rule.

$$\frac{\delta(\alpha F[\psi] + \beta G[\psi])}{\delta\psi(x)} = \alpha\frac{\delta F[\psi]}{\delta\psi(x)} + \beta\frac{\delta G[\psi]}{\delta\psi(x)}$$
$$\frac{\delta F[\psi]G[\psi]}{\delta\psi(x)} = \frac{\delta F[\psi]}{\delta\psi(x)}G[\psi] + F[\psi]\frac{\delta G[\psi]}{\delta\psi(x)}$$

- Normalization: for the functional $\psi \mapsto \psi(y)$,

$$\frac{\delta\psi(y)}{\delta\psi(x)} = \delta^{(4)}(x-y).$$

- Compatibility: for a functional of the form "composition" $\psi \mapsto g(\psi(y))$, the functional derivative is essentially the usual partial derivative

$$\frac{\delta g(\psi(y))}{\delta\psi(x)} = \frac{\partial g(\psi(y))}{\partial\psi}\delta^{(4)}(x-y).$$

We define the variation of $F[\psi]$ to be

$$\delta F[\psi] = F[\psi + \delta\psi] - F[\psi] = \int dx \frac{\delta F[\psi]}{\delta\psi(x)}\delta\psi(x).$$

We can now calculate, for example, the variation of $F[\psi] = \int dx\, g(\psi(x))$

$$\delta F[\psi] = \int dx\, \Big(g(\psi(x) + \delta\psi(x)) - g(\psi(x))\Big) = \int dx\, \frac{\partial g(y)}{\partial y}\Big|_{y=\psi(x)}\delta\psi(x),$$

so that

$$\frac{\delta F[\psi]}{\delta\psi(x)} = \frac{\partial g(y)}{\partial y}\Big|_{y=\psi(x)}.$$

The definitions can be generalised to the multiple variables case when $F[\psi_m]$ is a functional of the variables $\psi_m = \psi_m(x)\colon \mathbb{R}^4 \to \mathbb{R}$, $m = 1, \dots, M$. For the functional

$$F[\psi_1, \dots, \psi_M] = \int dx\, g(\psi_1(x), \dots, \psi_M(x)),$$

we find

$$\delta F[\psi_1, \dots, \psi_M] = \int dx \sum_m \frac{\partial g(y_1, \dots, y_M)}{\partial y_m}\Big|_{y=\psi(x)} \delta\psi_m(x),$$

so that

$$\frac{\delta F[\psi_1, \dots, \psi_M]}{\delta\psi_m(x)} = \frac{\partial g(y_1, \dots, y_M)}{\partial y_m}\Big|_{y=\psi(x)}.$$

a) Let us consider a Lagrangian

$$L(t) = L[\phi_n, \dot{\phi}_n],$$

where $\phi_n = \phi_n(\boldsymbol{x}, t)\colon \mathbb{R}^4 \to \mathbb{R}$ are fields ($n = 1, \dots, N$). So L is a functional on the function variables $\phi_n, \dot{\phi}_n$. The action is:

$$S[\phi_n, \dot{\phi}_n] = \int_{t_1}^{t_2} dt\, L(t).$$

The variation of the action with respect to $\phi_n, \dot{\phi}_n$ as functions of the space coordinates, is:

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \int d^3x \sum_n \left(\frac{\delta L}{\delta\phi_n}\delta\phi_n + \frac{\delta L}{\delta\dot{\phi}_n}\delta\dot{\phi}_n \right) \\ &= \int_{t_1}^{t_2} dt \int d^3x \sum_n \left(\frac{\delta L}{\delta\phi_n} - \frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\phi}_n} \right) \delta\phi_n; \end{aligned}$$

in the last line, we have integrated by parts with respect to time, using the fact that $\delta\dot{\phi}_n = \frac{\partial}{\partial t}\delta\phi_n$ and the boundary conditions $\delta\phi_n(\boldsymbol{x}, t_1) = \delta\phi_n(\boldsymbol{x}, t_2) = 0$. For the last action principle, the variation of the action along classical trajectories is zero. Since the variation of the fields is arbitrary, we have obtained the Euler-Lagrange equations

$$\frac{\delta L}{\delta\phi_n} - \frac{\partial}{\partial t}\frac{\delta L}{\delta\dot{\phi}_n} = 0.$$

Let us suppose now that the Lagrangian can be written as the space integral of a *Lagrangian density* of the form

$$L(t) = \int dx^3 \mathcal{L}\big(\phi_n(\boldsymbol{x}, t), \boldsymbol{\nabla}\phi_n(\boldsymbol{x}, t), \dot{\phi}_n(\boldsymbol{x}, t)\big).$$

Thus, for the Lagrangian

$$\begin{aligned} \delta L &= \int d^3x\, \delta\mathcal{L} \\ &= \int d^3x \sum_n \left(\frac{\partial\mathcal{L}}{\partial\phi_n}\delta\phi_n + \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\phi_n}\cdot\delta\boldsymbol{\nabla}\phi_n + \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n}\delta\dot{\phi}_n \right) \\ &= \int d^3x \sum_n \left[\left(\frac{\partial\mathcal{L}}{\partial\phi_n} - \boldsymbol{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\phi_n} \right)\delta\phi_n + \frac{\partial\mathcal{L}}{\partial\dot{\phi}_n}\delta\dot{\phi}_n \right], \end{aligned}$$

where we have integrated by parts with respect to the space coordinates, using the fact that $\delta\boldsymbol{\nabla}\phi_n = \boldsymbol{\nabla}\delta\phi_n$ and the boundary conditions $\delta\phi_n(\boldsymbol{x}, t) = 0$ for $\boldsymbol{x}$ on the boundary of the integration volume. On the other hand, by definition

$$\delta L = \int d^3x \sum_n \left(\frac{\delta L}{\delta\phi_n}\delta\phi_n + \frac{\delta L}{\delta\dot{\phi}_n}\delta\dot{\phi}_n \right).$$

Hence, we have obtained the functional derivatives for the Lagrangian in terms of the Lagrangian density as

$$\frac{\delta L}{\delta \phi_n} = \frac{\partial \mathcal{L}}{\partial \phi_n} - \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla} \phi_n}, \qquad \frac{\delta L}{\delta \dot{\phi}_n} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n}.$$

The Euler-Lagrange equations become

$$\frac{\partial \mathcal{L}}{\partial \phi_n} - \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla} \phi_n} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} = 0,$$

which can be written in a covariant form as

$$\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n} = 0.$$

We can see that the assumption on the form of the Lagrangian as an integral over space of a Lagrangian density (which is a function of ϕ_n and $\partial_\mu \phi_n$) is crucial for obtaining the covariant Euler-Lagrange equations.

For a general Lagrangian, let us introduce the conjugate momenta

$$\pi_n(\boldsymbol{x}, t) = \frac{\delta L}{\delta \dot{\phi}_n}$$

and the Hamiltonian

$$H(t) = \int d^3x \sum_n \pi_n(\boldsymbol{x}, t) \dot{\phi}_n(\boldsymbol{x}, t) - L(t).$$

Thus, we have

$$\dot{\pi}_n = \frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{\phi}_n} = \frac{\delta L}{\delta \phi_n},$$

because of the Euler-Lagrange equations. If L can be expressed in terms of a Lagrangian density, we have

$$H(t) = \int d^3x \, \mathcal{H}\big(\phi_n(\boldsymbol{x}, t), \boldsymbol{\nabla} \phi_n(\boldsymbol{x}, t), \pi_n(\boldsymbol{x}, t), \boldsymbol{\nabla} \pi_n(\boldsymbol{x}, t)\big),$$

where $\mathcal{H} = \sum_n \pi_n \dot{\phi}_n - \mathcal{L}$ is called the Hamiltonian density. The variation of the Hamiltonian is

$$\begin{aligned} \delta H &= \int d^3x \sum_n (\pi_n \delta \dot{\phi}_n + \dot{\phi}_n \delta \pi_n) - \delta L \\ &= \int d^3x \sum_n \left(\pi_n \delta \dot{\phi}_n + \dot{\phi}_n \delta \pi_n - \frac{\delta L}{\delta \phi_n} \delta \phi_n - \frac{\delta L}{\delta \dot{\phi}_n} \delta \dot{\phi}_n \right) \\ &= \int d^3x \sum_n (\dot{\phi}_n \delta \pi_n - \dot{\pi}_n \delta \phi_n). \end{aligned}$$

On the other hand, by definition,

$$\delta H = \int d^3x \sum_n \left(\frac{\delta H}{\delta \phi_n} \delta \phi_n + \frac{\delta H}{\delta \pi_n} \delta \pi_n \right),$$

which leads to the Hamiltonian system

$$\begin{cases} \dot{\phi}_n = \frac{\delta H}{\delta \pi_n} \\ \dot{\pi}_n = -\frac{\delta H}{\delta \phi_n}. \end{cases}$$

With the same procedure of above, we conclude that in the case of Hamiltonian density

$$\begin{cases} \dot{\phi}_n = \frac{\partial \mathcal{H}}{\partial \pi_n} - \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{\nabla} \pi_n} \\ \dot{\pi}_n = -\frac{\partial \mathcal{H}}{\partial \phi_n} + \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{\nabla} \phi_n}. \end{cases}$$

Given two functionals F, G of the functions ϕ_n, π_n, we can define the Poisson brackets to be

$$\{F,G\}_{\text{PB}} = \int d^3x \sum_n \left(\frac{\delta F}{\delta \phi_n(\boldsymbol{x})} \frac{\delta G}{\delta \pi_n(\boldsymbol{x})} - \frac{\delta F}{\delta \pi_n(\boldsymbol{x})} \frac{\delta G}{\delta \phi_n(\boldsymbol{x})} \right).$$

Then, if F does not depend explicitly on time, we have

$$\dot{F}(t) = \{F, H\}_{\text{PB}}.$$

In fact,

$$\begin{aligned} \dot{F}(t) &= \int d^3x \sum_n \left(\frac{\delta F}{\delta \phi_n(\boldsymbol{x})} \dot{\phi}_n + \frac{\delta F}{\delta \pi_n(\boldsymbol{x})} \dot{\pi}_n \right) \\ &= \int d^3x \sum_n \left(\frac{\delta F}{\delta \phi_n(\boldsymbol{x})} \frac{\delta H}{\delta \pi_n(\boldsymbol{x})} - \frac{\delta F}{\delta \pi_n(\boldsymbol{x})} \frac{\delta H}{\delta \phi_n(\boldsymbol{x})} \right) \\ &= \{F, H\}_{\text{PB}}. \end{aligned}$$

We have also the fundamental Poisson brackets for the functionals $\phi_n \mapsto \phi_n(\boldsymbol{x}, t)$ and $\pi_n \mapsto \pi_n(\boldsymbol{x}', t)$:

$$\begin{aligned} \{\phi_n(\boldsymbol{x},t), \pi_m(\boldsymbol{x}',t)\}_{\text{PB}} &= \int d^3x'' \sum_k \left(\frac{\delta \phi_n(\boldsymbol{x},t)}{\delta \phi_k(\boldsymbol{x}'')} \frac{\delta \pi_m(\boldsymbol{x}',t)}{\delta \pi_k(\boldsymbol{x}'')} - \frac{\delta \phi_n(\boldsymbol{x},t)}{\delta \pi_k(\boldsymbol{x}'')} \frac{\delta \pi_m(\boldsymbol{x}',t)}{\delta \phi_k(\boldsymbol{x}'')} \right) \\ &= \int d^3x'' \sum_k \delta_{nk} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'')\, \delta_{mk} \delta^{(3)}(\boldsymbol{x}' - \boldsymbol{x}'') \\ &= \delta_{nm}\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \end{aligned}$$

$$\{\phi_n(\boldsymbol{x},t), \phi_m(\boldsymbol{x}',t)\}_{\text{PB}} = \int d^3x'' \sum_k \left(\frac{\delta \phi_n(\boldsymbol{x},t)}{\delta \phi_k(\boldsymbol{x}'')} \frac{\delta \phi_m(\boldsymbol{x}',t)}{\delta \pi_k(\boldsymbol{x}'')} - \frac{\delta \phi_n(\boldsymbol{x},t)}{\delta \pi_k(\boldsymbol{x}'')} \frac{\delta \phi_m(\boldsymbol{x}',t)}{\delta \phi_k(\boldsymbol{x}'')} \right) = 0,$$

and similarly $\{\pi_n(\boldsymbol{x},t), \pi_m(\boldsymbol{x}',t)\}_{\text{PB}} = 0$. Furthermore, we can write the canonical system as

$$\begin{cases} \dot{\phi}_n = \{\phi_n, H\}_{\text{PB}} \\ \dot{\pi}_n = \{\pi_n, H\}_{\text{PB}}. \end{cases}$$

c) Let us consider the KGE (in natural units $\hbar = c = 1$)

$$\left(\Box + m^2\right) \phi(x) = 0.$$

It can be derived by the least action principle, from the Lagrangian density

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \left(\partial^\mu \phi\, \partial_\mu \phi - m^2 \phi^2 \right).$$

The Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$ immediately leads to the KGE, since

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \qquad \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \partial^\mu \partial_\mu \phi = \Box \phi.$$

Thus, we can define the conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi},$$

the Hamiltonian density

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \frac{1}{2} \left[\pi^2 + (\boldsymbol{\nabla} \phi)^2 + m^2 \phi^2 \right]$$

and the Hamiltonian

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \frac{1}{2} \left[\pi^2 + (\boldsymbol{\nabla} \phi)^2 + m^2 \phi^2 \right].$$

The canonical system can be written as

$$\begin{cases} \dot{\phi} = \frac{\partial \mathcal{H}}{\partial \pi} - \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{\nabla} \pi} = \pi \\ \dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \phi} + \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{H}}{\partial \boldsymbol{\nabla} \phi} = \nabla^2 \phi - m^2 \phi, \end{cases}$$

which is equivalent to the KGE. The fundamental Poisson brackets are

$$\{\phi(\boldsymbol{x}, t), \pi(\boldsymbol{x}', t)\}_{\mathrm{PB}} = \int d^3x'' \left(\frac{\delta \phi(\boldsymbol{x}, t)}{\delta \phi(\boldsymbol{x}'')} \frac{\delta \pi(\boldsymbol{x}', t)}{\delta \pi(\boldsymbol{x}'')} - \frac{\delta \phi(\boldsymbol{x}, t)}{\delta \pi(\boldsymbol{x}'')} \frac{\delta \pi(\boldsymbol{x}', t)}{\delta \phi(\boldsymbol{x}'')} \right) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')$$

and similarly

$$\{\phi(\boldsymbol{x}, t), \phi(\boldsymbol{x}', t)\}_{\mathrm{PB}} = \{\pi(\boldsymbol{x}, t), \pi(\boldsymbol{x}', t)\}_{\mathrm{PB}} = 0.$$

We quantize the field, following the prescription of canonical quantization

$$\begin{gathered} \phi, \pi \longrightarrow \hat{\phi}, \hat{\pi} \\ \{\,\cdot\,,\,\cdot\,\}_{\mathrm{PB}} \longrightarrow -i\,[\,\cdot\,,\,\cdot\,], \end{gathered}$$

Thus, we obtain the equal-time commutation relations

$$\begin{aligned} &[\hat{\phi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{x}', t)] = i\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \\ &[\hat{\phi}(\boldsymbol{x}, t), \hat{\phi}(\boldsymbol{x}', t)] = [\hat{\pi}(\boldsymbol{x}, t), \hat{\pi}(\boldsymbol{x}', t)] = 0. \end{aligned}$$

The Hamiltonian will be

$$\hat{H} = \int d^3x \, \frac{1}{2} \left[\hat{\pi}^2 + (\boldsymbol{\nabla} \hat{\phi})^2 + m^2 \hat{\phi}^2 \right]$$

and the canonical system

$$\begin{cases} \dot{\hat{\phi}} = -i\,[\hat{\phi}, \hat{H}] \\ \dot{\hat{\pi}} = -i\,[\hat{\pi}, \hat{H}]. \end{cases}$$

Let us compute the commutators using the fundamental commutation relations (omitting the equal-time dependence).

$$[\hat{\phi}(\boldsymbol{x}), \hat{H}] = \int d^3x' \frac{1}{2} \Big(\big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}^2(\boldsymbol{x}')\big] + \big[\hat{\phi}(\boldsymbol{x}), \big(\boldsymbol{\nabla}' \hat{\phi}(\boldsymbol{x}')\big)^2\big] + m^2 \big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^2(\boldsymbol{x}')\big] \Big)$$

On the other hand,

$$\begin{aligned} &\big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}^2(\boldsymbol{x}')\big] = \hat{\pi}(\boldsymbol{x}')\big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{x}')\big] + \big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{x}')\big]\hat{\pi}(\boldsymbol{x}') = 2i\,\hat{\pi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \\ &\big[\hat{\phi}(\boldsymbol{x}), \big(\boldsymbol{\nabla}' \hat{\phi}(\boldsymbol{x}')\big)^2\big] = \boldsymbol{\nabla}' \hat{\phi}(\boldsymbol{x}')\big(\boldsymbol{\nabla}' \big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big]\big) + \big(\boldsymbol{\nabla}' \big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big]\big)\boldsymbol{\nabla}' \hat{\phi}(\boldsymbol{x}') = 0 \\ &\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^2(\boldsymbol{x}')\big] = \hat{\phi}(\boldsymbol{x}')\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big] + \big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big]\hat{\phi}(\boldsymbol{x}') = 0, \end{aligned}$$

that leads to

$$[\hat{\phi}(\boldsymbol{x}), \hat{H}] = i \int d^3x' \, \hat{\pi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') = i\,\hat{\pi}(\boldsymbol{x}).$$

In a similar way, from the commutators

$$\begin{aligned} &\big[\hat{\pi}(\boldsymbol{x}), \hat{\pi}^2(\boldsymbol{x}')\big] = 0 \\ &\big[\hat{\pi}(\boldsymbol{x}), \big(\boldsymbol{\nabla}' \hat{\phi}(\boldsymbol{x}')\big)^2\big] = -2i\,\boldsymbol{\nabla}' \hat{\phi}(\boldsymbol{x}') \cdot \boldsymbol{\nabla}' \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') = 2i\,\nabla'^2 \hat{\phi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \\ &\big[\hat{\pi}(\boldsymbol{x}), \hat{\phi}^2(\boldsymbol{x}')\big] = -2i\,\hat{\phi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \end{aligned}$$

we can see that

$$[\hat{\pi}(\boldsymbol{x}), \hat{H}] = i \int d^3x' \left(\nabla'^2 \hat{\phi}(\boldsymbol{x}') - m^2 \hat{\phi}(\boldsymbol{x}') \right) \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') = i \left(\nabla^2 \hat{\phi}(\boldsymbol{x}) - m^2 \hat{\phi}(\boldsymbol{x}) \right).$$

Thus, the canonical system is formally as the KG one:

$$\begin{cases} \dot{\hat{\phi}} = \hat{\pi}(\boldsymbol{x}) \\ \dot{\hat{\pi}} = \nabla^2 \hat{\phi}(\boldsymbol{x}) - m^2 \hat{\phi}(\boldsymbol{x}). \end{cases}$$

d) A symmetry of a system is defined as a transformation of the coordinates and/or the fields such that the action does not change:

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu \\ \phi_n(x) &\longrightarrow \phi'_n(x') \end{aligned}$$

implies that

$$\Delta S = \int_{\Sigma'} d^4x' \, \mathcal{L}'(x') - \int_\Sigma d^4x \, \mathcal{L}(x) = 0$$

for an arbitrary integration volume. Noether's theorem can be expressed as follows: for every symmetry of a system, there exists a conserved quantity. We prove the theorem for infinitesimal transformations:

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu + \delta x^\mu \\ \phi_n(x) &\longrightarrow \phi'_n(x') = \phi_n(x) + \delta\phi_n(x). \end{aligned}$$

It is important to note that δ and ∂_μ do not commute, since δ is a variation of both the coordinates and the fields. Thus, it will be useful to define the modified variation

$$\widetilde{\delta}\phi_n(x) = \phi'_n(x) - \phi_n(x),$$

which has the property to commute with the derivative: $\partial_\mu \widetilde{\delta}\phi_n(x) = \widetilde{\delta}\partial_\mu\phi_n(x)$. The two types of variation are related at first order through

$$\begin{aligned} \widetilde{\delta}\phi_n(x) &= \phi'_n(x) - \phi_n(x) \\ &= \phi'_n(x) - \phi'_n(x') + \phi'_n(x') - \phi_n(x) \\ &= \delta\phi_n(x) - \partial_\mu\phi'_n(x)\,\delta x^\mu + \text{h.o.t.} \\ &= \delta\phi_n(x) - \partial_\mu\phi_n(x)\,\delta x^\mu + \text{h.o.t.} \end{aligned}$$

(here h.o.t. means "higher order terms"), where in the last step we have used the fact that at first order $\partial_\mu\phi'_n(x)\,\delta x^\mu = \partial_\mu\phi_n(x)\,\delta x^\mu$. Let us work out the conserved quantity, starting from the variation of the action. Introducing the variation of the Lagrangian density and with the change of variable in the integral, we find

$$\begin{aligned} \delta S &= \int_{\Sigma'} d^4x' \big(\mathcal{L}(x) + \delta\mathcal{L}(x)\big) - \int_\Sigma d^4x \, \mathcal{L}(x) \\ &= \int_\Sigma d^4x \left|\det \frac{\partial x'}{\partial x}\right| \big(\mathcal{L}(x) + \delta\mathcal{L}(x)\big) - \int_\Sigma d^4x \, \mathcal{L}(x) \end{aligned}$$

On the other hand, the Jacobian at the first order will be

$$\det\frac{\partial x'}{\partial x} = \begin{vmatrix} 1+\partial_0\delta x^0 & \partial_1\delta x^0 & \partial_2\delta x^0 & \partial_3\delta x^0 \\ \partial_0\delta x^1 & 1+\partial_1\delta x^1 & \partial_2\delta x^1 & \partial_3\delta x^1 \\ \partial_0\delta x^2 & \partial_1\delta x^2 & 1+\partial_2\delta x^2 & \partial_3\delta x^2 \\ \partial_0\delta x^3 & \partial_1\delta x^3 & \partial_2\delta x^3 & 1+\partial_3\delta x^3 \end{vmatrix} = 1 + \partial_\mu\delta x^\mu + \text{h.o.t.},$$

which is positive. Hence,

$$\begin{aligned} \delta S &= \int_\Sigma d^4x \, (1+\partial_\mu\delta x^\mu)\big(\mathcal{L}(x) + \delta\mathcal{L}(x)\big) - \int_\Sigma d^4x \, \mathcal{L}(x) \\ &= \int_\Sigma d^4x \, \delta\mathcal{L}(x) + \int_\Sigma d^4x \, \mathcal{L}(x)\,\partial_\mu\delta x^\mu \\ &= \int_\Sigma d^4x \, \Big(\widetilde{\delta}\mathcal{L}(x) + \partial_\mu\mathcal{L}(x)\,\delta x^\mu\Big) + \int_\Sigma d^4x \, \mathcal{L}(x)\,\partial_\mu\delta x^\mu \\ &= \int_\Sigma d^4x \, \Big(\widetilde{\delta}\mathcal{L}(x) + \partial_\mu\left(\mathcal{L}(x)\delta x^\mu\right)\Big). \end{aligned}$$

Finally, we can express the modified variation of the Lagrangian density in terms of the variation of the coordinates and the fields.

$$\begin{aligned}\widetilde{\delta}\mathcal{L}(x) &= \frac{\partial\mathcal{L}(x)}{\partial\phi_n}\widetilde{\delta}\phi_n(x) + \frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\widetilde{\delta}\partial^\mu\phi_n(x)\\ &= \frac{\partial\mathcal{L}(x)}{\partial\phi_n}\widetilde{\delta}\phi_n(x) + \frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\partial^\mu\widetilde{\delta}\phi_n(x)\\ &= \left(\frac{\partial\mathcal{L}(x)}{\partial\phi_n} - \partial^\mu\frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\right)\widetilde{\delta}\phi_n(x) + \partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\widetilde{\delta}\phi_n(x)\right).\end{aligned}$$

Here we have used the summation convention also for the index n. Thus, if the fields satisfy the Euler-Lagrange equations, we obtain

$$\begin{aligned}\delta S &= \int_\Sigma d^4x\,\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\widetilde{\delta}\phi_n(x) + \mathcal{L}(x)\delta x_\mu\right)\\ &= \int_\Sigma d^4x\,\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\left(\delta\phi_n(x) - \partial_\nu\phi_n(x)\delta x^\nu\right) + \mathcal{L}(x)\delta x_\mu\right)\\ &= \int_\Sigma d^4x\,\partial^\mu\left(\frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\delta\phi_n(x) - \left(\frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\partial_\nu\phi_n(x) - \eta_{\mu\nu}\mathcal{L}(x)\right)\delta x^\nu\right).\end{aligned}$$

Now, the variation of the action must be zero, since we are dealing with a symmetry. Thus the arbitrariness of the volume Σ implies that the integrand must be zero:

$$\partial^\mu j_\mu = 0, \qquad j_\mu = \frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\delta\phi_n(x) - \left(\frac{\partial\mathcal{L}(x)}{\partial\partial^\mu\phi_n}\partial_\nu\phi_n(x) - \eta_{\mu\nu}\mathcal{L}(x)\right)\delta x^\nu.$$

This is a continuity equation for the Noether charge j_μ, which expresses a conservation law as we can see by integrating over space and by using Gauss' theorem:

$$0 = \int_V d^3x\left(\frac{\partial j_0}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}\right) = \frac{\partial}{\partial t}\int_V d^3x\, j_0 + \int_{\partial V} d\boldsymbol{a}\cdot\boldsymbol{j} = \frac{\partial}{\partial t}\int_V d^3x\, j_0.$$

Since the fields are assumed to fall off fast at infinity, we obtain the conservation of the Noether charge

$$Q = \int d^3x\, j_0(x).$$

e) Let us consider a system invariant under (infinitesimal) translations

$$x^\mu \longrightarrow x^\mu + \epsilon^\mu.$$

Here the values ϵ^μ are independent from each other. Thus, the variation of fields is zero and the Noether current, having taken away the constant factors ϵ^ν, becomes

$$\Theta_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi_n}\partial_\nu\phi_n(x) - \eta_{\mu\nu}\mathcal{L},$$

which is called the canonical energy-momentum tensor.

The four conserved quantities are expressed by the continuity equations $\partial^\mu\Theta_{\mu\nu} = 0$. The four conserved quantities will be the energy and the momentum vectors

$$P_\nu = \int d^3x\,\Theta_{0\nu} = \int d^3x\left(\sum_n \pi_n\,\partial_\nu\phi_n - \eta_{0\nu}\mathcal{L}\right).$$

Note that

$$P_0 = \int d^3x\left(\sum_n \pi_n\,\dot{\phi}_n - \mathcal{L}\right) = H$$

is the Hamiltonian, while

$$\boldsymbol{P} = -\int d^3x \sum_n \pi_n \, \boldsymbol{\nabla}\phi_n,$$

Let us consider now a system invariant under (infinitesimal) phase rotation

$$\phi_n(x) \longrightarrow \phi'_n(x) = \phi_n(x) + i\epsilon \sum_m \lambda_{nm}\phi_m(x).$$

Here the values λ_{nm} are independent from each other. Thus, the variation of the coordinates is zero and the Noether current, having split off the constant factor ϵ, becomes

$$j_\mu = i\frac{\partial \mathcal{L}}{\partial \partial^\mu \phi_n}\sum_m \lambda_{nm}\phi_m = i\sum_{m,n}\frac{\partial \mathcal{L}}{\partial \partial^\mu \phi_n}\lambda_{nm}\phi_m,$$

where in the last step we have made the sum over n explicit. Hence, the conserved Noether charge will be

$$Q = i\int d^3x \sum_{m,n} \pi_n \, \lambda_{nm} \, \phi_m.$$

5 Second quantization of the Schrödinger equation

a) Write the Lagrangian for the Schrödinger field. Using Euler-Lagrange equations, show that it gives the Schrödinger equation. Determine the conjugate momentum and write the Hamiltonian. Write down the Poisson brackets.

b) Following the prescription of canonical quantization, quantize the Schrödinger field. Write the Heisenberg equations for the field and its conjugate momentum, and show that they give again the Schrödinger equation. What is the difference between the Schrödinger equation now derived, and that derived in (a)?

c) Consider the expansion of the Schrödinger field

$$\hat{\psi}(\boldsymbol{x}, t) = \sum_n \hat{a}_n(t)u_n(\boldsymbol{x}),$$

where $u_n(\boldsymbol{x})$ form a complete orthonormal set of the Hilbert space, while $\hat{a}_n(t)$ are suitable time-dependent operators. Starting from the equal-time commutation relations among the field and its conjugate momentum, derive the commutation relation between $\hat{a}_n(t)$ and its adjoint.

d) In the particular case where $u_n(\boldsymbol{x})$ are the eigenstates of the stationary Schrödinger equation (with a time-independent potential), show how the operators $\hat{a}_n(t)$ and their adjoints evolve in time. In this case, how do the commutation relations between $\hat{a}_n$ and their adjoints simplify?

e) Define the vacuum state and, starting from it, construct the Fock space. Explain its structure, and the role of the operators $\hat{a}_n$ and their adjoints. Why is QFT is a many-body theory?

---

a) The Lagrangian density of the Schrödinger field is

$$\mathcal{L}(\psi, \psi^*, \partial^\mu\psi, \partial^\mu\psi^*) = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - V(x)\psi^*\psi.$$

We have that

$$\frac{\partial \mathcal{L}}{\partial \psi} = -V(x)\psi^* \qquad \partial_t\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\hbar\dot{\psi}^* \qquad \boldsymbol{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\nabla^2\psi^*$$

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\hbar\dot{\psi} - V(x)\psi \qquad \partial_t\frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = 0 \qquad \boldsymbol{\nabla}\cdot\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\nabla^2\psi.$$

So the Euler-Lagrange equations

$$
\begin{aligned}
&-V(x)\psi^* - i\hbar\dot{\psi}^* + \frac{\hbar^2}{2m}\nabla^2\psi^* = 0 \\
&-V(x)\psi + i\hbar\dot{\psi} + \frac{\hbar^2}{2m}\nabla^2\psi = 0
\end{aligned}
$$

are the Schrödinger equation and its conjugate.
The canonical conjugate momenta are

$$
\begin{aligned}
\pi_\psi = \pi &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^* \\
\pi_{\psi^*} &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = 0.
\end{aligned}
$$

Since $\mathcal{L}$ has two independent dynamical variables, it could be expected that one of the conjugate momenta is zero. Setting $\pi_\psi = \pi$, the Hamiltonian density will be

$$
\begin{aligned}
\mathcal{H} = \pi\dot{\psi} - \mathcal{L} &= i\hbar\psi^*\dot{\psi} - i\hbar\psi^*\dot{\psi} + \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi + V(x)\psi^*\psi \\
&= \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi + V(x)\psi^*\psi.
\end{aligned}
$$

Then the Hamiltonian will be

$$
\begin{aligned}
H = \int d^3x\,\mathcal{H}(x) &= \int d^3x\left(\frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi + V(x)\psi^*\psi\right) \\
&= \int d^3x\left(\frac{\hbar^2}{2m}\boldsymbol{\nabla}\cdot(\psi^*\boldsymbol{\nabla}\psi) - \frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V(x)\psi^*\psi\right) \\
&= \int d^3x\,\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)\psi,
\end{aligned}
$$

since the fields are assumed to vanish at the boundary.
The Poisson brackets are defined as:

$$
\begin{aligned}
&\{\psi(\boldsymbol{x},t),\pi(\boldsymbol{x}',t)\}_{\text{PB}} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \\
&\{\psi(\boldsymbol{x},t),\psi(\boldsymbol{x}',t)\}_{\text{PB}} = \{\pi(\boldsymbol{x},t),\pi(\boldsymbol{x}',t)\}_{\text{PB}} = 0.
\end{aligned}
$$

b) We quantize the Schrödinger field, following the rules for bosons:

$$
\begin{aligned}
\psi,\pi &\longrightarrow \hat{\psi},\hat{\pi} \\
\{\,\cdot\,,\,\cdot\,\}_{\text{PB}} &\longrightarrow \frac{1}{i\hbar}[\,\cdot\,,\,\cdot\,].
\end{aligned}
$$

Thus, we obtain the equal-time commutation relations

$$
\begin{aligned}
&[\hat{\psi}(\boldsymbol{x},t),\hat{\pi}(\boldsymbol{x}',t)] = i\hbar\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \\
&[\hat{\psi}(\boldsymbol{x},t),\hat{\psi}(\boldsymbol{x}',t)] = [\hat{\pi}(\boldsymbol{x},t),\hat{\pi}(\boldsymbol{x}',t)] = 0.
\end{aligned}
$$

The Heisenberg equations of motion are:

$$
\begin{cases}
\dot{\hat{\psi}} = \frac{1}{i\hbar}[\hat{\psi},\hat{H}] \\
\dot{\hat{\pi}} = \frac{1}{i\hbar}[\hat{\pi},\hat{H}],
\end{cases}
$$

where the second one is the hermitian conjugate of the first one (up to a factor $i\hbar$). Thus, we need only to compute $[\hat{\psi},\hat{H}]$, where

$$
\hat{H}(t) = \int d^3x\,\hat{\psi}^\dagger(\boldsymbol{x},t)\underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)}_{=\mathcal{D}_{\boldsymbol{x}}}\hat{\psi}(\boldsymbol{x},t)
$$

and $\hat{\psi}^\dagger = \frac{1}{i\hbar}\hat{\pi}$.

$$\begin{aligned}[\hat{\psi}(\boldsymbol{x},t),\hat{H}(t)] &= \int d^3x' \left(\hat{\psi}^\dagger(\boldsymbol{x}',t)\mathcal{D}_{\boldsymbol{x}'}[\hat{\psi}(\boldsymbol{x},t),\hat{\psi}(\boldsymbol{x}',t)] + [\hat{\psi}(\boldsymbol{x},t),\hat{\psi}^\dagger(\boldsymbol{x}',t)]\mathcal{D}_{\boldsymbol{x}'}\hat{\psi}(\boldsymbol{x}',t)\right)\\ &= \int d^3x' \frac{1}{i\hbar}[\hat{\psi}(\boldsymbol{x},t),\hat{\pi}(\boldsymbol{x}',t)]\mathcal{D}_{\boldsymbol{x}'}\hat{\psi}(\boldsymbol{x}',t)\\ &= \int d^3x' \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\mathcal{D}_{\boldsymbol{x}'}\hat{\psi}(\boldsymbol{x}',t)\\ &= \mathcal{D}_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x},t),\end{aligned}$$

where we have used the fact that $\mathcal{D}_{\boldsymbol{x}'}$ commutes with $\hat{\psi}(\boldsymbol{x},t)$. Thus, the Heisenberg equation becomes

$$i\hbar\dot{\hat{\psi}} = \mathcal{D}_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x},t),$$

which is formally the Schrödinger equation. The difference between the the one derived in (a) is that this one involves operators, and therefore it is the Heisenberg equation.

c) Let us plug the expression for the Fourier coefficients of the Schrödinger field

$$\hat{a}_n(t) = \int d^3x\, \hat{\psi}(\boldsymbol{x},t)u_n^*(\boldsymbol{x})$$

into the commutation relations.

$$\begin{aligned}[\hat{a}_n(t),\hat{a}_m(t)] &= \int d^3x\, d^3x'[\hat{\psi}(\boldsymbol{x},t)u_n^*(t),\hat{\psi}(\boldsymbol{x}',t)u_m^*(t)]\\ &= \int d^3x\, d^3x' \underbrace{[\hat{\psi}(\boldsymbol{x},t),\hat{\psi}(\boldsymbol{x}',t)]}_{=0} u_n^*(t)u_m^*(t) = 0.\end{aligned}$$

Similarly, $[\hat{a}_n^\dagger(t),\hat{a}_m^\dagger(t)] = 0$. Further,

$$\begin{aligned}[\hat{a}_n(t),\hat{a}_m^\dagger(t)] &= \int d^3x\, d^3x'[\hat{\psi}(\boldsymbol{x},t)u_n^*(\boldsymbol{x}),\hat{\psi}^\dagger(\boldsymbol{x}',t)u_m(\boldsymbol{x}')]\\ &= \int d^3x\, d^3x'[\hat{\psi}(\boldsymbol{x},t),\hat{\psi}^\dagger(\boldsymbol{x}',t)]u_n^*(\boldsymbol{x})u_m(\boldsymbol{x}')\\ &= \int d^3x\, d^3x'\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')u_n^*(\boldsymbol{x})u_m(\boldsymbol{x}')\\ &= \int d^3x\, u_n^*(\boldsymbol{x})u_m(\boldsymbol{x}) = \delta_{nm}.\end{aligned}$$

Here (and above) we have used the Heisenberg picture, where the operators evolve in time, while the states, like $u_n(\boldsymbol{x})$, do not depend on t.

d) Since the potential does not depend on time, we can speak of stationary eigenstates of the Schrödinger differential operator. We insert the expansion of the Schrödinger field into the Heisenberg equation. On the left-hand side, we find

$$i\hbar\dot{\hat{\psi}}(\boldsymbol{x},t) = \sum_n i\hbar\,\dot{\hat{a}}_n(t)u_n(\boldsymbol{x}),$$

while on the right-hand side we have

$$\mathcal{D}_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x},t) = \sum_n \hat{a}_n(t)\mathcal{D}_{\boldsymbol{x}}u_n(\boldsymbol{x}).$$

Using the fact that $\mathcal{D}_{\boldsymbol{x}}u_n(\boldsymbol{x}) = E_n u_n(\boldsymbol{x})$, we find

$$\mathcal{D}_{\boldsymbol{x}}\hat{\psi}(\boldsymbol{x},t) = \sum_n E_n\hat{a}_n(t)u_n(\boldsymbol{x}).$$

since the functions $u_n(\boldsymbol{x})$ form an orthonormal basis, the coefficients of the expansion must be the same:

$$i\hbar\,\dot{\hat{a}}_n(t) = E_n\hat{a}_n(t),$$

so we have

$$\hat{a}_n(t) = e^{-\frac{i}{\hbar}E_n t}\,\hat{a}_n(0)$$

and the adjoint equation

$$\hat{a}_n^\dagger(t) = e^{\frac{i}{\hbar}E_n t}\,\hat{a}_n^\dagger(0).$$

Let us set $\hat{a}_n(0) = \hat{a}_n$ and $\hat{a}_n^\dagger(0) = \hat{a}_n^\dagger$. Then the commutation relations for $\hat{a}_n$ and their adjoints become equivalent to those of $\hat{a}_n(t)$ and $\hat{a}_m^\dagger(t)$. One implication is trivial, since $\hat{a}_n = \hat{a}_n(0)$. For the second one, we find

$$\begin{aligned}[\hat{a}_n(t), \hat{a}_m(t)] &= [e^{-\frac{i}{\hbar}E_n t}\hat{a}_n, e^{-\frac{i}{\hbar}E_m t}\hat{a}_m] \\ &= e^{-\frac{i}{\hbar}(E_n+E_m)t}[\hat{a}_n, \hat{a}_m] = 0\end{aligned}$$

and similarly $[\hat{a}_n^\dagger(t), \hat{a}_m^\dagger(t)] = 0$, while

$$\begin{aligned}[\hat{a}_n(t), \hat{a}_m^\dagger(t)] &= [e^{-\frac{i}{\hbar}E_n t}\hat{a}_n, e^{\frac{i}{\hbar}E_m t}\hat{a}_m^\dagger] \\ &= e^{-\frac{i}{\hbar}(E_n-E_m)t}[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}.\end{aligned}$$

e) The operators $\hat{a}_n$ and their adjoint operators satisfy the harmonic oscillator algebra

$$\begin{aligned}&[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm} \\ &[\hat{a}_n, \hat{a}_m] = [\hat{a}_n^\dagger, \hat{a}_m^\dagger] = 0.\end{aligned}$$

The above algebra implies the existence of a vacuum state, defined by

$$\hat{a}_n\,|0\rangle = 0 \qquad \forall n.$$

This is a consequence of the fact that the ith number operators

$$\hat{n}_i = \hat{a}_i^\dagger\hat{a}_i$$

commute among themselves and that, in a Hilbert space, the norm is positive definite. The Hilbert space, called Fock space, is spanned by the basis of common eigenstates $|n_1\,n_2\ldots\rangle$. They are orthonormalized, in the sense that

$$\langle n_1'\,n_2'\ldots|n_1\,n_2\ldots\rangle = \delta_{n_1 n_1'}\delta_{n_2 n_2'}\cdots\,.$$

This ket represents a state with n_i particles of energy E_i.

The $\hat{a}_i^\dagger$ operators act on the vacuum as

$$C_{n_1 n_2\ldots}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\cdots|0\rangle = |n_1\,n_2\ldots\rangle\,,$$

where $C_{n_1 n_2\ldots}$ is the normalization factor, while the $\hat{a}_i$ operators act on $|n_1\,n_2\ldots\rangle$ as

$$\hat{a}_i\,|n_1\,n_2\ldots\rangle = \begin{cases} D_{n_i}\,|n_1\,n_2\ldots n_i-1\ldots\rangle & \text{if } n_i > 0 \\ 0 & \text{if } n_i = 0,\end{cases}$$

where D_{n_i} is a normalization factor. Furthermore, we can see that

$$\begin{aligned}\hat{H} &= \int d^3x\,\hat{\psi}^\dagger(x)\mathcal{D}_{\boldsymbol{x}}\hat{\psi}(x) = \sum_{i,j}\int d^3x\,\hat{a}_j^\dagger(t)u_j^*(\boldsymbol{x})\mathcal{D}_{\boldsymbol{x}}\hat{a}_i(t)u_i(\boldsymbol{x}) \\ &= \sum_{i,j}\int d^3x\,E_i\,\hat{a}_j^\dagger(t)\hat{a}_i(t)\,u_j^*(\boldsymbol{x})u_i(\boldsymbol{x}) = \sum_{i,j}E_i\,\hat{a}_j^\dagger\hat{a}_i\,\delta_{ij} \\ &= \sum_i E_i\,\hat{a}_i^\dagger\hat{a}_i.\end{aligned}$$

Hence,

$$\hat{H}\,|n_1\,n_2\ldots\rangle = \left(\sum_i E_i\,n_i\right)|n_1\,n_2\ldots\rangle\,.$$

Thus, the operator $\hat{a}_n^\dagger$ can be seen as a creation operator for particles of energy E_n, while its adjoint is an annihilation operator for the same energetic particles. In this sense, QFT can be seen as a many-body theory.

6 Second quantization of the Schrödinger equation: Bose and Fermi statistics

a) Consider the second-quantized Schrödinger field and its decomposition

$$\hat{\psi}(\boldsymbol{x},t) = \sum_n \hat{a}_n(t)u_n(\boldsymbol{x}),$$

where $u_n(\boldsymbol{x})$ are the eigenstates of the stationary Schrödinger equation. Write down the equal-time commutation relations between the Schrödinger field and its conjugate, and the corresponding commutation relations between the operators $\hat{a}_n$ and their conjugate (note the absence of time-dependence here on $\hat{a}_n$; explain why). Consider the normalized state of the Fock space

$$|n_1,n_2,\ldots\rangle = C_{n_1,n_2,\ldots}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\ldots|0\rangle.$$

Explain what it represents, and the physical role of the operators $\hat{a}_n$ and their adjoints. Determine the normalization constant $C_{n_1,n_2,\ldots}$.

b) Consider the state

$$|\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots\boldsymbol{x}_n\rangle = \frac{1}{\sqrt{n!}}\hat{\psi}^\dagger(\boldsymbol{x}_1)\hat{\psi}^\dagger(\boldsymbol{x}_2)\ldots\hat{\psi}^\dagger(\boldsymbol{x}_n)|0\rangle.$$

Explain what it represents, and the physical role of the operator $\hat{\psi}(\boldsymbol{x})$ and its adjoint. Compute the explicit expression of

$$\begin{aligned}
\phi_n^{(1)}(\boldsymbol{x}) &= \langle\boldsymbol{x}|1n\rangle,\\
\phi_{n,m}^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2) &= \langle\boldsymbol{x}_1,\boldsymbol{x}_2|1n,1m\rangle,\\
\phi_{n_1,n_2,\ldots n_k}^{(k)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots\boldsymbol{x}_k) &= \langle\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots\boldsymbol{x}_k|1n_1,1n_2,\ldots1n_k\rangle.
\end{aligned}$$

What is the significance of having chosen the commutation relations, to quantize the Schrödinger field?

c) Prove that $\phi_{n_1,n_2,\ldots n_k}^{(k)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots\boldsymbol{x}_k)$ satisfies the many-particle Schrödinger equation.

d) Now write down the equal-time *anti*-commutation relations between the Schrödinger field and its conjugate, and the corresponding anticommutation relations between the operators $\hat{a}_n$ and their conjugate. Describe the structure of the Fock space in this case: start from the ground state and arrive at describing the generic state. What is the significance of having chosen the anticommutation relations, to quantize the Schrödinger field?

e) Compute the explicit expression of

$$\begin{aligned}
\phi_n^{(1)}(\boldsymbol{x}) &= \langle\boldsymbol{x}|1n\rangle,\\
\phi_{n,m}^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2) &= \langle\boldsymbol{x}_1,\boldsymbol{x}_2|1n,1m\rangle,\\
\phi_{n_1,n_2,\ldots n_k}^{(k)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots\boldsymbol{x}_k) &= \langle\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots\boldsymbol{x}_k|1n_1,1n_2,\ldots1n_k\rangle,
\end{aligned}$$

also in the form of a Slater determinant. What can be said about the fact that the Schrödinger field can be quantized either as a Bose field or as a Fermi field? What happens in a relativistic setting?

---

a) The equal-time commutation relations for the Schrödinger field are

$$[\hat{\psi}(\boldsymbol{x},t),\hat{\pi}(\boldsymbol{x}',t)] = i\hbar\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$$
$$[\hat{\psi}(\boldsymbol{x},t),\hat{\psi}(\boldsymbol{x}',t)] = [\hat{\pi}(\boldsymbol{x},t),\hat{\pi}(\boldsymbol{x}',t)] = 0,$$

since $\hat{\pi}=\hat{\pi}_\psi = i\hbar\hat{\psi}^\dagger$ and $\hat{\pi}_{\psi^\dagger}=0$, while the commutation relations for $\hat{a}_n(t)$ and $\hat{a}_n^\dagger(t)$ are

$$[\hat{a}_n(t),\hat{a}_m^\dagger(t)] = \delta_{nm}$$
$$[\hat{a}_n(t),\hat{a}_m(t)] = [\hat{a}_n^\dagger(t),\hat{a}_m^\dagger(t)] = 0.$$

Further, since $\hat{\psi}$ still satisfies the Schrödinger equation

$$i\hbar\dot{\hat{\psi}}(x) = \mathcal{D}_{\boldsymbol{x}}\hat{\psi}(x),$$

the dependence of $\hat{a}_n(t)$ on time is trivial:

$$\hat{a}_n(t) = e^{-\frac{i}{\hbar}E_n t}\hat{a}_n(0).$$

Setting $\hat{a}_n(0)=\hat{a}_n$, we obtain that the commutation relations for $\hat{a}_n(t)$ and $\hat{a}_n^\dagger(t)$ are equivalent to those of $\hat{a}_n$ and $\hat{a}_n^\dagger$. The normalized states of the Fock space

$$C_{n_1 n_2\ldots}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\cdots|0\rangle = |n_1\,n_2\ldots\rangle\,,$$

which actually form a basis of the Hilbert space, represent a system of n_1 particles of energy E_1, n_2 particles of energy E_2, etcetera. They are also eigenstates of the density number operators $\hat{n}_i=\hat{a}_i^\dagger\hat{a}_i$.

The operators $\hat{a}_i$ and their adjoints are interpreted as annihilation and creator operators respectively of particles of energy E_i. In fact,

$$\hat{a}_i\,|n_1\,n_2\ldots\rangle \propto \begin{cases} |n_1\,n_2\ldots n_i-1\ldots\rangle & \text{if } n_i>0\\ 0 & \text{if } n_i=0,\end{cases}$$

while

$$\hat{a}_i^\dagger\,|n_1\,n_2\ldots\rangle \propto |n_1\,n_2\ldots n_i+1\ldots\rangle\,.$$

Let us evaluate the normalization factor in the first case. Taking the norm of $\hat{a}_i\,|n_1\,n_2\ldots\rangle$, we have

$$\langle n_1\,n_2\ldots|\hat{a}_i^\dagger\hat{a}_i|n_1\,n_2\ldots\rangle = \langle n_1\,n_2\ldots|\hat{n}_i|n_1\,n_2\ldots\rangle = n_i\,\langle n_1\,n_2\ldots|n_1\,n_2\ldots\rangle = n_i,$$

so that $\hat{a}_i\,|n_1\,n_2\ldots\rangle = \sqrt{n_i}\,|n_1\ldots n_i-1\ldots\rangle$. This is useful to evaluate the normalization factors $C_{n_1n_2\ldots}$. Indeed, taking the norm of $|n_1\,n_2\ldots\rangle$, we find

$$\begin{aligned}1 = \langle n_1\,n_2\ldots|n_1\,n_2\ldots\rangle &= C_{n_1n_2\ldots}\,\langle n_1\,n_2\ldots|(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\cdots|0\rangle\\ &= C_{n_1n_2\ldots}\sqrt{n_1}\,\langle n_1-1\,n_2\ldots|(\hat{a}_1^\dagger)^{n_1-1}(\hat{a}_2^\dagger)^{n_2}\cdots|0\rangle\\ &= C_{n_1n_2\ldots}\sqrt{n_1!}\,\langle 0\,n_2\ldots|(\hat{a}_2^\dagger)^{n_2}(\hat{a}_3^\dagger)^{n_3}\cdots|0\rangle\\ &= C_{n_1n_2\ldots}\sqrt{n_1!\,n_2!\cdots}\,\langle 0|0\rangle\\ &= C_{n_1n_2\ldots}\sqrt{n_1!\,n_2!\cdots}.\end{aligned}$$

Thus, $C_{n_1\,n_2\ldots} = (n_1!\,n_2!\cdots)^{-\frac{1}{2}}$.

b) The state

$$|\boldsymbol{x}_1\ldots\boldsymbol{x}_n\rangle = \frac{1}{\sqrt{n!}}\hat{\psi}^\dagger(\boldsymbol{x}_1)\cdots\hat{\psi}^\dagger(\boldsymbol{x}_n)\,|0\rangle$$

represents a system of n particles in the positions $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n$ (in a fixed instant t). Thus, the operator $\hat{\psi}^\dagger(\boldsymbol{x})$ creates a particle in the position $\boldsymbol{x}$, while its adjoint $\hat{\psi}(\boldsymbol{x})$ destroys such a particle. This fact can be seen by using the commutation relation

$$\begin{aligned}\hat{\psi}(\boldsymbol{x})\,|\boldsymbol{x}_1\ldots\boldsymbol{x}_n\rangle &= \frac{1}{\sqrt{n!}}\hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}_1)\cdots\hat{\psi}^\dagger(\boldsymbol{x}_n)\,|0\rangle \\ &= \frac{1}{\sqrt{n!}}\Big(\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_1)\underbrace{\hat{\psi}^\dagger(\boldsymbol{x}_2)\cdots\hat{\psi}^\dagger(\boldsymbol{x}_n)\,|0\rangle}_{=\sqrt{(n-1)!}|\boldsymbol{x}_2\ldots\boldsymbol{x}_n\rangle} + \hat{\psi}^\dagger(\boldsymbol{x}_1)\hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}_2)\cdots\hat{\psi}^\dagger(\boldsymbol{x}_n)\,|0\rangle\Big) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_i)\,|\boldsymbol{x}_1\ldots\boldsymbol{x}_{i-1}\,\boldsymbol{x}_{i+1}\ldots\boldsymbol{x}_n\rangle + \frac{1}{\sqrt{n!}}\hat{\psi}^\dagger(\boldsymbol{x}_1)\cdots\hat{\psi}^\dagger(\boldsymbol{x}_n)\hat{\psi}(\boldsymbol{x})\,|0\rangle\,.\end{aligned}$$

Expanding $\hat{\psi}(\boldsymbol{x})$ in the basis of $u_n(\boldsymbol{x})$, it can be easily seen that $\hat{\psi}(\boldsymbol{x})\,|0\rangle = 0$, so that

$$\hat{\psi}(\boldsymbol{x})\,|\boldsymbol{x}_1\ldots\boldsymbol{x}_n\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_i)\,|\boldsymbol{x}_1\ldots\boldsymbol{x}_{i-1}\,\boldsymbol{x}_{i+1}\ldots\boldsymbol{x}_n\rangle\,.$$

The usual QM wave functions can be obtained by projecting a k-particles state formed of one particle of energy E_{n_1}, one particle of energy E_{n_2}, etcetera onto a position state $|\boldsymbol{x}_1\ldots\boldsymbol{x}_k\rangle$:

$$\phi^{(k)}_{n_1\ldots n_k}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k) = \langle\boldsymbol{x}_1\ldots\boldsymbol{x}_k|1n_1\,\ldots\,1n_k\rangle\,.$$

We can see that

$$\begin{aligned}\phi^{(1)}_n(\boldsymbol{x}) &= \langle 0|\hat{\psi}(\boldsymbol{x})\hat{a}^\dagger_n|0\rangle \\ &= \sum_m \langle 0|\hat{a}_m\hat{a}^\dagger_n|0\rangle\,u_m(\boldsymbol{x}) \\ &= \sum_m \big(\delta_{nm}\,\langle 0|0\rangle + \langle 0|\hat{a}^\dagger_n\hat{a}_m|0\rangle\big)\,u_m(\boldsymbol{x}) = u_n(\boldsymbol{x}).\end{aligned}$$

Similarly, if $n\neq m$,

$$\begin{aligned}\phi^{(2)}_{n,m}(\boldsymbol{x}_1,\boldsymbol{x}_2) &= \frac{1}{\sqrt{2}}\,\langle 0|\hat{\psi}(\boldsymbol{x}_2)\hat{\psi}(\boldsymbol{x}_1)\hat{a}^\dagger_n\hat{a}^\dagger_m|0\rangle = \frac{1}{\sqrt{2}}\sum_{i,j}\langle 0|\hat{a}_j\hat{a}_i\hat{a}^\dagger_n\hat{a}^\dagger_m|0\rangle\,u_i(\boldsymbol{x}_1)u_j(\boldsymbol{x}_2) \\ &= \frac{1}{\sqrt{2}}\sum_{i,j}\Big(\langle 0|\hat{a}_j\hat{a}^\dagger_n\hat{a}_i\hat{a}^\dagger_m|0\rangle + \delta_{in}\,\langle 0|\hat{a}_j\hat{a}^\dagger_m|0\rangle\Big)u_i(\boldsymbol{x}_1)u_j(\boldsymbol{x}_2) \\ &= \frac{1}{\sqrt{2}}\sum_{i,j}\Big(\underbrace{\langle 0|\hat{a}_j\hat{a}^\dagger_n\hat{a}^\dagger_m\hat{a}_i|0\rangle}_{=0} + \delta_{im}\,\langle 0|\hat{a}_j\hat{a}^\dagger_n|0\rangle + \delta_{in}\,\underbrace{\langle 0|\hat{a}^\dagger_m\hat{a}_j|0\rangle}_{=0} + \delta_{in}\delta_{jm}\Big)u_i(\boldsymbol{x}_1)u_j(\boldsymbol{x}_2) \\ &= \frac{1}{\sqrt{2}}\sum_{i,j}\Big(\delta_{im}\,\underbrace{\langle 0|\hat{a}^\dagger_n\hat{a}_j|0\rangle}_{=0} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\Big)u_i(\boldsymbol{x}_1)u_j(\boldsymbol{x}_2) \\ &= \frac{1}{\sqrt{2}}\sum_{i,j}\Big(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\Big)u_i(\boldsymbol{x}_1)u_j(\boldsymbol{x}_2) = \frac{1}{\sqrt{2}}\Big(u_n(\boldsymbol{x}_1)u_m(\boldsymbol{x}_2) + u_m(\boldsymbol{x}_1)u_n(\boldsymbol{x}_2)\Big).\end{aligned}$$

In the case $n=m$, that is $|1n\,1m\rangle = |2n\rangle$, we should consider the normalization factor $C_{2n} = \frac{1}{\sqrt{2}}$ too, so that

$$\begin{aligned}\phi^{(2)}_{n,n}(\boldsymbol{x}_1,\boldsymbol{x}_2) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Big(u_n(\boldsymbol{x}_1)u_n(\boldsymbol{x}_2) + u_n(\boldsymbol{x}_1)u_n(\boldsymbol{x}_2)\Big) \\ &= u_n(\boldsymbol{x}_1)u_n(\boldsymbol{x}_2).\end{aligned}$$

In general,

$$\begin{aligned}\phi^{(k)}_{n_1\ldots n_k}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k) &= \frac{1}{\sqrt{n_1!\ldots n_k!}}\frac{1}{\sqrt{k!}}\sum_{\sigma\in\mathfrak{S}_k}u_{n_{\sigma(1)}}(\boldsymbol{x}_1)\cdots u_{n_{\sigma(k)}}(\boldsymbol{x}_k) \\ &= \frac{1}{\sqrt{n_1!\ldots n_k!}}\frac{1}{\sqrt{k!}}\ \mathrm{perm}\begin{pmatrix}u_{n_1}(\boldsymbol{x}_1) & u_{n_1}(\boldsymbol{x}_2) & \cdots & u_{n_1}(\boldsymbol{x}_k)\\ u_{n_2}(\boldsymbol{x}_1) & u_{n_2}(\boldsymbol{x}_2) & \cdots & u_{n_2}(\boldsymbol{x}_k)\\ \vdots & \vdots & \ddots & \vdots\\ u_{n_k}(\boldsymbol{x}_1) & u_{n_k}(\boldsymbol{x}_2) & \cdots & u_{n_k}(\boldsymbol{x}_k)\end{pmatrix},\end{aligned}$$

which is a complete symmetric wave function. This means that such theory treats identical bosonic particles and this is a consequence of having chosen the commutator to quantize the fields.

c) Let us show now that the wave function $\phi^{(k)}_{n_1\ldots n_k}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k,t)$ satisfies the many-particle Schrödinger equation.

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\phi^{(k)}_{n_1\ldots n_k}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k,t) &= i\hbar\frac{1}{\sqrt{k!}}\frac{\partial}{\partial t}\langle 0|\hat\psi(\boldsymbol{x}_k,t)\cdots\hat\psi(\boldsymbol{x}_1,t)|\,1n_1\ldots 1n_k\rangle \\
&= i\hbar\frac{1}{\sqrt{k!}}\,\langle 0|\sum_{l=1}^{k}\hat\psi(\boldsymbol{x}_k,t)\cdots\frac{\partial\hat\psi(\boldsymbol{x}_l,t)}{\partial t}\cdots\hat\psi(\boldsymbol{x}_1,t)|\,1n_1\ldots 1n_k\rangle\,.
\end{aligned}
$$

Since $\hat\psi(\boldsymbol{x}_l,t)$ satisfies the Schrödinger equation

$$
i\hbar\frac{\partial}{\partial t}\hat\psi(\boldsymbol{x}_l,t)=\mathcal{D}_{\boldsymbol{x}_l}\hat\psi(\boldsymbol{x}_l,t),\qquad \mathcal{D}_{\boldsymbol{x}_l}=-\frac{\hbar^2}{2m}\nabla^2_{\boldsymbol{x}_l}+V(\boldsymbol{x}_l),
$$

we have

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\phi^{(k)}_{n_1\ldots n_k}(\boldsymbol{x}_1,\ldots,\boldsymbol{x},t) &= \frac{1}{\sqrt{k!}}\,\langle 0|\sum_{l=1}^{k}\hat\psi(\boldsymbol{x}_k,t)\cdots\mathcal{D}_{\boldsymbol{x}_l}\hat\psi(\boldsymbol{x}_l,t)\cdots\hat\psi(\boldsymbol{x}_1,t)|\,1n_1\ldots 1n_k\rangle \\
&= \sum_{l=1}^{k}\frac{1}{\sqrt{k!}}\mathcal{D}_{\boldsymbol{x}_l}\langle 0|\hat\psi(\boldsymbol{x}_k,t)\cdots\hat\psi(\boldsymbol{x}_1,t)|\,1n_1\ldots 1n_k\rangle \\
&= \sum_{l=1}^{k}\mathcal{D}_{\boldsymbol{x}_l}\phi^{(k)}_{n_1\ldots n_k}(\boldsymbol{x}_1,\ldots,\boldsymbol{x},t)
\end{aligned}
$$

since $\mathcal{D}_{\boldsymbol{x}_l}$ commutes with $\hat\psi(\boldsymbol{x}_j,t)$ for $j\neq l$.

d) We now quantize the Schrödinger field with the equal-time anticommutation relations:

$$
\begin{aligned}
&\{\hat\psi(\boldsymbol{x},t),\hat\psi(\boldsymbol{x}',t)\}=\{\hat\psi^\dagger(\boldsymbol{x},t),\hat\psi^\dagger(\boldsymbol{x}',t)\}=0\\
&\{\hat\pi(\boldsymbol{x},t),\hat\pi(\boldsymbol{x}',t)\}=0\\
&\{\hat\psi^\dagger(\boldsymbol{x},t),\hat\pi(\boldsymbol{x}',t)\}=0\\
&\{\hat\psi(\boldsymbol{x},t),\hat\pi(\boldsymbol{x}',t)\}=i\hbar\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'),
\end{aligned}
$$

since $\hat\pi=\hat\pi_\psi=i\hbar\hat\psi^\dagger$ and $\hat\pi_{\psi^\dagger}=0$, while the anticommutation relations for $\hat a_n$ and $\hat a_n^\dagger$ are

$$
\begin{aligned}
&\{\hat a_n,\hat a_m\}=\{\hat a_n^\dagger,\hat a_m^\dagger\}=0\\
&\{\hat a_n,\hat a_m^\dagger\}=\delta_{nm}.
\end{aligned}
$$

An immediate consequence of the anticommutation relations is that the square of a creation and annihilator operator is the zero operator: $\hat a_n^2=(\hat a_n^\dagger)^2=0$. Thus, the number density operators are idempotent:

$$
\hat n_k^2=\hat a_k^\dagger\hat a_k\hat a_k^\dagger\hat a_k=\hat n_k-(\hat a_k^\dagger)^2\hat a_k^2=\hat n_k.
$$

An idempotent operator has eigenvalues 0 or 1. The Fock space is generated by the eigenstates of the number density operators $|n_1\,n_2\ldots\rangle$, where $n_k=0,1$. Starting from the vacuum, which is the state destroyed by every annihilator operator

$$
\hat a_n\,|0\rangle=0,
$$

we can create the states applying the creator operators

$$
|n_1\,n_2\ldots\rangle=(\hat a_1^\dagger)^{n_1}(\hat a_2^\dagger)^{n_2}\cdots|0\rangle\,,\qquad n_k=0,1.
$$

The creation operator creates a particle with energy E_k if the kth level is empty:

$$\begin{aligned}\hat{a}_k^\dagger |n_1 \dots 0_k \dots\rangle &= \hat{a}_k^\dagger (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} \cdots |0\rangle \\ &= (-1)^{n_1} (\hat{a}_1^\dagger)^{n_1} \hat{a}_k^\dagger (\hat{a}_2^\dagger)^{n_2} \cdots |0\rangle \\ &= (-1)^{\sum_{i=1}^{k-1} n_i} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} \cdots \hat{a}_k^\dagger \cdots |0\rangle \\ &= (-1)^{\sum_{i=1}^{k-1} n_i} |n_1 \dots 1_k \dots\rangle ,\end{aligned}$$

while it destroys states with the kth level already occupied

$$\hat{a}_k^\dagger |n_1 \dots 1_k \dots\rangle = (-1)^{\sum_{i=1}^{k-1} n_i} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} \cdots (\hat{a}_k^\dagger)^2 \cdots |0\rangle = 0.$$

Thus, the Pauli exclusion principle is satisfied. Hence, by using the anticommutation relation, we find fermionic identical particles.

e)

$$\begin{aligned}\phi_n^{(1)}(\boldsymbol{x}) &= \langle 0|\hat{\psi}(\boldsymbol{x})\hat{a}_n^\dagger|0\rangle \\ &= \sum_m \langle 0|\hat{a}_m \hat{a}_n^\dagger|0\rangle\, u_m(\boldsymbol{x}) \\ &= \sum_m \big(\delta_{nm} \langle 0|0\rangle - \langle 0|\hat{a}_n^\dagger \hat{a}_m|0\rangle\big)\, u_m(\boldsymbol{x}) = u_n(\boldsymbol{x}).\end{aligned}$$

Similarly,

$$\begin{aligned}\phi_{n,m}^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) &= \frac{1}{\sqrt{2}} \langle 0|\hat{\psi}(\boldsymbol{x}_2)\hat{\psi}(\boldsymbol{x}_1)\hat{a}_n^\dagger \hat{a}_m^\dagger|0\rangle \\ &= \frac{1}{\sqrt{2}} \sum_{i,j} \langle 0|\hat{a}_j \hat{a}_i \hat{a}_n^\dagger \hat{a}_m^\dagger|0\rangle\, u_i(\boldsymbol{x}_1) u_j(\boldsymbol{x}_2).\end{aligned}$$

Since $n \neq m$, the only non-zero contribution to the sum is given by the terms with $i = n$, $j = m$ with a plus sign or $i = m$, $j = n$ with a minus sign (otherwise, we can anticommute the operators and apply the annihilator operators on the vacuum). Hence

$$\phi_{n,m}^{(2)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{\sqrt{2}} \Big(u_n(\boldsymbol{x}_1) u_m(\boldsymbol{x}_2) - u_m(\boldsymbol{x}_1) u_n(\boldsymbol{x}_2) \Big).$$

In general,

$$\begin{aligned}\phi_{n_1 \dots n_k}^{(k)}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) &= \frac{1}{\sqrt{k!}} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}\sigma\, u_{n_{\sigma(1)}}(\boldsymbol{x}_1) \cdots u_{n_{\sigma(k)}}(\boldsymbol{x}_k) \\ &= \frac{1}{\sqrt{k!}} \det \begin{pmatrix} u_{n_1}(\boldsymbol{x}_1) & u_{n_1}(\boldsymbol{x}_2) & \cdots & u_{n_1}(\boldsymbol{x}_k) \\ u_{n_2}(\boldsymbol{x}_1) & u_{n_2}(\boldsymbol{x}_2) & \cdots & u_{n_2}(\boldsymbol{x}_k) \\ \vdots & \vdots & \ddots & \vdots \\ u_{n_k}(\boldsymbol{x}_1) & u_{n_k}(\boldsymbol{x}_2) & \cdots & u_{n_k}(\boldsymbol{x}_k) \end{pmatrix},\end{aligned}$$

which is a complete antisymmetric wave function. This means that such theory treats identical fermionic particles and this is a consequence of having chosen the anticommutator to quantize the fields.
The fact that the Schrödinger field can be quantized with either the Bose or Fermi rules is a consequence of the non-relativistic framework of the equation. In a relativistic theory, the spin-statistics applies: only one type of quantization rule can lead to a consistent theory, *i.e.* Bose rules for integer spin and Fermi rules for half-integer spin.

Quantum Field Theory: Klein-Gordon, Dirac and Maxwell fields

7 Quantization of the Klein-Gordon field

a) Write down the Lagrangian for the free Klein-Gordon field $\phi(x)$, with all constants explicitly expressed. Then set $\hbar = c = 1$ for the subsequent calculations. Compute the conjugate momentum $\pi(x)$ and the Hamiltonian density $\mathcal{H}(x)$. Quantize the field following the prescription of canonical quantization. Write down the Heisenberg equations of motion for the field and its conjugate momentum and show that they reproduce the Klein-Gordon equation.

b) Show that the quantized field $\hat{\phi}(x)$ can be decomposed in plane waves as follows:

$$\hat{\phi}(\boldsymbol{x}, t) = \int d^3p\, N_p \left(\hat{a}_{\boldsymbol{p}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - \omega_p t)} + \hat{a}^\dagger_{\boldsymbol{p}} e^{-i(\boldsymbol{p}\cdot\boldsymbol{x} - \omega_p t)} \right).$$

Derive the corresponding expression for the conjugate field $\hat{\pi}(x)$. Derive the commutation relations among the operators $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}^\dagger_{\boldsymbol{p}}$ (and among themselves) and compute the explicit expression of the normalization factor N_p, in order for the commutation relations among $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}^\dagger_{\boldsymbol{p}}$ to be in the standard form.

c) Starting from the quantized Hamiltonian $\hat{H}$ expressed in terms of the field $\hat{\phi}(x)$ and its conjugate $\hat{\pi}(x)$, and using the plane wave decomposition, derive the corresponding expression in terms of $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}^\dagger_{\boldsymbol{p}}$. Discuss the meaning of the expression thus derived, and why one has to introduce the normal ordering prescription (explain what it consists of).

d) Repeat the same procedure of point (c), for the total momentum $\hat{\boldsymbol{P}}$. Why is it not necessary to use the normal ordering in this case?

e) Introduce the vacuum state $|0\rangle$ and, starting from it, construct the Fock space for the quantized Klein-Gordon field. Explain the role of the operators $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}^\dagger_{\boldsymbol{p}}$.

---

a) A possible Lagrangian density for the neutral Klein-Gordon (KG) field is

$$\mathcal{L}(x) = \frac{\hbar^2}{2}\partial^\mu \phi\, \partial_\mu \phi - \frac{m^2c^2}{2}\phi^2.$$

Setting $\hbar = c = 1$, we can compute the conjugate momentum and the Hamiltonian density

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi},$$
$$\mathcal{H}(x) = \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right).$$

Let us quantize the fields by using the commutation relations:

$$\phi, \pi \longrightarrow \hat{\phi}, \hat{\pi}$$
$$\{\,\cdot\,,\,\cdot\,\}_{\text{PB}} \longrightarrow -i\,[\,\cdot\,,\,\cdot\,].$$

The Heisenberg equations will be

$$\begin{cases} \dot{\hat{\phi}}(x) = -i\,[\hat{\phi}(x), \hat{H}(t)] \\ \dot{\hat{\pi}}(x) = -i\,[\hat{\pi}(x), \hat{H}(t)], \end{cases}$$

where $\hat{H}$ is the total Hamiltonian

$$\hat{H} = \int dx^3\, \hat{\mathcal{H}}(x) = \frac{1}{2}\int d^3x \left(\hat{\pi}^2 + (\boldsymbol{\nabla}\hat{\phi})^2 + m^2\hat{\phi}^2\right).$$

Then the commutators will be (omitting the equal-time dependence)

$$[\hat{\pi}(\boldsymbol{x}), \hat{H}] = \frac{1}{2}\int d^3x' \left([\hat{\phi}(\boldsymbol{x}), \hat{\pi}^2(\boldsymbol{x}')] + [\hat{\phi}(\boldsymbol{x}), \left(\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')\right)^2] + m^2[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^2(\boldsymbol{x}')]\right).$$

Using the fundamental commutation relations, we find

$$\begin{aligned}
\big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}^2(\boldsymbol{x}')\big] &= \hat{\pi}(\boldsymbol{x}')\big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{x}')\big] + \big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{x}')\big]\hat{\pi}(\boldsymbol{x}') = 2i\,\hat{\pi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \\
\big[\hat{\phi}(\boldsymbol{x}), \big(\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')\big)^2\big] &= \boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')\big(\boldsymbol{\nabla}'\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big]\big) + \big(\boldsymbol{\nabla}'\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big]\big)\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}') = 0 \\
\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^2(\boldsymbol{x}')\big] &= \hat{\phi}(\boldsymbol{x}')\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big] + \big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big]\hat{\phi}(\boldsymbol{x}') = 0,
\end{aligned}$$

which lead to

$$[\hat{\phi}(\boldsymbol{x}), \hat{H}] = i\int d^3x'\,\hat{\pi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') = i\,\hat{\pi}(\boldsymbol{x}).$$

In a similar way, from the commutators

$$\begin{aligned}
&\big[\hat{\pi}(\boldsymbol{x}), \hat{\pi}^2(\boldsymbol{x}')\big] = 0, \\
&\big[\hat{\pi}(\boldsymbol{x}), \big(\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')\big)^2\big] = -2i\,\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')\cdot\boldsymbol{\nabla}'\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') = 2i\,\nabla'^2\hat{\phi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'), \\
&\big[\hat{\pi}(\boldsymbol{x}), \hat{\phi}^2(\boldsymbol{x}')\big] = -2i\,\hat{\phi}(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'),
\end{aligned}$$ [1]

we can see that

$$[\hat{\pi}(\boldsymbol{x}), \hat{H}] = i\int d^3x'\left(\nabla'^2\hat{\phi}(\boldsymbol{x}') - m^2\hat{\phi}(\boldsymbol{x}')\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') = i\left(\nabla^2\hat{\phi}(\boldsymbol{x}) - m^2\hat{\phi}(\boldsymbol{x})\right).$$

Thus, the equations of motion become:

$$\begin{cases}\dot{\hat{\phi}}(x) = \hat{\pi}(x) \\ \dot{\hat{\pi}}(x) = \nabla^2\hat{\phi}(x) - m^2\hat{\phi}(x),\end{cases}$$

which are equivalent to the KGE

$$\left(\Box + m^2\right)\hat{\phi}(x) = 0.$$

b) Let us expand the solution in the basis of plane waves

$$\hat{\phi}(x) = \int d^3p\,N_p\,\hat{a}_{\boldsymbol{p}}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

where N_p is a (real) normalization factor which is supposed to depends only on the module of $\boldsymbol{p}$ (we will see that this assumption is not restrictive). Since $\hat{\phi}$ satisfies the KGE, we find

$$\begin{aligned}
0 &= \int d^3p\,N_p\left(\ddot{\hat{a}}_{\boldsymbol{p}}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - (i\boldsymbol{p})^2\hat{a}_{\boldsymbol{p}}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + m^2\hat{a}_{\boldsymbol{p}}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \\
&= \int d^3p\,N_p\left(\ddot{\hat{a}}_{\boldsymbol{p}}(t) + p^2\hat{a}_{\boldsymbol{p}}(t) + m^2\hat{a}_{\boldsymbol{p}}(t)\right)e^{i\boldsymbol{p}\cdot\boldsymbol{x}},
\end{aligned}$$

which leads to

$$\ddot{\hat{a}}_{\boldsymbol{p}} = -\omega_p^2\,\hat{a}_{\boldsymbol{p}} \qquad \omega_p = \sqrt{p^2+m^2}.$$

The general solution of the equation is

$$\hat{a}_{\boldsymbol{p}}(t) = \hat{a}^{(1)}_{\boldsymbol{p}}e^{-i\omega_p t} + \hat{a}^{(2)}_{\boldsymbol{p}}e^{i\omega_p t}.$$

Since the classical field ϕ is real, then the quantized one is hermitian: $\hat{\phi} = \hat{\phi}^\dagger$. With this fact, we find

$$\begin{aligned}
\hat{\phi}(x) &= \int d^3p\,N_p\left(\hat{a}^{(1)}_{\boldsymbol{p}}e^{-i\omega_p t} + \hat{a}^{(2)}_{\boldsymbol{p}}e^{i\omega_p t}\right)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \\
&= \hat{\phi}^\dagger(x) = \int d^3p\,N_p\left((\hat{a}^{(1)}_{\boldsymbol{p}})^\dagger e^{i\omega_p t} + (\hat{a}^{(2)}_{\boldsymbol{p}})^\dagger e^{-i\omega_p t}\right)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}},
\end{aligned}$$

so that $\hat{a}^{(1)}_{\boldsymbol{p}} = (\hat{a}^{(2)}_{-\boldsymbol{p}})^\dagger$. Setting $\hat{a}^{(1)}_{\boldsymbol{p}} = \hat{a}_{\boldsymbol{p}}$ and with the replacement $\boldsymbol{p} \to -\boldsymbol{p}$ in the second integral for $\hat{\phi}$, we find

$$\hat{\phi}(x) = \int d^3p\, N_p \left(\hat{a}^\dagger_{\boldsymbol{p}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} + \hat{a}_{\boldsymbol{p}}\, e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} \right).$$

Since $\hat{\pi} = \dot{\hat{\phi}}$, we find

$$\hat{\pi}(x) = i \int d^3p\, N_p\, \omega_p \left(\hat{a}^\dagger_{\boldsymbol{p}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} - \hat{a}_{\boldsymbol{p}}\, e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} \right).$$

In order to obtain the commutation relation between the operators $\hat{a}_{\boldsymbol{p}}$ and their adjoints it is useful to isolate the operators using some Fourier transforms

$$\begin{aligned} N_p \left(\hat{a}^\dagger_{\boldsymbol{p}}\, e^{i\omega_p t} + \hat{a}_{-\boldsymbol{p}}\, e^{-i\omega_p t} \right) &= \int \frac{d^3x}{(2\pi)^3}\, \hat{\phi}(x)\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \\ iN_p\, \omega_p \left(\hat{a}^\dagger_{\boldsymbol{p}}\, e^{i\omega_p t} - \hat{a}_{-\boldsymbol{p}}\, e^{-i\omega_p t} \right) &= \int \frac{d^3x}{(2\pi)^3}\, \hat{\pi}(x)\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}, \end{aligned}$$

here we have replaced $\boldsymbol{p} \to -\boldsymbol{p}$ in the $\hat{a}_{\boldsymbol{p}}$ terms of the $\hat{\phi}(x)$ and $\hat{\pi}(x)$ integrals and used the fact that $\omega_p = \omega_{-p}$. So that

$$2N_p\, \hat{a}^\dagger_{\boldsymbol{p}}\, e^{i\omega_p t} = \int \frac{d^3x}{(2\pi)^3} \left(\hat{\phi}(x) + \frac{\hat{\pi}(x)}{i\omega_p} \right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}}.$$

We can finally isolate $\hat{a}^\dagger_{\boldsymbol{p}}$ and by dagging it we can obtain $\hat{a}_{\boldsymbol{p}}$

$$\begin{aligned} \hat{a}^\dagger_{\boldsymbol{p}} &= \frac{1}{2N_p} \int \frac{d^3x}{(2\pi)^3} \left(\hat{\phi}(x) + \frac{\hat{\pi}(x)}{i\omega_p} \right) e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} \\ \hat{a}_{\boldsymbol{p}} &= \frac{1}{2N_p} \int \frac{d^3x}{(2\pi)^3} \left(\hat{\phi}(x) - \frac{\hat{\pi}(x)}{i\omega_p} \right) e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}. \end{aligned}$$

We can now compute the commutators, *i.e.*

$$\begin{aligned} [\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = \frac{1}{4N_p N_{p'}} \int \frac{d^3x}{(2\pi)^3} \int \frac{d^3y}{(2\pi)^3} \Big(&[\hat{\phi}(x), \hat{\phi}(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{-i(\omega_{p'} t - \boldsymbol{p}'\cdot\boldsymbol{y})} + \\ &+ \frac{1}{i\omega_{p'}} [\hat{\phi}(x), \hat{\pi}(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{-i(\omega_{p'} t - \boldsymbol{p}'\cdot\boldsymbol{y})} + \\ &- \frac{1}{i\omega_p} [\hat{\pi}(x), \hat{\phi}(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{-i(\omega_{p'} t - \boldsymbol{p}'\cdot\boldsymbol{y})} + \\ &+ \frac{1}{\omega_p\, \omega_{p'}} [\hat{\pi}(x), \hat{\pi}(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{-i(\omega_{p'} t - \boldsymbol{p}'\cdot\boldsymbol{y})} \Big), \end{aligned}$$

using the equal-time commutation relations we find

$$\begin{aligned} [\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] &= \frac{1}{4N_p N_{p'}} \int \frac{d^3x}{(2\pi)^6} \Big(\frac{1}{\omega_{p'}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{-i(\omega_{p'} t - \boldsymbol{p}'\cdot\boldsymbol{x})} + \\ &\qquad + \frac{1}{\omega_p}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{-i(\omega_{p'} t - \boldsymbol{p}'\cdot\boldsymbol{x})} \Big) \\ &= \frac{1}{4N_p N_{p'}} \frac{1}{(2\pi)^3} \left(\frac{1}{\omega_p} + \frac{1}{\omega_{p'}} \right) \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \\ &= \frac{1}{N_p^2} \frac{1}{(2\pi)^3} \frac{1}{2\omega_p}\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}').^1 \end{aligned}$$

(The equalities are such if integrated over momentum space.) Clearly a wise choice for the normalization factor N_p is

$$N_p = \frac{1}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega_p}},$$

this leads to the result

$$[\hat a_{\boldsymbol p},\hat a^\dagger_{\boldsymbol p'}]=\delta^{(3)}(\boldsymbol p-\boldsymbol p').$$

With analogous calculation we can compute *i.e.* $[\hat a_{\boldsymbol p},\hat a_{\boldsymbol p'}]$

$$\begin{aligned}[\hat a_{\boldsymbol p},\hat a_{\boldsymbol p'}]=\frac{1}{4N_pN_{p'}}\int\frac{d^3x}{(2\pi)^3}\int\frac{d^3y}{(2\pi)^3}\Big([\hat\phi(x),\hat\phi(y)]\,e^{i(\omega_pt-\boldsymbol p\cdot\boldsymbol x)}\,e^{i(\omega_{p'}t-\boldsymbol p'\cdot\boldsymbol y)}+&\\ -\frac{1}{i\omega_{p'}}[\hat\phi(x),\hat\pi(y)]\,e^{i(\omega_pt-\boldsymbol p\cdot\boldsymbol x)}\,e^{i(\omega_{p'}t-\boldsymbol p'\cdot\boldsymbol y)}+&\\ -\frac{1}{i\omega_{p}}[\hat\pi(x),\hat\phi(y)]\,e^{i(\omega_pt-\boldsymbol p\cdot\boldsymbol x)}\,e^{i(\omega_{p'}t-\boldsymbol p'\cdot\boldsymbol y)}+&\\ -\frac{1}{\omega_p\,\omega_{p'}}[\hat\pi(x),\hat\pi(y)]\,e^{i(\omega_pt-\boldsymbol p\cdot\boldsymbol x)}\,e^{i(\omega_{p'}t-\boldsymbol p'\cdot\boldsymbol y)}\Big),&\end{aligned}$$

using the equal-time commutation relations we find

$$\begin{aligned}[\hat a_{\boldsymbol p},\hat a_{\boldsymbol p'}]&=\frac{1}{4N_pN_{p'}}\int\frac{d^3x}{(2\pi)^6}\Big(-\frac{1}{\omega_{p'}}\,e^{i(\omega_pt-\boldsymbol p\cdot\boldsymbol x)}\,e^{i(\omega_{p'}t-\boldsymbol p'\cdot\boldsymbol x)}+\\ &\qquad+\frac{1}{\omega_p}\,e^{i(\omega_pt-\boldsymbol p\cdot\boldsymbol x)}\,e^{i(\omega_{p'}t-\boldsymbol p'\cdot\boldsymbol x)}\Big)\\ &=\frac{1}{4N_pN_{p'}}\frac{1}{(2\pi)^3}\Big(\frac{1}{\omega_p}-\frac{1}{\omega_{p'}}\Big)\,e^{i(\omega_p+\omega_{p'})t}\,\delta^{(3)}(\boldsymbol p+\boldsymbol p')=0,\end{aligned}$$

for symmetry we have $[\hat a^\dagger_{\boldsymbol p},\hat a^\dagger_{\boldsymbol p'}]=0$.

We have finally found the usual harmonic oscillator algebra

$$\begin{aligned}[\hat a_{\boldsymbol p},\hat a^\dagger_{\boldsymbol p'}]&=\delta^{(3)}(\boldsymbol p-\boldsymbol p')\\ [\hat a_{\boldsymbol p},\hat a_{\boldsymbol p'}]&=[\hat a^\dagger_{\boldsymbol p},\hat a^\dagger_{\boldsymbol p'}]=0.\end{aligned}$$

c) Since

$$\boldsymbol\nabla\hat\phi=i\int d^3p\,\boldsymbol p\Big(\hat a_{\boldsymbol p}\,u_{\boldsymbol p}-\hat a^\dagger_{\boldsymbol p}\,u^*_{\boldsymbol p}\Big),\qquad\qquad u_{\boldsymbol p}=N_pe^{i(\boldsymbol p\cdot\boldsymbol x-\omega_pt)},$$

the Hamiltonian will be

$$\begin{aligned}\hat H&=\frac12\int d^3x\,\Big(\hat\pi^2+(\boldsymbol\nabla\hat\phi)^2+m^2\hat\phi^2\Big)\\ &=\frac12\int d^3x\,d^3p\,d^3p'\Big(-\omega_p\omega_{p'}\,\Big(\hat a^\dagger_{\boldsymbol p}\,u^*_{\boldsymbol p}(x)-\hat a_{\boldsymbol p}\,u_{\boldsymbol p}(x)\Big)\Big(\hat a^\dagger_{\boldsymbol p'}\,u^*_{\boldsymbol p'}(x)-\hat a_{\boldsymbol p'}\,u_{\boldsymbol p'}(x)\Big)\\ &\qquad-\boldsymbol p\cdot\boldsymbol p'\Big(\hat a^\dagger_{\boldsymbol p}\,u^*_{\boldsymbol p}(x)-\hat a_{\boldsymbol p}\,u_{\boldsymbol p}(x)\Big)\Big(\hat a^\dagger_{\boldsymbol p'}\,u^*_{\boldsymbol p'}(x)-\hat a_{\boldsymbol p'}\,u_{\boldsymbol p'}(x)\Big)\\ &\qquad+m^2\Big(\hat a^\dagger_{\boldsymbol p}\,u^*_{\boldsymbol p}(x)+\hat a_{\boldsymbol p}\,u_{\boldsymbol p}(x)\Big)\Big(\hat a^\dagger_{\boldsymbol p'}\,u^*_{\boldsymbol p'}(x)+\hat a_{\boldsymbol p'}\,u_{\boldsymbol p'}(x)\Big)\Big).\end{aligned}$$

The integration over space can be done thanks to the relations

$$\begin{aligned}\int d^3x\,u_{\boldsymbol p}(x)u^*_{\boldsymbol p'}(x)&=\frac{1}{2\sqrt{\omega_p\omega_{p'}}}\int\frac{d^3x}{(2\pi)^3}e^{-i(p-p')\cdot x}\\ &=\frac{1}{2\sqrt{\omega_p\omega_{p'}}}e^{-i(\omega_p-\omega_{p'})t}\,\delta^{(3)}(\boldsymbol p-\boldsymbol p')\\ &=\frac{1}{2\omega_p}\delta^{(3)}(\boldsymbol p-\boldsymbol p'),\\ \int d^3x\,u_{\boldsymbol p}(x)u_{\boldsymbol p'}(x)&=\frac{1}{2\sqrt{\omega_p\omega_{p'}}}\int\frac{d^3x}{(2\pi)^3}e^{-i(p+p')\cdot x}\\ &=\frac{1}{2\sqrt{\omega_p\omega_{p'}}}e^{-i(\omega_p+\omega_{p'})t}\,\delta^{(3)}(\boldsymbol p+\boldsymbol p')\\ &=\frac{1}{2\omega_p}e^{-2i\omega_pt}\,\delta^{(3)}(\boldsymbol p+\boldsymbol p'),\end{aligned}$$

and those for their adjoints. We get:

$$\begin{aligned}\hat{H} = \frac{1}{2}\int d^3p\Big(&-\omega_p^2\,\frac{1}{2\omega_p}\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t} - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t}\Big)\\ &- p^2\,\frac{1}{2\omega_p}\Big(-\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t} - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - \hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t}\Big)\\ &+ m^2\,\frac{1}{2\omega_p}\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t}\Big)\Big)\\ = \frac{1}{2}\int d^3p\frac{1}{2\omega_p}\Big(&(-\omega_p^2 + p^2 + m^2)(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t} + \hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t})\\ &+ (\omega_p^2 + p^2 + m^2)(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}})\Big).\end{aligned}$$

Using the dispersion relation and writing $\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} = \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \delta^{(3)}(\mathbf{0})$, we find

$$\hat{H} = \int d^3p\;\omega_p\,\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \frac{1}{2}\delta^{(3)}(\mathbf{0})\right).$$

The Hamiltonian is positive definite (as it can be directly seen from the first expression), but the expectation value on every state is infinite. In fact the contribution to the energy is given by ω_p for every particle of momentum $\boldsymbol{p}$ via the operators $\hat{n}_{\boldsymbol{p}}$ plus the zero-point energy, which is independent of the occupation number. This becomes even more obvious if we discretise the momentum space:

$$\hat{H} = \sum_{\boldsymbol{p}_i}\omega_{p_i}\left(\hat{n}_{\boldsymbol{p}_i} + \frac{1}{2}\right)$$

and the zero-point energy $E_0 = \frac{1}{2}\sum_{\boldsymbol{p}_i}\omega_{p_i}$ is divergent.

Since the problem arises from the terms of the form $\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}$, we 'solve' the problem by forcing them by hand to become $\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}$ without picking up the divergent contribution. We do this by introducing the normal ordering. For every operator we separate the contribution given by the positive frequencies $e^{-i\omega_p t}$ and the negative frequencies $e^{i\omega_p t}$

$$\hat{\alpha}(x) = \hat{\alpha}^{(+)}(x) + \hat{\alpha}^{(-)}(x), \qquad \hat{\beta}(x) = \hat{\beta}^{(+)}(x) + \hat{\beta}^{(-)}(x)$$

and define the normal ordered product to be

$$:\hat{\alpha}\hat{\beta}: = \hat{\alpha}^{(+)}\hat{\beta}^{(+)} + \hat{\alpha}^{(-)}\hat{\beta}^{(+)} \pm \hat{\beta}^{(-)}\hat{\alpha}^{(+)} + \hat{\alpha}^{(-)}\hat{\beta}^{(-)},$$

where the sign is + if the fields are bosonic and − if the fields are fermionic. In this way the negative frequencies contributions are forced to stay on the left. With this prescription, the Hamiltonian becomes

$$:\hat{H}: = \frac{1}{2}\int d^3x\,:\!\Big(\hat{\pi}^2 + (\boldsymbol{\nabla}\hat{\phi})^2 + m^2\hat{\phi}^2\Big)\!: = \int d^3p\;\omega_p\,\hat{n}_{\boldsymbol{p}},$$

which is the previous Hamiltonian with the zero-point energy removed.

d) Since the Lagrangian density of the KG field is invariant under translations, we have the conserved classical 4-momentum

$$P_\mu = \int d^3x\Big(\pi(x)\partial_\mu\phi(x) - \eta_{0\mu}\mathcal{L}(x)\Big).$$

In particular, we can define the momentum operator to be the symmetrized version of $\boldsymbol{P}$:

$$\hat{\boldsymbol{P}} = -\frac{1}{2}\int d^3x\big(\hat{\pi}\boldsymbol{\nabla}\hat{\phi} + \boldsymbol{\nabla}\hat{\phi}\hat{\pi}\big).$$

[1]The equalities have to be intended as such if integrated over momentum space.

In this way, $\hat{\boldsymbol{P}}$ is hermitian. Expanding the fields in plane waves, we find

$$\begin{aligned}
\hat{\boldsymbol{P}} &= -\frac{1}{2}\int d^3x\, d^3p\, d^3p' \Big(\omega_p \boldsymbol{p}' \Big(\hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x) - \hat{a}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x)\Big)\Big(\hat{a}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}'}(x) - \hat{a}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}'}(x)\Big) \\
&\qquad \omega_{p'} \boldsymbol{p}\Big(\hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x) - \hat{a}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x)\Big)\Big(\hat{a}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}'}(x) - \hat{a}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}'}(x)\Big)\Big) \\
&= -\frac{1}{2}\int d^3p\, d^3p' \Big(\omega_p \boldsymbol{p}' \Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x) - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x) \\
&\qquad - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x) + \hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x)\Big) \\
&\qquad + \omega_{p'} \boldsymbol{p}\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x) - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x) \\
&\qquad - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x) + \hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x)\Big)\Big) \\
&= -\frac{1}{2}\int d^3p \Big(\omega_p \boldsymbol{p}\frac{1}{2\omega_p}\Big(-\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}} e^{2i\omega_p t} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - \hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} e^{-2i\omega_p t}\Big) \\
&\qquad + \omega_p \boldsymbol{p}\frac{1}{2\omega_p}\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}} e^{2i\omega_p t} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} e^{-2i\omega_p t}\Big)\Big) \\
&= \frac{1}{2}\int d^3p\, \boldsymbol{p}\, \big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}\big).
\end{aligned}$$

The vacuum momentum does not give any problem, since the integral is zero for symmetry reasons

$$\hat{\boldsymbol{P}} = \int d^3p\, \boldsymbol{p}\, \hat{n}_{\boldsymbol{p}}.$$

Hence, in this particular case the normal ordering prescription does not make any difference:

$$:\hat{\boldsymbol{P}}: = \hat{\boldsymbol{P}}.$$

This is a consequence of the isotropy of space. The same argument does not apply to the energy, since it is always positive: $\omega_p \geq 0$. It can be easily seen that the momentum operator is still a conserved quantity, since $[\hat{\boldsymbol{P}}, \hat{H}] = 0$ by applying the commutation relations among the creation and annihilation operators.

e) Since we have an harmonic oscillator algebra, we can introduce the vacuum, which is destroyed by every annihilator operator

$$\hat{a}_{\boldsymbol{p}} \left|0\right\rangle = 0.$$

Then we can construct the eigenstates $\left|n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots\right\rangle$ of the number density operators $\hat{n}_{\boldsymbol{p}_i}$ applying the creator operators to the vacuum:

$$\left|n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots\right\rangle = C_{n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots}(\hat{a}^\dagger_{\boldsymbol{p}_1})^{n_1}(\hat{a}^\dagger_{\boldsymbol{p}_2})^{n_2}\cdots\left|0\right\rangle,$$

where $C_{n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots}$ is a normalization factor. The number density operators count particles with definite momentum

$$\hat{n}_{\boldsymbol{p}_i}\left|n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots\right\rangle = n_i\left|n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots\right\rangle$$

while $\hat{a}^\dagger_{\boldsymbol{p}_i}$ and $\hat{a}_{\boldsymbol{p}_i}$ create and destroy respectively a particle with definite 4-momentum $(\omega_{p_i}, \boldsymbol{p}_i)$

$$\begin{aligned}
\hat{a}^\dagger_{\boldsymbol{p}_i}\left|n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots\right\rangle &= C\left|n_1\boldsymbol{p}_1\ldots(n_i+1)\boldsymbol{p}_i\ldots\right\rangle \\
\hat{a}_{\boldsymbol{p}_i}\left|n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2\ldots\right\rangle &= \begin{cases} D\left|n_1\boldsymbol{p}_1\ldots(n_i-1)\boldsymbol{p}_i\ldots\right\rangle & \text{if } n_i \geq 1 \\ 0 & \text{if } n_i = 0. \end{cases}
\end{aligned}$$

Here C and D are normalization factors. We can say that such particles have mass m and definite (relativistic) momentum, since the states are eigenstates of the 4-momentum operators too (the 4-momentum operator $:\hat{P}_\mu:$ commutes with $\hat{n}_{\boldsymbol{p}}$). In fact

$$
\begin{aligned}
:\hat{H}: |n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2 \ldots\rangle &= \int d^3p\, \omega_p\, \hat{n}_{\boldsymbol{p}}\, |n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2 \ldots\rangle \\
&= \int d^3p\, \omega_p \sum_i n_i\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}_i)\, |n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2 \ldots\rangle \\
&= \left(\sum_i \omega_{p_i}\, n_i\right) |n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2 \ldots\rangle
\end{aligned}
$$

and similarly

$$
\hat{\boldsymbol{P}}\, |n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2 \ldots\rangle = \left(\sum_i \boldsymbol{p}_i\, n_i\right) |n_1\boldsymbol{p}_1\, n_2\boldsymbol{p}_2 \ldots\rangle\,.
$$

8 Quantization of the charged Klein-Gordon field

a) Write down the Lagrangian for the free charged Klein-Gordon field $\phi(x)$, with all constants explicitly expressed. Then set $\hbar = c = 1$ for the subsequent calculations. Compute the conjugate momenta and the Hamiltonian density $\mathcal{H}(x)$. Quantize the field following the prescription of canonical quantization. Write down the Heisenberg equations of motion for the fields and their conjugate momenta and show that they reproduce the Klein-Gordon equation.

b) Show that the quantized field $\hat{\phi}(x)$ can be decomposed in plane waves as follows:

$$
\hat{\phi}(\boldsymbol{x}, t) = \int d^3p\, N_p \left(\hat{a}_{\boldsymbol{p}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_p t)} + \hat{b}^\dagger_{\boldsymbol{p}} e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_p t)}\right).
$$

Derive the corresponding expression for the conjugate field $\hat{\pi}(x)$. Derive the commutation relations among the operators $\hat{a}_{\boldsymbol{p}}$, $\hat{a}^\dagger_{\boldsymbol{p}}$, $\hat{b}_{\boldsymbol{p}}$ and $\hat{b}^\dagger_{\boldsymbol{p}}$ and compute the explicit expression of the normalization factor N_p, in order for these commutation relations to be in the standard form.

c) Starting from the quantized Hamiltonian $\hat{H}$ expressed in terms of the fields $\hat{\phi}(x)$, $\hat{\pi}(x)$ and their adjoints, and using the plane wave decomposition, derive the corresponding expression in terms of $\hat{a}_{\boldsymbol{p}}$, $\hat{a}^\dagger_{\boldsymbol{p}}$, $\hat{b}_{\boldsymbol{p}}$ and $\hat{b}^\dagger_{\boldsymbol{p}}$. Discuss the meaning of the expression thus derived, and why one has to introduce the normal ordering prescription (explain what it consists of).

d) Repeat the same procedure of point (c), for the total momentum $\hat{\boldsymbol{P}}$. Why is it not necessary to use the normal ordering in this case?

e) Discuss the gauge symmetry of the Lagrangian and derive the conserved charge Q its leads to. Write it in terms of the fields $\phi(x)$, $\pi(x)$ and their complex conjugates. From that, derive the quantized expression of $\hat{Q}$ in terms of $\hat{a}_{\boldsymbol{p}}$, $\hat{a}^\dagger_{\boldsymbol{p}}$, $\hat{b}_{\boldsymbol{p}}$ and $\hat{b}^\dagger_{\boldsymbol{p}}$. Comment on the physical meaning of the operators $\hat{a}_{\boldsymbol{p}}$, $\hat{a}^\dagger_{\boldsymbol{p}}$, $\hat{b}_{\boldsymbol{p}}$ and $\hat{b}^\dagger_{\boldsymbol{p}}$.

---

a) The Lagrangian density for the charged Klein-Gordon (KG) field is

$$
\mathcal{L}(x) = \hbar^2\, \partial^\mu\phi^*\, \partial_\mu\phi - m^2c^2\phi^*\phi.
$$

We set $\hbar = c = 1$. The conjugate momenta are

$$
\begin{aligned}
\pi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^* \\
\pi^* &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi}
\end{aligned}
$$

and the Hamiltonian density

$$
\mathcal{H}(x) = 2\pi^*\pi - \pi^*\pi + \boldsymbol{\nabla}\phi^*\cdot\boldsymbol{\nabla}\phi + m^2\phi^*\phi = \pi^*\pi + \boldsymbol{\nabla}\phi^*\cdot\boldsymbol{\nabla}\phi + m^2\phi^*\phi.
$$

Let us quantize the fields using the commutation relations.

$$\phi, \phi^*, \pi, \pi^* \longrightarrow \hat{\phi}, \hat{\phi}^\dagger, \hat{\pi}, \hat{\pi}^\dagger$$
$$\{\,\cdot\,,\,\cdot\,\}_{\text{PB}} \longrightarrow -i\,[\,\cdot\,,\,\cdot\,].$$

The Heisenberg equations will be

$$\begin{cases} \dot{\hat{\phi}}(x) = -i\,[\hat{\phi}(x), \hat{H}(t)] \\ \dot{\hat{\phi}}^\dagger(x) = -i\,[\hat{\phi}^\dagger(x), \hat{H}(t)] \\ \dot{\hat{\pi}}(x) = -i\,[\hat{\pi}(x), \hat{H}(t)] \\ \dot{\hat{\pi}}^\dagger(x) = -i\,[\hat{\pi}^\dagger(x), \hat{H}(t)], \end{cases}$$

where $\hat{H}$ is the total Hamiltonian:

$$\hat{H} = \int d^3x\, \hat{\mathcal{H}}(x) = \int d^3x \left(\hat{\pi}^\dagger \hat{\pi} + \boldsymbol{\nabla} \hat{\phi}^\dagger \boldsymbol{\nabla} \hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi} \right).$$

Note that $\hat{H}$ is hermitian. The commutators become (omitting the equal-time dependence)

$$[\hat{\phi}(\boldsymbol{x}), \hat{H}] = \int d^3x' \Big([\hat{\phi}(\boldsymbol{x}), \hat{\pi}^\dagger(\boldsymbol{x}')\hat{\pi}(\boldsymbol{x}')] + [\hat{\phi}(\boldsymbol{x}), \boldsymbol{\nabla}'\hat{\phi}^\dagger(\boldsymbol{x}')\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')] + m^2[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^\dagger(\boldsymbol{x}')\hat{\phi}(\boldsymbol{x}')] \Big).$$

Using the fundamental commutation relations, we find

$$\begin{aligned} \big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}^\dagger(\boldsymbol{x}')\hat{\pi}(\boldsymbol{x}')\big] &= \hat{\pi}^\dagger(\boldsymbol{x}')\big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{x}')\big] + \big[\hat{\phi}(\boldsymbol{x}), \hat{\pi}^\dagger(\boldsymbol{x}')\big]\hat{\pi}(\boldsymbol{x}') = i\,\hat{\pi}^\dagger(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \\ \big[\hat{\phi}(\boldsymbol{x}), \boldsymbol{\nabla}'\hat{\phi}^\dagger(\boldsymbol{x}')\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')\big] &= \boldsymbol{\nabla}'\hat{\phi}^\dagger(\boldsymbol{x}')\big(\boldsymbol{\nabla}'\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big]\big) + \big(\boldsymbol{\nabla}'\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^\dagger(\boldsymbol{x}')\big]\big)\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}') = 0 \\ \big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^\dagger(\boldsymbol{x}')\hat{\phi}(\boldsymbol{x}')\big] &= \hat{\phi}^\dagger(\boldsymbol{x}')\big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')\big] + \big[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^\dagger(\boldsymbol{x}')\big]\hat{\phi}(\boldsymbol{x}') = 0. \end{aligned}$$

Using the previous relations, we obtain

$$[\hat{\phi}(\boldsymbol{x}), \hat{H}] = \int d^3x'\, i\hat{\pi}^\dagger(\boldsymbol{x}')\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') = i\hat{\pi}^\dagger(\boldsymbol{x}').$$

In a similar way, from the commutators

$$\begin{aligned} &\big[\hat{\pi}(\boldsymbol{x}), \hat{\pi}^\dagger(\boldsymbol{x}')\hat{\pi}(\boldsymbol{x}')\big] = 0 \\ &\big[\hat{\pi}(\boldsymbol{x}), \boldsymbol{\nabla}'\hat{\phi}^\dagger(\boldsymbol{x}')\boldsymbol{\nabla}'\hat{\phi}(\boldsymbol{x}')\big] = -i\,\boldsymbol{\nabla}'\hat{\phi}^\dagger(\boldsymbol{x}')\cdot\boldsymbol{\nabla}'\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') = i\,\nabla'^2\hat{\phi}^\dagger(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'),\text{[2]} \\ &\big[\hat{\pi}(\boldsymbol{x}), \hat{\phi}^\dagger(\boldsymbol{x}')\hat{\phi}(\boldsymbol{x}')\big] = -i\,\hat{\phi}^\dagger(\boldsymbol{x}')\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \end{aligned}$$

we can see that

$$[\hat{\pi}(\boldsymbol{x}), \hat{H}] = \int d^3x'\, i\left(\nabla'^2\hat{\phi}^\dagger(\boldsymbol{x}') - m^2\hat{\phi}^\dagger(\boldsymbol{x}')\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') = i\left(\nabla^2\hat{\phi}^\dagger(\boldsymbol{x}) - m^2\hat{\phi}^\dagger(\boldsymbol{x})\right).$$

Thus, taking the adjoint equations, we obtain

$$\begin{cases} \dot{\hat{\phi}}(x) = \hat{\pi}^\dagger(x) \\ \dot{\hat{\phi}}^\dagger(x) = \hat{\pi}(x) \\ \dot{\hat{\pi}}(x) = \nabla^2\hat{\phi}^\dagger(x) - m^2\hat{\phi}^\dagger(x) \\ \dot{\hat{\pi}}^\dagger(x) = \nabla^2\hat{\phi}(x) - m^2\hat{\phi}(x), \end{cases}$$

[2] The last equality is meaningful only under integration over space.

which is equivalent to the KGE and its adjoint

$$\left(\Box + m^2\right)\hat{\phi}(x) = 0$$
$$\left(\Box + m^2\right)\hat{\phi}^\dagger(x) = 0.$$

b) Let us expand the solution in the basis of plane waves

$$\hat{\phi}(x) = \int d^3p\, N_p\, \hat{a}_{\boldsymbol{p}}(t) e^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

where N_p is a (real) normalization factor, which depends only on the modulus of $\boldsymbol{p}$ (we will see that this assumption is not restrictive). Since $\hat{\phi}$ satisfies the KGE, we find

$$\begin{aligned} 0 &= \int d^3p\, N_p\, \left(\ddot{\hat{a}}_{\boldsymbol{p}}(t) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - (i\boldsymbol{p})^2 \hat{a}_{\boldsymbol{p}}(t) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + m^2 \hat{a}_{\boldsymbol{p}}(t) e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \\ &= \int d^3p\, N_p\, \left(\ddot{\hat{a}}_{\boldsymbol{p}}(t) + p^2 \hat{a}_{\boldsymbol{p}}(t) + m^2 \hat{a}_{\boldsymbol{p}}(t)\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}}, \end{aligned}$$

which leads to

$$\ddot{\hat{a}}_{\boldsymbol{p}} = -\omega_p^2\, \hat{a}_{\boldsymbol{p}} \qquad \omega_p = \sqrt{p^2 + m^2}.$$

The general solution of the equation is

$$\hat{a}_{\boldsymbol{p}}(t) = \hat{a}^{(1)}_{\boldsymbol{p}} e^{-i\omega_p t} + \hat{a}^{(2)}_{\boldsymbol{p}} e^{i\omega_p t}.$$

With the same expansion for $\hat{\phi}^\dagger$

$$\hat{\phi}^\dagger(x) = \int d^3p\, N_p\, \hat{b}_{\boldsymbol{p}}(t) e^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

we find

$$\hat{b}_{\boldsymbol{p}}(t) = \hat{b}^{(1)}_{\boldsymbol{p}} e^{-i\omega_p t} + \hat{b}^{(2)}_{\boldsymbol{p}} e^{i\omega_p t}.$$

Then we find

$$\begin{aligned} \left(\hat{\phi}\,(x)\right)^\dagger &= \int d^3p\, N_p \left((\hat{a}^{(1)}_{\boldsymbol{p}})^\dagger e^{i\omega_p t} + (\hat{a}^{(2)}_{\boldsymbol{p}})^\dagger e^{-i\omega_p t}\right) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \\ &= \int d^3p\, N_p \left(\hat{b}^{(1)}_{\boldsymbol{p}} e^{-i\omega_p t} + \hat{b}^{(2)}_{\boldsymbol{p}} e^{i\omega_p t}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = \hat{\phi}^\dagger(x), \end{aligned}$$

so that $\hat{b}^{(1)}_{\boldsymbol{p}} = (\hat{a}^{(2)}_{-\boldsymbol{p}})^\dagger$ and $\hat{b}^{(2)}_{\boldsymbol{p}} = (\hat{a}^{(1)}_{-\boldsymbol{p}})^\dagger$. Setting $\hat{a}^{(1)}_{\boldsymbol{p}} = \hat{a}_{\boldsymbol{p}}$, $\hat{b}^{(1)}_{\boldsymbol{p}} = \hat{b}_{\boldsymbol{p}}$ and with a substitution $\boldsymbol{p} \to -\boldsymbol{p}$ in the second integral of the expansion, we find

$$\begin{aligned} \hat{\phi}(x) &= \int d^3p\, N_p \left(\hat{a}_{\boldsymbol{p}}\, e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} + \hat{b}^\dagger_{\boldsymbol{p}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\right) \\ \hat{\phi}^\dagger(x) &= \int d^3p\, N_p \left(\hat{b}_{\boldsymbol{p}}\, e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} + \hat{a}^\dagger_{\boldsymbol{p}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\right). \end{aligned}$$

Since $\hat{\pi} = \dot{\hat{\phi}}^\dagger$ and $\hat{\pi}^\dagger = \dot{\hat{\phi}}$, we find

$$\begin{aligned} \hat{\pi}(x) &= -i \int d^3p\, N_p\, \omega_p \left(\hat{b}_{\boldsymbol{p}}\, e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} - \hat{a}^\dagger_{\boldsymbol{p}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\right) \\ \hat{\pi}^\dagger(x) &= -i \int d^3p\, N_p\, \omega_p \left(\hat{a}_{\boldsymbol{p}}\, e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})} - \hat{b}^\dagger_{\boldsymbol{p}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\right). \end{aligned}$$

In order to obtain the commutation relation between the operators $\hat{a}_{\boldsymbol{p}}$, $\hat{b}_{\boldsymbol{p}}$ and their adjoints it is useful to isolate the operators using some Fourier transforms

$$\begin{aligned} N_p\left(\hat{b}^\dagger_{\boldsymbol{p}}\, e^{i\omega_p t} + \hat{a}_{-\boldsymbol{p}}\, e^{-i\omega_p t}\right) &= \int \frac{d^3x}{(2\pi)^3}\, \hat{\phi}(x)\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \\ N_p\left(\hat{a}^\dagger_{\boldsymbol{p}}\, e^{i\omega_p t} + \hat{b}_{-\boldsymbol{p}}\, e^{-i\omega_p t}\right) &= \int \frac{d^3x}{(2\pi)^3}\, \hat{\phi}^\dagger(x)\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \end{aligned}$$

$$iN_p\,\omega_p\Big(\hat{a}^\dagger_{\boldsymbol{p}}\,e^{i\omega_p t} - \hat{b}_{-\boldsymbol{p}}\,e^{-i\omega_p t}\Big) = \int \frac{d^3x}{(2\pi)^3}\,\hat{\pi}(x)\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$
$$iN_p\,\omega_p\Big(\hat{b}^\dagger_{\boldsymbol{p}}\,e^{i\omega_p t} - \hat{a}_{-\boldsymbol{p}}\,e^{-i\omega_p t}\Big) = \int \frac{d^3x}{(2\pi)^3}\,\hat{\pi}^\dagger(x)\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

here we have replaced $\boldsymbol{p} \to -\boldsymbol{p}$ in the $\hat{a}_{\boldsymbol{p}}$ and $\hat{b}_{\boldsymbol{p}}$ terms of the $\hat{\phi}(x)\,,\hat{\phi}^\dagger(x)\,,\hat{\pi}(x)$ and $\hat{\pi}^\dagger(x)$ integrals and used the fact that $\omega_p = \omega_{-p}$. So that

$$2N_p\,\hat{a}^\dagger_{\boldsymbol{p}}\,e^{i\omega_p t} = \int \frac{d^3x}{(2\pi)^3}\,\Big(\hat{\phi}^\dagger(x) + \frac{\hat{\pi}(x)}{i\omega_p}\Big)\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$$
$$2N_p\,\hat{b}^\dagger_{\boldsymbol{p}}\,e^{i\omega_p t} = \int \frac{d^3x}{(2\pi)^3}\,\Big(\hat{\phi}(x) + \frac{\hat{\pi}^\dagger(x)}{i\omega_p}\Big)\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}}.$$

We can finally isolate $\hat{a}^\dagger_{\boldsymbol{p}}\,,\hat{b}^\dagger_{\boldsymbol{p}}$ and by dagging it we can obtain $\hat{a}_{\boldsymbol{p}}$ and $\hat{b}_{\boldsymbol{p}}$

$$\hat{a}^\dagger_{\boldsymbol{p}} = \frac{1}{2N_p}\int \frac{d^3x}{(2\pi)^3}\,\Big(\hat{\phi}^\dagger(x) + \frac{\hat{\pi}(x)}{i\omega_p}\Big)\,e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}$$
$$\hat{a}_{\boldsymbol{p}} = \frac{1}{2N_p}\int \frac{d^3x}{(2\pi)^3}\,\Big(\hat{\phi}(x) - \frac{\hat{\pi}^\dagger(x)}{i\omega_p}\Big)\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}$$
$$\hat{b}^\dagger_{\boldsymbol{p}} = \frac{1}{2N_p}\int \frac{d^3x}{(2\pi)^3}\,\Big(\hat{\phi}(x) + \frac{\hat{\pi}^\dagger(x)}{i\omega_p}\Big)\,e^{-i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}$$
$$\hat{b}_{\boldsymbol{p}} = \frac{1}{2N_p}\int \frac{d^3x}{(2\pi)^3}\,\Big(\hat{\phi}^\dagger(x) - \frac{\hat{\pi}(x)}{i\omega_p}\Big)\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}.$$

We can now compute the commutators, *i.e.*

$$\begin{aligned}[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p'}}] = \frac{1}{4N_pN_{p'}}\int \frac{d^3x}{(2\pi)^3}\int \frac{d^3y}{(2\pi)^3}\,\Big(&[\hat{\phi}(x), \hat{\phi}^\dagger(y)]\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\,e^{-i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})} + \\ &+ \frac{1}{i\omega_{p'}}[\hat{\phi}(x), \hat{\pi}(y)]\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\,e^{-i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})} + \\ &- \frac{1}{i\omega_p}[\hat{\pi}^\dagger(x), \hat{\phi}^\dagger(y)]\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\,e^{-i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})} + \\ &+ \frac{1}{\omega_p\,\omega_{p'}}[\hat{\pi}^\dagger(x), \hat{\pi}(y)]\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\,e^{-i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})}\Big),\end{aligned}$$

using the equal-time commutation relations we find

$$\begin{aligned}[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p'}}] &= \frac{1}{4N_pN_{p'}}\int \frac{d^3x}{(2\pi)^6}\,\Big(\frac{1}{\omega_{p'}}\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\,e^{-i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{x})} + \\ &\qquad + \frac{1}{\omega_p}\,e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\,e^{-i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{x})}\Big) \\ &= \frac{1}{4N_pN_{p'}}\frac{1}{(2\pi)^3}\,\Big(\frac{1}{\omega_p} + \frac{1}{\omega_{p'}}\Big)\,\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \\ &= \frac{1}{N_p^2}\frac{1}{(2\pi)^3}\frac{1}{2\omega_p}\,\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}').\end{aligned}$$

(The equalities are such if integrated over momentum space.) Clearly a wise choice for the normalization factor N_p is

$$N_p = \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_p}},$$

this leads to the result

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p'}}] = \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}').$$

For symmetry we have $[\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'})$.
We can now compute *i.e.* $[\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}]$. So that

$$\begin{aligned}[\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}] = \frac{1}{4N_pN_{p'}} \int \frac{d^3x}{(2\pi)^3} \int \frac{d^3y}{(2\pi)^3} \Big(&[\hat{\phi}(x), \hat{\phi}^\dagger(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})} + \\ &- \frac{1}{i\omega_{p'}}[\hat{\phi}(x), \hat{\pi}(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})} + \\ &- \frac{1}{i\omega_p}[\hat{\pi}^\dagger(x), \hat{\phi}^\dagger(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})} + \\ &- \frac{1}{\omega_p\, \omega_{p'}}[\hat{\pi}^\dagger(x), \hat{\pi}(y)]\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{y})}\Big),\end{aligned}$$

using the equal-time commutation relations we find

$$\begin{aligned}[\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}] &= \frac{1}{4N_pN_{p'}} \int \frac{d^3x}{(2\pi)^6} \Big(-\frac{1}{\omega_{p'}}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{x})} + \\ &\qquad + \frac{1}{\omega_p}\, e^{i(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})}\, e^{i(\omega_{p'} t - \boldsymbol{p'}\cdot\boldsymbol{x})}\Big) \\ &= \frac{1}{4N_pN_{p'}} \frac{1}{(2\pi)^3} \Big(\frac{1}{\omega_p} - \frac{1}{\omega_{p'}} \Big)\, e^{i(\omega_p + \omega_{p'})t}\, \delta^{(3)}(\boldsymbol{p}+\boldsymbol{p'}) = 0,\end{aligned}$$ [2]

for symmetry we have $[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = 0$. With analogous calculation we can find $[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p'}}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p'}}] = [\hat{b}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}] = [\hat{b}^\dagger_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = 0$.

We have finally found the usual harmonic oscillator algebra

$$\begin{aligned}&[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p'}}] = [\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'}) \\ &[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{p'}}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p'}}] = [\hat{b}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}] = [\hat{b}^\dagger_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = 0 \\ &[\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}] = [\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p'}}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p'}}] = 0.\end{aligned}$$

c) Starting from

$$\begin{aligned}\boldsymbol{\nabla}\hat{\phi} &= i \int d^3p\, \boldsymbol{p} \Big(\hat{a}_{\boldsymbol{p}}\, u_{\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}} \Big) \\ \boldsymbol{\nabla}\hat{\phi}^\dagger &= i \int d^3p\, \boldsymbol{p} \Big(\hat{b}_{\boldsymbol{p}}\, u_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}} \Big),\end{aligned}$$

the Hamiltonian will be

$$\begin{aligned}\hat{H} &= \int d^3x \left(\hat{\pi}^\dagger \hat{\pi} + \boldsymbol{\nabla}\hat{\phi}^\dagger \cdot \boldsymbol{\nabla}\hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi} \right) \\ &= \int d^3x\, d^3p\, d^3p' \Big(-\omega_p \omega_{p'} \Big(\hat{a}_{\boldsymbol{p'}}\, u_{\boldsymbol{p'}}(x) - \hat{b}^\dagger_{\boldsymbol{p'}}\, u^*_{\boldsymbol{p'}}(x) \Big) \Big(\hat{b}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x) - \hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x) \Big) \\ &\qquad - \boldsymbol{p}\cdot\boldsymbol{p'} \Big(\hat{b}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x) - \hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x) \Big) \Big(\hat{a}_{\boldsymbol{p'}}\, u_{\boldsymbol{p'}}(x) - \hat{b}^\dagger_{\boldsymbol{p'}}\, u^*_{\boldsymbol{p'}}(x) \Big) \\ &\qquad + m^2 \Big(\hat{b}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x) + \hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x) \Big) \Big(\hat{a}_{\boldsymbol{p'}}\, u_{\boldsymbol{p'}}(x) + \hat{b}^\dagger_{\boldsymbol{p'}}\, u^*_{\boldsymbol{p'}}(x) \Big) \Big).\end{aligned}$$

The integration over space can be done by using the relations

$$\begin{aligned}\int d^3x\, u_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p'}}(x) &= \frac{1}{2\sqrt{\omega_p \omega_{p'}}} \int \frac{d^3x}{(2\pi)^3} e^{-i(p-p')\cdot x} \\ &= \frac{1}{2\sqrt{\omega_p \omega_{p'}}} e^{-i(\omega_p - \omega_{p'})t}\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'}) \\ &= \frac{1}{2\omega_p} \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'}),\end{aligned}$$

$$\begin{aligned}\int d^3x\, u_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x) &= \frac{1}{2\sqrt{\omega_p\omega_{p'}}}\int \frac{d^3x}{(2\pi)^3}e^{-i(p+p')\cdot x}\\ &= \frac{1}{2\sqrt{\omega_p\omega_{p'}}}e^{-i(\omega_p+\omega_{p'})t}\,\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}')\\ &= \frac{1}{2\omega_p}e^{-2i\omega_p t}\,\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}'),\end{aligned}$$

(The equalities are meaningful under integration over momentum space.) and their adjoints. We then have (using the Dirac deltas to kill one of the integrals)

$$\begin{aligned}\hat{H} = \int d^3p\Big(&-\omega_p^2\,\frac{1}{2\omega_p}\Big(\hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t} - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t}\Big)\\ &- p^2\,\frac{1}{2\omega_p}\Big(-\hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t} - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t}\Big)\\ &+ m^2\,\frac{1}{2\omega_p}\Big(\hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t} + \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t}\Big)\Big)\\ = \int d^3p\frac{1}{2\omega_p}\Big(&(-\omega_p^2 + p^2 + m^2)(\hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p t} + \hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p t})\\ &+ (\omega_p^2 + p^2 + m^2)(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}})\Big).\end{aligned}$$

Using the dispersion relation, we find

$$\hat{H} = \int d^3p\,\omega_p\,\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}}\right).$$

Using the commutation relation, we obtain

$$\hat{H} = \int d^3p\,\omega_p\,\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + \delta^{(3)}(0)\right).$$

The Hamiltonian is positive definite (as it can be seen from the first expression), but the expectation value on every state is infinite. In fact the contribution to the energy is given by ω_p for every type-a and type-b of particle of momentum $\boldsymbol{p}$ via the operators $\hat{n}_{\boldsymbol{p}} = \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}$ and $\hat{\bar{n}}_{\boldsymbol{p}} = \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}$ respectively and by the zero-point energy, which is independent of the occupation number. This can be clearly seen by discretising the momentum space:

$$\hat{H} = \sum_{\boldsymbol{p}_i}\omega_{p_i}\left(\hat{n}_{\boldsymbol{p}_i} + \hat{\bar{n}}_{\boldsymbol{p}_i} + 1\right)$$

and the zero-point energy $E_0 = \sum_{\boldsymbol{p}_i}\omega_{p_i}$ is divergent.

The problem can be solved by introducing the normal ordering. Since the difficulty is originated by the terms $\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}$ and $\hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}}$ in the various products, for every operator we separate the contribution given by the positive frequencies $e^{-i\omega_p t}$ and the negative frequencies $e^{i\omega_p t}$

$$\hat{\alpha}(x) = \hat{\alpha}^{(+)}(x) + \hat{\alpha}^{(-)}(x), \qquad \hat{\beta}(x) = \hat{\beta}^{(+)}(x) + \hat{\beta}^{(-)}(x)$$

and define the normal ordered product to be

$$:\hat{\alpha}\hat{\beta}: = \hat{\alpha}^{(+)}\hat{\beta}^{(+)} + \hat{\alpha}^{(-)}\hat{\beta}^{(+)} \pm \hat{\beta}^{(-)}\hat{\alpha}^{(+)} + \hat{\alpha}^{(-)}\hat{\beta}^{(-)},$$

where the sign is $+$ if the fields are bosonic and $-$ if the fields are fermionic. Note that the negative frequencies contributions are moved to the left by hand. With this prescription, the Hamiltonian becomes

$$:\hat{H}: = \int d^3p\,\omega_p\,:\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}}\right): = \int d^3p\,\omega_p\,\left(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}\right),$$

that is the previous Hamiltonian with the zero-point energy removed.

d) Since the Lagrangian density of the charged KG field is invariant under translations, we have the conserved classical 4-momentum

$$P_\mu = \int d^3x \Big(\pi(x)\partial_\mu \phi(x) + \pi^*(x)\partial_\mu \phi^*(x) - \eta_{0\mu}\mathcal{L}(x) \Big).$$

In particular, we can define the momentum operator to be the symmetrized version of $\boldsymbol{P}$:

$$\hat{\boldsymbol{P}} = \underbrace{-\frac{1}{2}\int d^3x \big(\hat{\pi}\boldsymbol{\nabla}\hat{\phi} + \hat{\pi}^\dagger \boldsymbol{\nabla}\hat{\phi}^\dagger\big)}_{=\hat{\boldsymbol{T}}} \underbrace{-\frac{1}{2}\int d^3x \big(\boldsymbol{\nabla}\hat{\phi}\hat{\pi} + \boldsymbol{\nabla}\hat{\phi}^\dagger \hat{\pi}^\dagger\big)}_{=\hat{\boldsymbol{T}}^\dagger} .$$

In this way, $\hat{\boldsymbol{P}}$ is hermitian. Expanding the fields in plane waves, we find

$$\begin{aligned}
\hat{\boldsymbol{T}} = -\frac{1}{2}\int d^3x\, d^3p\, d^3p' \Big(& \omega_p \boldsymbol{p}' \Big(\hat{b}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x) - \hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x)\Big)\Big(\hat{a}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}'}(x) - \hat{b}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}'}(x)\Big) \\
& + \omega_p \boldsymbol{p}' \Big(\hat{a}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x) - \hat{b}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x)\Big)\Big(\hat{b}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}'}(x) - \hat{a}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}'}(x)\Big)\Big) \\
= -\frac{1}{2}\int d^3p\, d^3p'\, \omega_p \boldsymbol{p}' \Big(& \hat{b}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x) u_{\boldsymbol{p}'}(x) - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p}'}(x) \\
& - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x) u_{\boldsymbol{p}'}(x) + \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p}'}(x) \\
& + \hat{a}_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x) u_{\boldsymbol{p}'}(x) - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p}'}(x) \\
& - \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x) u_{\boldsymbol{p}'}(x) + \hat{b}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p}'}(x)\Big) \\
= -\frac{1}{2}\int d^3p\, \omega_p \boldsymbol{p} \frac{1}{2\omega_p}\Big(& -\hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} e^{-2i\omega_p t} - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}} e^{2i\omega_p t} \\
& - \hat{a}_{\boldsymbol{p}}\hat{b}_{-\boldsymbol{p}} e^{-2i\omega_p t} - \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} - \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}} e^{2i\omega_p t}\Big) \\
= \frac{1}{4}\int d^3p\, \boldsymbol{p}\, & \big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}\big) \\
& + \frac{1}{4}\int d^3p\, \boldsymbol{p}\, \big((\hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{b}_{-\boldsymbol{p}}) e^{-2i\omega_p t} + (\hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}}) e^{2i\omega_p t}\big).
\end{aligned}$$

Taking the adjoint and summing the two terms, we find

$$\hat{\boldsymbol{P}} = \frac{1}{2}\int d^3p\, \boldsymbol{p}\, \big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} + \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}}\big),$$

since the mixed terms cancelled due to symmetry reasons. In this case, the vacuum momentum does not give any problem, since the integral is zero, again for symmetry reasons. Thus we obtain

$$\hat{\boldsymbol{P}} = \int d^3p\, \boldsymbol{p}\, \big(\hat{n}_{\boldsymbol{p}} + \hat{\bar{n}}_{\boldsymbol{p}}\big) ,$$

that is, the contribution to the total momentum is given by $\boldsymbol{p}$ for every type-a and type-b of particle via the operators $\hat{n}_{\boldsymbol{p}}$ and $\hat{\bar{n}}_{\boldsymbol{p}}$ respectively. In this particular case the normal ordering prescription does not make any difference:

$$:\hat{\boldsymbol{P}}: = \hat{\boldsymbol{P}}.$$

This is a consequence of space isotropy. The same argument does not apply to the energy, since it is always positive: $\omega_p \geq 0$. It can be easily seen that the momentum operator is still a conserved quantity, since $[\hat{\boldsymbol{P}}, \hat{H}] = 0$ by applying the commutation relations among the creation and annihilation operators.

e) We can see that the Lagrangian density of the charged KG field is invariant under internal phase rotations:

$$\begin{aligned}
\phi &\longrightarrow \phi' = e^{i\alpha}\phi \\
\phi^* &\longrightarrow \phi'^* = e^{-i\alpha}\phi^* .
\end{aligned}$$

Thanks to Noether's theorem, we know that there is conserved charge

$$Q \propto \int d^3x\, j_0,$$

where

$$j_\mu = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi^*}\delta\phi^*.$$

Since $\delta\phi = i\alpha\,\phi$ and $\delta\phi^* = -i\alpha\,\phi^*$, we find

$$Q = -ie\int d^3x\,\left(\pi\phi - \pi^*\phi^*\right),$$

where e is a real constant. The corresponding operator is the symmetrized version of Q

$$\hat{Q} = \underbrace{-\frac{ie}{2}\int d^3x\,\left(\hat{\pi}\hat{\phi} - \hat{\pi}^\dagger\hat{\phi}^\dagger\right)}_{=\hat{q}}\underbrace{-\frac{ie}{2}\int d^3x\,\left(\hat{\phi}\,\hat{\pi} - \hat{\phi}^\dagger\hat{\pi}^\dagger\right)}_{=\hat{q}^\dagger}.$$

Expanding the fields in plane waves, we find

$$\begin{aligned}
\hat{q} = &-\frac{ie}{2}\int d^3x\,d^3p\,d^3p'\Big(-i\omega_p\Big(\hat{b}_{\boldsymbol{p}}u_{\boldsymbol{p}}(x) - \hat{a}^\dagger_{\boldsymbol{p}}u^*_{\boldsymbol{p}}(x)\Big)\Big(\hat{a}_{\boldsymbol{p}'}u_{\boldsymbol{p}'}(x) + \hat{b}^\dagger_{\boldsymbol{p}'}u^*_{\boldsymbol{p}'}(x)\Big)\\
&\qquad + i\omega_p\Big(\hat{a}_{\boldsymbol{p}}u_{\boldsymbol{p}}(x) - \hat{b}^\dagger_{\boldsymbol{p}}u^*_{\boldsymbol{p}}(x)\Big)\Big(\hat{b}_{\boldsymbol{p}'}u_{\boldsymbol{p}'}(x) + \hat{a}^\dagger_{\boldsymbol{p}'}u^*_{\boldsymbol{p}'}(x)\Big)\Big)\\
= &\,\frac{e}{2}\int d^3x\,d^3p\,d^3p'\,\omega_p\Big(-\hat{b}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}u_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x) - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}u_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x)\\
&\qquad + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}u^*_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x) + \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}u^*_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x)\\
&\qquad + \hat{a}_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}'}u_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x) + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}u_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x)\\
&\qquad - \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}'}u^*_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(x) - \hat{b}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}u^*_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(x)\Big)\\
= &\,\frac{e}{2}\int d^3p\,\omega_p\frac{1}{2\omega_p}\Big(-\hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}}e^{-2i\omega_p} - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p}\\
&\qquad + \hat{a}_{\boldsymbol{p}}\hat{b}_{-\boldsymbol{p}}e^{-2i\omega_p} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}}e^{2i\omega_p}\Big)\\
= &\,\frac{e}{4}\int d^3p\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}}\Big)\\
&\qquad + \frac{e}{4}\int d^3p\Big((\hat{a}_{\boldsymbol{p}}\hat{b}_{-\boldsymbol{p}} - \hat{b}_{\boldsymbol{p}}\hat{a}_{-\boldsymbol{p}})e^{-2i\omega_p} + (\hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{-\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{-\boldsymbol{p}})e^{2i\omega_p}\Big).
\end{aligned}$$

Taking the adjoint and summing the two terms, we find

$$\hat{Q} = \frac{e}{2}\int d^3p\Big(\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}} + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}} - \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} - \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}}\Big),$$

since the mixed terms cancelled due to symmetry reasons. Now, applying the commutation relation, we find that the vacuum charge contains two infinite contributions. We can remove this problem with the normal ordering prescription:

$$:\hat{Q}: = e\int d^3p\Big(\hat{n}_{\boldsymbol{p}} - \hat{\bar{n}}_{\boldsymbol{p}}\Big).$$

It can be easily seen that the charge operator is still a conserved quantity, since $[\hat{Q}, \hat{H}] = 0$, by applying the commutation relations among the creation and annihilation operators.

Since we have an harmonic oscillator algebra, we can construct a Fock space with the eigenstates of the number operators $\hat{n}_{\boldsymbol{p}}$ and $\hat{\bar{n}}_{\boldsymbol{p}}$. Then, $\hat{a}^\dagger_{\boldsymbol{p}}$ and $\hat{a}_{\boldsymbol{p}}$ represent a creation and annihilation operator respectively for particle with definite momentum $\boldsymbol{p}$ and charge e, while $\hat{b}^\dagger_{\boldsymbol{p}}$ and $\hat{b}_{\boldsymbol{p}}$ represent a creation and annihilation operator respectively for particle with the same momentum $\boldsymbol{p}$, but opposite charge $-e$. The first are called particles, the second anti-aprticles.

9 Commutation relations, propagators, microcausality and spin-statistics theorem

a) Consider the free charged Klein-Gordon field $\hat{\phi}(x)$, and the commutation relations $[\hat{\phi}(x), \hat{\phi}^\dagger(y)] = i\Delta(x-y)$. Show that

$$\Delta(x-y) = -\int \frac{d^3p}{(2\pi)^3} \frac{\sin p(x-y)}{\omega_p}.$$

and that $\Delta(x-y) = 0$ for $(x-y)^2 < 0$.

b) Starting from the previous result, state what the *microcausality* condition is, and prove it for the present case.

c) State the *spin-statistics* theorem, and prove it for the present case.

d) Consider the *Feynman* propagator $i\Delta_F(x-y) = \langle 0|T(\hat{\phi}(x)\hat{\phi}^\dagger(y))|0\rangle$, and prove that it can be written as follows:

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{1}{p^2 - m^2 + i\epsilon}$$

e) What is the difference between a commutation function and a propagation function?

---

a) Using the expansion

$$\hat{\phi}(x) = \int d^3p \left(\hat{a}_{\boldsymbol{p}}\, u_{\boldsymbol{p}}(x) + \hat{b}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x)\right),$$

$$u_{\boldsymbol{p}}(x) = \frac{1}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{2\omega_p}} e^{-ipx},$$

the Pauli-Jordan function $\Delta(x-y)$ will be

$$\begin{aligned}
\Delta(x-y) = -i \int d^3p\, d^3p' \Big(&[\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p}'}] u_{\boldsymbol{p}}(x) u_{\boldsymbol{p}'}(y) + [\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] u_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p}'}(y) + \\
&+ [\hat{b}^\dagger_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p}'}] u^*_{\boldsymbol{p}}(x) u_{\boldsymbol{p}'}(y) + [\hat{b}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] u^*_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p}'}(y) \Big) \\
= -i \int d^3p\, &(u_{\boldsymbol{p}}(x) u^*_{\boldsymbol{p}}(y) - u^*_{\boldsymbol{p}}(x) u_{\boldsymbol{p}}(y)) \\
= -i \int \frac{d^3p}{(2\pi)^3} &\frac{1}{2\omega_p} \left(e^{-ip(x-y)} - e^{ip(x-y)}\right) \\
= -\int \frac{d^3p}{(2\pi)^3} &\frac{\sin p(x-y)}{\omega_p},
\end{aligned}$$

where we have used the commutation relations

$$[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{p}'}] = [\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p}'}] = \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}'),$$

$$[\hat{a}_{\boldsymbol{p}}, \hat{b}_{\boldsymbol{p}'}] = [\hat{a}^\dagger_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{p}'}] = 0.$$

Let us write now the function in an explicit covariant form. Setting $z = x - y$, we will have

$$\begin{aligned}
\Delta(z) &= -i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{-ipz} - e^{ipz}\right) \\
&= -i \int \frac{d^4p}{(2\pi)^3} \frac{1}{2\omega_p} \Big(\delta(p^0 - \omega_p) - \delta(p^0 + \omega_p)\Big) e^{-ipz} \\
&= -i \int \frac{d^4p}{(2\pi)^3} \frac{1}{2\omega_p} \epsilon(p^0) \Big(\delta(p^0 - \omega_p) + \delta(p^0 + \omega_p)\Big) e^{-ipz}
\end{aligned}$$

where

$$\epsilon(p^0) = \begin{cases} +1 & \text{if } p^0 > 0 \\ -1 & \text{if } p^0 < 0. \end{cases}$$

Using the fact that

$$\delta\big((p^0)^2 - \omega_p^2\big) = \frac{1}{2\omega_p}\Big(\delta(p^0 - \omega_p) + \delta(p^0 + \omega_p)\Big),$$

we obtain

$$\begin{aligned}\Delta(z) &= -i \int \frac{d^4p}{(2\pi)^3}\, \epsilon(p^0)\, \delta\big((p^0)^2 - \omega_p^2\big) e^{-ipz} \\ &= -i \int \frac{d^4p}{(2\pi)^3}\, \epsilon(p^0)\, \delta\big(p^2 - m^2\big) e^{-ipz},\end{aligned}$$

where we have used the dispersion relation $\omega_p^2 = |\boldsymbol{p}|^2 + m^2$. Now the expression is manifestly invariant under proper Lorentz transformations. In fact, if

$$z \longrightarrow z' = \Lambda z \qquad \Lambda \text{ a orthochronous Lorentz transformation,}$$

we can perform the same change of variable in the integral ($p \longrightarrow p' = \Lambda p$). Then $\epsilon(p^0) = \epsilon(p'^0)$ since Λ is orthochronous, $p^2 = p'^2$, $pz = p'z'$ and finally the measure does not change, since $|\det \Lambda| = 1$.

Now,

$$\Delta(0, \boldsymbol{z}) = -\int \frac{d^3p}{(2\pi)^3} \frac{\sin \boldsymbol{p}\cdot\boldsymbol{z}}{\omega_p} = 0,$$

since the integrand is odd. Thus, if $(x-y)^2 < 0$, we can find a proper Lorentz transformation $x - y \to z$ with $z^0 = 0$, *i.e.* in the new frame of reference the two events are simultaneous. Then, thanks to Lorentz invariance,

$$\Delta(x-y) = \Delta(0, \boldsymbol{z}) = 0.$$

In addition, with the covariant expression for Δ, we can see that it is a Green function for the Klein-Gordon (KG) equation with no source. In fact,

$$(\Box_x + m^2)\Delta(x-y) = -i \int \frac{d^4p}{(2\pi)^3}\, \epsilon(p^0)\, \delta\big(p^2 - m^2\big)\, (-p^2 + m^2)\, e^{-ip(x-y)} = 0.$$

b) Every observable can be written as

$$\hat{O}(x) = \hat{\phi}^\dagger(x) O(x) \hat{\phi}(x).$$

Thus, we have

$$\begin{aligned}[\hat{O}(x), \hat{O}(y)] &= O(x)O(y)[\hat{\phi}^\dagger(x)\hat{\phi}(x), \hat{\phi}^\dagger(y)\hat{\phi}(y)] \\ &= O(x)O(y)\left(\hat{\phi}^\dagger(x)[\hat{\phi}(x), \hat{\phi}^\dagger(y)\hat{\phi}(y)] + [\hat{\phi}^\dagger(x), \hat{\phi}^\dagger(y)\hat{\phi}(y)]\hat{\phi}(x)\right) \\ &= O(x)O(y)\Big(\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)[\hat{\phi}(x), \hat{\phi}(y)] + \hat{\phi}^\dagger(x)[\hat{\phi}(x), \hat{\phi}^\dagger(y)]\hat{\phi}(y) \\ &\qquad + \hat{\phi}^\dagger(y)[\hat{\phi}^\dagger(x), \hat{\phi}(y)]\hat{\phi}(x) + [\hat{\phi}^\dagger(x), \hat{\phi}^\dagger(y)]\hat{\phi}(y)\hat{\phi}(x)\Big) \\ &= O(x)O(y)\left(i\Delta(x-y)\hat{\phi}^\dagger(x)\hat{\phi}(y) - i\Delta(y-x)\hat{\phi}^\dagger(y)\hat{\phi}(x)\right) \\ &= O(x)O(y)\left(\hat{\phi}^\dagger(x)\hat{\phi}(y) + \hat{\phi}^\dagger(y)\hat{\phi}(x)\right) i\Delta(x-y).\end{aligned}$$

So if $(x-y)^2 < 0$, we have that $[\hat{O}(x), \hat{O}(y)] = 0$, *i.e.* measurements of $\hat{O}$ in points with a space-like separation do not interfere with each other. This is the so-called microcausality principle.

c) The spin-statistics theorem states that, in the assumptions of

- Lorentz invariance,
- Microcausality
- Positive-definite energy,

then every boson has integer spin, while every fermion has semi-integer spin. In our case, quantizing the KG field by following the Fermi rules contradicts the microcausality principle. In fact, the propagator becomes

$$i\Delta_1(x-y) = \{\hat{\phi}(x), \hat{\phi}^\dagger(y)\} = \int \frac{d^3p}{(2\pi)^3}\frac{\cos p(x-y)}{\omega_p},$$

since in the previous calculations

$$\{\hat{b}_{\boldsymbol{p}'}, \hat{b}^\dagger_{\boldsymbol{p}}\} = \{\hat{a}_{\boldsymbol{p}'}, \hat{a}^\dagger_{\boldsymbol{p}}\} = \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}').$$

In this case the integrand of $\Delta_1(0, \boldsymbol{z})$ is not an odd function. In particular, $\Delta_1(0, \boldsymbol{z}) \neq 0$. Then, using the relation

$$[\hat{A}, \hat{B}\hat{C}] = \{\hat{A}, \hat{B}\}\hat{C} - \hat{B}\{\hat{A}, \hat{C}\},$$

we obtain

$$\begin{aligned}
[\hat{O}(x), \hat{O}(y)] &= O(x)O(y)[\hat{\phi}^\dagger(x)\hat{\phi}(x), \hat{\phi}^\dagger(y)\hat{\phi}(y)] \\
&= O(x)O(y)\left(\hat{\phi}^\dagger(x)[\hat{\phi}(x), \hat{\phi}^\dagger(y)\hat{\phi}(y)] + [\hat{\phi}^\dagger(x), \hat{\phi}^\dagger(y)\hat{\phi}(y)]\hat{\phi}(x)\right) \\
&= O(x)O(y)\Big(\hat{\phi}^\dagger(x)\{\hat{\phi}(x), \hat{\phi}^\dagger(y)\}\hat{\phi}(y) - \hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)\{\hat{\phi}(x), \hat{\phi}(y)\} \\
&\qquad + \{\hat{\phi}^\dagger(x), \hat{\phi}^\dagger(y)\}\hat{\phi}(y)\hat{\phi}(x) - \hat{\phi}^\dagger(y)\{\hat{\phi}^\dagger(x), \hat{\phi}(y)\}\hat{\phi}(x)\Big) \\
&= O(x)O(y)\left(i\Delta_1(x-y)\hat{\phi}^\dagger(x)\hat{\phi}(y) - i\Delta_1(y-x)\hat{\phi}^\dagger(y)\hat{\phi}(x)\right) \\
&= O(x)O(y)\left(\hat{\phi}^\dagger(x)\hat{\phi}(y) - \hat{\phi}^\dagger(y)\hat{\phi}(x)\right) i\Delta_1(x-y),
\end{aligned}$$

which in general is not zero for space-like separated points. Thus, microcausality is violated, which means that the KG field, which describes 0-spin particles, cannot describe fermions.

d) The Feynman propagator is defined as

$$\Delta_F(x-y) = -i\langle 0|T\left(\hat{\phi}(x)\hat{\phi}^\dagger(y)\right)|0\rangle,$$

where $T(\,\cdot\,)$ is the time-ordered product[3]

$$T\left(\hat{\phi}(x)\hat{\phi}^\dagger(y)\right) = \Theta(x^0-y^0)\hat{\phi}(x)\hat{\phi}^\dagger(y) + \Theta(y^0-x^0)\hat{\phi}^\dagger(y)\hat{\phi}(x).$$

Let us write the Feynman propagator in manifestly covariant form. Plugging the expansion of $\hat{\phi}$ into the definition of Δ_F, we obtain, for $x^0 > y^0$,

$$\begin{aligned}
i\Delta_F(x-y) = \int d^3p\, d^3p' \langle 0|&\hat{a}_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(y) + \hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}\, u_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(y) \\
&\hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}'}\, u^*_{\boldsymbol{p}}(x)u_{\boldsymbol{p}'}(y) + \hat{b}^\dagger_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}\, u^*_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(y)|0\rangle.
\end{aligned}$$

Using the fact that

$$\hat{b}_{\boldsymbol{p}'}|0\rangle = \langle 0|\,\hat{b}^\dagger_{\boldsymbol{p}} = 0,$$

[3] The plus sign is repalced by a minus sign in case of Fermi particles.

we obtain

$$\begin{aligned} i\Delta_F(x-y) &= \int d^3p\, d^3p'\, \langle 0|\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}|0\rangle\ u_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}'}(y) \\ &= \int d^3p\, u_{\boldsymbol{p}}(x)u^*_{\boldsymbol{p}}(y) \\ &= \frac{1}{(2\pi)^3}\int d^3p\, \frac{1}{2\omega_p}e^{-ip(x-y)}, \end{aligned}$$

where we have used the relation

$$\langle 0|\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}'}|0\rangle = \langle 0|\hat{a}^\dagger_{\boldsymbol{p}'}\hat{a}_{\boldsymbol{p}}|0\rangle + \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')\,\langle 0|0\rangle = \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}').$$

Analogously, if $y^0 > x^0$, we have

$$\begin{aligned} i\Delta_F(x-y) = \int d^3p\, d^3p' \langle 0|\hat{b}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\ u_{\boldsymbol{p}}(y)u_{\boldsymbol{p}'}(x) + \hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}\ u_{\boldsymbol{p}}(y)u^*_{\boldsymbol{p}'}(x) \\ \hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}'}\ u^*_{\boldsymbol{p}}(y)u_{\boldsymbol{p}'}(x) + \hat{a}^\dagger_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}\ u^*_{\boldsymbol{p}}(y)u^*_{\boldsymbol{p}'}(x)|0\rangle. \end{aligned}$$

Using the fact that

$$\hat{a}_{\boldsymbol{p}'}\,|0\rangle = \langle 0|\,\hat{a}^\dagger_{\boldsymbol{p}} = 0,$$

we obtain

$$\begin{aligned} i\Delta_F(x-y) &= \int d^3p\, d^3p'\, \langle 0|\hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}|0\rangle\ u_{\boldsymbol{p}}(y)u^*_{\boldsymbol{p}'}(x) \\ &= \int d^3p\, u_{\boldsymbol{p}}(y)u^*_{\boldsymbol{p}}(x) \\ &= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_p}e^{ip(x-y)}, \end{aligned}$$

where we have used the relation

$$\langle 0|\hat{b}_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}'}|0\rangle = \langle 0|\hat{b}^\dagger_{\boldsymbol{p}'}\hat{b}_{\boldsymbol{p}}|0\rangle + \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')\,\langle 0|0\rangle = \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}').$$

Thus, we have the expression

$$i\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_p}\left(\Theta(x^0-y^0)e^{-ip(x-y)} + \Theta(y^0-x^0)e^{ip(x-y)}\right).$$

With the change of variable $\boldsymbol{p}\to -\boldsymbol{p}$ in the second integral, we obtain

$$i\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_p}\left(\Theta(x^0-y^0)e^{-i\omega_p(x^0-y^0)} + \Theta(y^0-x^0)e^{i\omega_p(x^0-y^0)}\right)e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}.$$

We can write the integral as a four-dimensional integral (in d^4p), by looking at a term inside the integral as a residue. In fact, by setting

$$f(p^0) = -\frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)}$$

we have two cases. If $y^0 > x^0$ we can integrate f along the contour Γ^+ shown in the figure and, using the Residue theorem, we have:

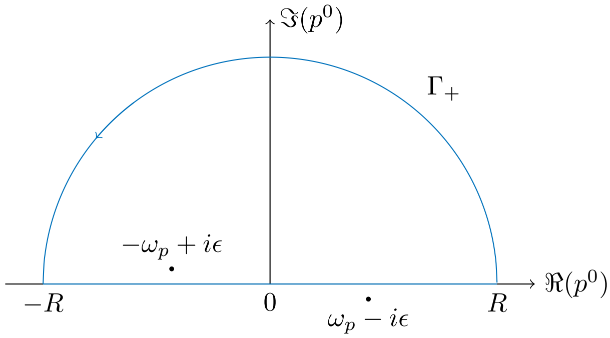
$$\begin{aligned} \frac{1}{2\pi i}\int_{\Gamma^+} dp^0\, f(p^0) &= \mathrm{Res}(-\omega_p+i\epsilon) \\ &= \lim_{p^0\to -\omega_p+i\epsilon}(p^0+\omega_p-i\epsilon)f(p^0) \\ &= \frac{e^{i\omega_p(x^0-y^0)+\epsilon(x^0-y^0)}}{2\omega_p-2i\epsilon} \\ &= \frac{e^{i\omega_p(x^0-y^0)}}{2\omega_p}. \end{aligned}$$

On the other hand, the contribution along the semicircle is zero, since

$$\lim_{|p^0|\to\infty} p^0 f(p^0) = 0.$$

Thus,

$$-\frac{1}{2\pi i}\int dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)} = \Theta(y^0-x^0)\frac{e^{i\omega_p(x^0-y^0)}}{2\omega_p}.$$



With analogous calculations for $x^0 > y^0$, integrating along a contour Γ^- in the lower half plane, we can find

$$-\frac{1}{2\pi i}\int dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)} = \Theta(x^0-y^0)\frac{e^{-i\omega_p(x^0-y^0)}}{2\omega_p}.$$

On the other hand,

$$\begin{aligned} f(p^0) &= -\frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)} \\ &= -\frac{e^{-ip^0(x^0-y^0)}}{(p^0)^2-\omega_p^2+2i\omega_p\epsilon+\epsilon^2} \\ &= -\frac{e^{-ip^0(x^0-y^0)}}{p^2-m^2+2i\omega_p\epsilon} \end{aligned}$$

Writing ϵ instead of $2\omega_p\epsilon$, we have

$$\begin{aligned} \Delta_F(x-y) &= -i\int\frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\frac{-e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &= \int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon}. \end{aligned}$$

With this expression, it can be shown that the Feynman propagator is a Green function for the Klein-Gordon equation (KGE) with a Dirac delta as a source term:

$$\begin{aligned} (\Box_x+m^2)\Delta_F(x-y) &= \int\frac{d^4p}{(2\pi)^4}\left(-p^2+m^2\right)\frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon} \\ &= -\delta^{(4)}(x-y). \end{aligned}$$

e) The Pauli-Jordan function Δ and the anticommutator function Δ_1 are examples of commutator functions, while the Feynman propagator Δ_F belongs to the class of propagator functions. Other commutator functions are

$$\Delta^+(z) = -i\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ipz} \qquad \text{positive frequency function}$$

$$\Delta^-(z) = i\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{ipz} \qquad \text{negative frequency function,}$$

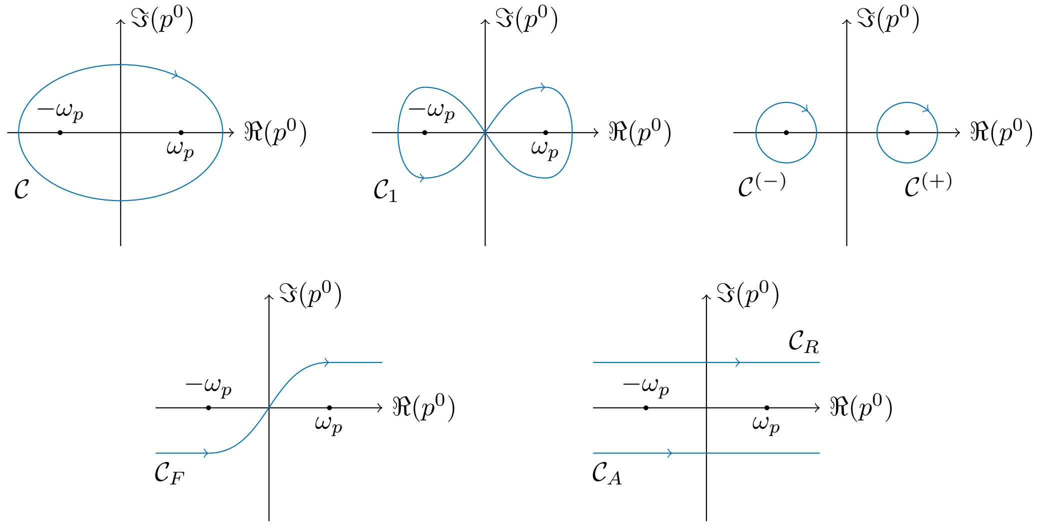
while other propagator functions are

$$\Delta_R(z) = \Theta(z^0)\Delta(z) \qquad \text{retarded propagator}$$
$$\Delta_A(z) = -\Theta(-z^0)\Delta(z) \qquad \text{advanced propagator.}$$

The main difference between the two classes of functions is that the former are solutions of the homogeneous KGE, as we have seen for Δ, while the latter are solutions of the inhomogeneous KGE, as we have seen for Δ_F:

$$(\Box_x + m^2)\Delta(x-y) = 0$$
$$(\Box_x + m^2)\Delta_F(x-y) = -\delta^{(4)}(x-y).$$

Furthermore, all these functions can be expressed as contour integrals in the complex p^0-plane. The difference between them is that the contours for the commutator functions are closed and limited, while those of the propagators extend to the infinity.



10 Quantization of the Dirac field

a) Write down the Lagrangian for the free Dirac field $\psi(x)$, with all constants explicitly expressed. Then set $\hbar = c = 1$ for the subsequent calculations. Compute the conjugate momenta and the Hamiltonian density $\mathcal{H}(x)$. Quantize the fields following the prescription of canonical quantization. Write down the Heisenberg equations of motion for the fields and their conjugate momenta and show that they reproduce the Dirac equation.

b) Given the plane wave solutions of the Dirac equation:

$$\psi_{\boldsymbol{p}}^{(r)}(\boldsymbol{x},t) = \frac{1}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}w_r(\boldsymbol{p})e^{-i\epsilon_r(\omega_p t - \boldsymbol{p}\cdot\boldsymbol{x})},$$

with all terms as defined during classes, write down the plane wave expansion of the quantized field $\hat{\psi}(x)$, by introducing the operators $\hat{a}(\mathbf{p},r)$ and their adjoints. Derive the anticommutation relations between $\hat{a}(\mathbf{p},r)$ and their adjoints, starting from those of $\hat{\psi}(x)$ and its conjugate.

c) Starting from the quantized Hamiltonian $\hat{H}$ expressed in terms of the fields $\hat{\psi}(x)$ and its conjugate, and using the plane wave decomposition, derive the corresponding expression in terms of $\hat{a}(\boldsymbol{p},r)$ and their adjoints. Discuss the problem with the expression thus derived, and how the Dirac-sea's picture solves the problem. In accordance with the Dirac-sea's picture, re-define the operators and spinors such that the field decomposition takes the form:

$$\hat{\psi}(\boldsymbol{x},t) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}\left(\hat{b}(\boldsymbol{p},s)u(\boldsymbol{p},s)e^{-ip\cdot x} + \hat{d}^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s)e^{+ip\cdot x}\right)$$

d) Prove the spin-statistics theorem for the Dirac field.

e) Introduce the vacuum state $|0\rangle$ and, starting from it, construct the Fock space for the quantized Dirac field. Explain the role of the operators $\hat{b}(p,s)$, $\hat{d}(p,s)$ and their adjoints.

---

a) The Lagrangian density for the Dirac field is

$$\begin{aligned}\mathcal{L}(x) &= \bar{\psi}(i\hbar c\,\gamma^\mu\partial_\mu - mc^2)\psi \\ &= i\hbar c\,\psi^\dagger\dot{\psi} + i\hbar c\,\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi - mc^2\,\psi^\dagger\beta\psi.\end{aligned}$$

We set $\hbar = c = 1$. The conjugate momenta are

$$\begin{aligned}\pi_\psi &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger \\ \pi_{\psi^\dagger} &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}^\dagger} = 0.\end{aligned}$$

Setting $\pi_\psi = \pi$, we get the Hamiltonian density

$$\mathcal{H}(x) = \dot{\psi}\pi - \mathcal{L}(x) = -\pi\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + im\beta)\,\psi.$$

Let us quantize the fields using the anticommutation relations.

$$\begin{aligned}\psi, \pi &\longrightarrow \hat{\psi}, \hat{\pi} \\ \{\,\cdot\,,\,\cdot\,\}_{\text{PB}} &\longrightarrow -i\,\{\,\cdot\,,\,\cdot\,\}.\end{aligned}$$

The Heisenberg equations will be

$$\begin{cases}\dot{\hat{\psi}}(x) = -i\,[\hat{\psi}(x), \hat{H}(t)] \\ \dot{\hat{\pi}}(x) = -i\,[\hat{\pi}(x), \hat{H}(t)],\end{cases}$$

where $\hat{H}$ is the total Hamiltonian

$$\hat{H} = \int dx^3\,\hat{\mathcal{H}}(x) = -\int d^3x\,\hat{\pi}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + im\beta)\,\hat{\psi}.$$

Note that the Hamiltonian density (and thus the Hamiltonian) is not an hermitian operator. However, the following results would be the same as those we would obtain by starting with a symmetrized Lagrangian. The commutators for a general component will be (omitting the equal-time dependence and the sum over repeated indices)

$$\begin{aligned}[\hat{\psi}_\lambda(\boldsymbol{x}), \hat{H}] &= -\int d^3x'\Big([\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\pi}_\sigma(\boldsymbol{x}')\boldsymbol{\alpha}_{\sigma\rho}\cdot\boldsymbol{\nabla}'\hat{\psi}_\rho(\boldsymbol{x}')] + im[\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\pi}_\sigma(\boldsymbol{x}')\beta_{\sigma\rho}\hat{\psi}_\rho(\boldsymbol{x}')] \\ &= -\int d^3x'\Big(\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\pi}_\sigma(\boldsymbol{x}')\}\boldsymbol{\alpha}_{\sigma\rho}\cdot\boldsymbol{\nabla}'\hat{\psi}_\rho(\boldsymbol{x}') - \hat{\pi}_\sigma(\boldsymbol{x}')\boldsymbol{\alpha}_{\sigma\rho}\cdot\boldsymbol{\nabla}'\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\psi}_\rho(\boldsymbol{x}')\} \\ &\qquad + im\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\pi}_\sigma(\boldsymbol{x}')\}\beta_{\sigma\rho}\hat{\psi}_\rho(\boldsymbol{x}') - im\,\hat{\pi}_\sigma(\boldsymbol{x}')\beta_{\sigma\rho}\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\psi}_\rho(\boldsymbol{x}')\}\Big).\end{aligned}$$

Here we have used the relation

$$[\hat{A}, \hat{B}\hat{C}] = \{\hat{A}, \hat{B}\}\hat{C} - \hat{B}\{\hat{A}, \hat{C}\}$$

and denoted $\boldsymbol{\alpha}_{\sigma\rho}$ the vector with components $\alpha^i_{\sigma\rho}$. Using the fundamental anticommutation relations

$$\begin{aligned}&\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\pi}_\sigma(\boldsymbol{x}')\} = i\,\delta_{\lambda\sigma}\,\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \\ &\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\psi}_\rho(\boldsymbol{x}')\} = 0,\end{aligned}$$

we find

$$
\begin{aligned}
[\hat{\psi}_\lambda(\boldsymbol{x}), \hat{H}] &= \int d^3x' \Big(-i\,\delta_{\lambda\sigma}\boldsymbol{\alpha}_{\sigma\rho} \cdot \boldsymbol{\nabla}'\hat{\psi}_\rho(\boldsymbol{x}')\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') + m\,\delta_{\lambda\sigma}\beta_{\sigma\rho}\hat{\psi}_\rho(\boldsymbol{x}')\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \Big) \\
&= -i\boldsymbol{\alpha}_{\lambda\rho} \cdot \boldsymbol{\nabla}\hat{\psi}_\rho(\boldsymbol{x}) + m\,\beta_{\lambda\rho}\hat{\psi}_\rho(\boldsymbol{x}),
\end{aligned}
$$

that leads to an equation formally equivalent to the Dirac equation

$$
\dot{\hat{\psi}} = -\boldsymbol{\alpha} \cdot \boldsymbol{\nabla}\hat{\psi} - im\,\beta\hat{\psi}
$$

or in explicitly covariant form

$$
(i\gamma^\mu\partial_\mu - m)\hat{\psi} = 0.
$$

b) Let us expand a general solution in the basis of plane waves

$$
\hat{\psi}(x) = \sum_{r=1}^{4} \int d^3p\,\hat{a}(\boldsymbol{p},r)\psi_{\boldsymbol{p}}^{(r)}(x).
$$

We can obtain the operators $\hat{a}(\boldsymbol{p},r)$ by projecting the state onto the basis:

$$
\begin{aligned}
\langle \psi_{\boldsymbol{p}}^{(r)} | \hat{\psi} \rangle &= \int d^3x\,\psi_{\boldsymbol{p}}^{(r)\dagger}(x)\hat{\psi}(x) \\
&= \sum_{r'=1}^{4} \int d^3p'\,\hat{a}(\boldsymbol{p}',r') \int d^3x\,\psi_{\boldsymbol{p}}^{(r)\dagger}(x)\psi_{\boldsymbol{p}'}^{(r')}(x) \\
&= \sum_{r'=1}^{4} \int d^3p'\,\hat{a}(\boldsymbol{p}',r')\delta_{rr'}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') = \hat{a}(\boldsymbol{p},r).
\end{aligned}
$$

Thus, the anticommutation relations between $\hat{a}(\boldsymbol{p},r)$ and their adjoints will be (omitting equal-time dependence and the sum over repeated indices)

$$
\begin{aligned}
\{\hat{a}(\boldsymbol{p},r), \hat{a}(\boldsymbol{p}',r')\} &= \int d^3x\,d^3x'\,\{\psi_{\boldsymbol{p}\lambda}^{(r)\dagger}(\boldsymbol{x})\hat{\psi}_\lambda(\boldsymbol{x}), \psi_{\boldsymbol{p}'\sigma}^{(r')\dagger}(\boldsymbol{x}')\hat{\psi}_\sigma(\boldsymbol{x}')\} \\
&= \int d^3x\,d^3x'\,\psi_{\boldsymbol{p}\lambda}^{(r)\dagger}(\boldsymbol{x})\psi_{\boldsymbol{p}'\sigma}^{(r')\dagger}(\boldsymbol{x}')\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\psi}_\sigma(\boldsymbol{x}')\} = 0.
\end{aligned}
$$

Similarly, $\{\hat{a}^\dagger(\boldsymbol{p},r), \hat{a}^\dagger(\boldsymbol{p}',r')\} = 0$. Last:

$$
\begin{aligned}
\{\hat{a}(\boldsymbol{p},r), \hat{a}^\dagger(\boldsymbol{p}',r')\} &= \int d^3x\,d^3x'\,\{\psi_{\boldsymbol{p}\lambda}^{(r)\dagger}(\boldsymbol{x})\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\psi}_\sigma^\dagger(\boldsymbol{x}')\psi_{\boldsymbol{p}'\sigma}^{(r')}(\boldsymbol{x}')\} \\
&= \int d^3x\,d^3x'\,\psi_{\boldsymbol{p}\lambda}^{(r)\dagger}(\boldsymbol{x})\psi_{\boldsymbol{p}'\sigma}^{(r')}(\boldsymbol{x}')\{\hat{\psi}_\lambda(\boldsymbol{x}), \hat{\psi}_\sigma^\dagger(\boldsymbol{x}')\} \\
&= \int d^3x\,d^3x'\,\psi_{\boldsymbol{p}\lambda}^{(r)\dagger}(\boldsymbol{x})\psi_{\boldsymbol{p}'\sigma}^{(r')}(\boldsymbol{x}')\,\delta_{\lambda\sigma}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \\
&= \int d^3x\,\psi_{\boldsymbol{p}}^{(r)\dagger}(\boldsymbol{x})\psi_{\boldsymbol{p}'}^{(r')}(\boldsymbol{x}) \\
&= \delta_{rr'}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}').
\end{aligned}
$$

c, e) Since $\hat{\psi}$ solves the Dirac equation, we can write

$$
\begin{aligned}
\hat{H} &= -\int d^3x\,\hat{\pi}\,(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + im\beta)\,\hat{\psi} \\
&= i\int d^3x\,\hat{\psi}^\dagger\dot{\hat{\psi}}.
\end{aligned}
$$

Plugging the plane wave decomposition into the Hamiltonian, we find

$$\begin{aligned}
\hat{H} &= i\int dx^3\, d^3p\, d^3p' \sum_{r,r'=1}^{4} \hat{a}^\dagger(\boldsymbol{p},r)\psi_{\boldsymbol{p}}^{(r)\dagger}(\boldsymbol{x})\hat{a}(\boldsymbol{p}',r')\dot{\psi}_{\boldsymbol{p}'}^{(r')}(\boldsymbol{x}) \\
&= i\int dx^3\, d^3p\, d^3p' \sum_{r,r'=1}^{4} \hat{a}^\dagger(\boldsymbol{p},r)\psi_{\boldsymbol{p}}^{(r)\dagger}(\boldsymbol{x})\hat{a}(\boldsymbol{p}',r')(-i\epsilon_{r'}\omega_{\boldsymbol{p}'})\psi_{\boldsymbol{p}'}^{(r')}(\boldsymbol{x}) \\
&= \sum_{r,r'=1}^{4}\int d^3p\, d^3p'\, \epsilon_{r'}\, \omega_{\boldsymbol{p}'}\, \hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p}',r')\int d^3x\, \psi_{\boldsymbol{p}}^{(r)\dagger}(\boldsymbol{x})\psi_{\boldsymbol{p}'}^{(r')}(\boldsymbol{x}) \\
&= \sum_{r,r'=1}^{4}\int d^3p\, d^3p'\, \epsilon_{r'}\, \omega_{\boldsymbol{p}'}\, \hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p}',r')\, \delta_{rr'}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') \\
&= \sum_{r=1}^{4}\int d^3p\, \epsilon_r\, \omega_p\, \hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r).
\end{aligned}$$

Separating the positive part from the negative one, we get

$$\hat{H} = \sum_{r=1,2}\int d^3p\, \omega_p\, \hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r) - \sum_{r=3,4}\int d^3p\, \omega_p\, \hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r).$$

Such an Hamiltonian is not bounded from below: as the number of particles in the lower continuum ($r = 3, 4$) grows, the expectation value of the Hamiltonian plummets. The issue can be solved with the Dirac sea picture: the vacuum state $|0\rangle$ is defined as the state in which all negative-energy levels are occupied (in accordance with Pauli exclusion principle). In this way,

$$\begin{aligned}
\hat{a}(\boldsymbol{p},r)\,|0\rangle &= 0 \qquad \text{for } r = 1,2 \\
\hat{a}^\dagger(\boldsymbol{p},r)\,|0\rangle &= 0 \qquad \text{for } r = 3,4.
\end{aligned}$$

The zero-point energy obviously diverges:

$$\langle 0|\hat{H}|0\rangle = -\sum_{r=3,4}\int d^3p\, \omega_p\, \langle 0|\hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r)|0\rangle = -\sum_{r=3,4}\int d^3p\, \omega_p\, \delta^{(3)}(\boldsymbol{0}).$$ [4]

Then we redefine the Hamiltonian in such a way that the divergent vacuum energy is removed:

$$\hat{H} \longrightarrow \hat{H} + \sum_{r=3,4}\int d^3p\, \omega_p\, \delta^{(3)}(\boldsymbol{0}).$$

The same result can be achieved with the normal ordering prescription. In this way we find

$$:\hat{H}: = \sum_{r=1,2}\int d^3p\, \omega_p\, \hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r) + \sum_{r=3,4}\int d^3p\, \omega_p\, \hat{a}(\boldsymbol{p},r)\hat{a}^\dagger(\boldsymbol{p},r).$$

and $\langle 0|\hat{H}|0\rangle = 0$. With the Dirac sea picture in mind, we can view

$\hat{a}^\dagger(\boldsymbol{p},r), \hat{a}(\boldsymbol{p},r)$ as creation and annihilation operators respectively of particles for $r = 1,2$

$\hat{a}(\boldsymbol{p},r), \hat{a}^\dagger(\boldsymbol{p},r)$ as creation and annihilation operators respectively of holes in Dirac's sea for $r = 3,4$.

Thus, the holes in the Dirac sea can be interpreted as antiparticles. In this way, the vacuum will be the state in which there are neither particles nor antiparticles. To avoid confusion, we introduce the notation

$$\begin{aligned}
\hat{a}(\boldsymbol{p},1) &= \hat{b}(\boldsymbol{p},+s) \\
\hat{a}(\boldsymbol{p},2) &= \hat{b}(\boldsymbol{p},-s) \\
\hat{a}(\boldsymbol{p},3) &= \hat{d}^\dagger(\boldsymbol{p},-s) \\
\hat{a}(\boldsymbol{p},4) &= \hat{d}^\dagger(\boldsymbol{p},+s),
\end{aligned}$$

[4] Here $\delta^{(3)}(\boldsymbol{0})$ is "morally" one, as we can see by discretising the momentum space.

so that we can view

$\hat{b}^\dagger, \hat{b}$ as creation and annihilation operators respectively of particles
$\hat{d}^\dagger, \hat{d}$ as creation and annihilation operators respectively of antiparticles.

Changing the notation for the spinors too

$$\begin{aligned} w_1(\boldsymbol{p}) &= u(\boldsymbol{p}, +s) \\ w_2(\boldsymbol{p}) &= u(\boldsymbol{p}, -s) \\ w_3(\boldsymbol{p}) &= v(\boldsymbol{p}, -s) \\ w_4(\boldsymbol{p}) &= v(\boldsymbol{p}, +s), \end{aligned}$$

we can rewrite the plane wave decomposition as

$$\hat{\psi}(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}} \sqrt{\frac{m}{\omega_p}} \left(\hat{b}(\boldsymbol{p}, s) u(\boldsymbol{p}, s) e^{-ipx} + \hat{d}^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s) e^{ipx} \right).$$

Thus, the vacuum now is defined by the relations

$$\begin{aligned} \hat{b}(\boldsymbol{p}, s) \left|0\right\rangle &= 0 \\ \hat{d}(\boldsymbol{p}, s) \left|0\right\rangle &= 0 \end{aligned}$$

and the Fock space is constructed from the orthonormal eigenstates $\left|n_1^+ \boldsymbol{p}_1\, n_1^- \boldsymbol{p}_1 \ldots \bar{n}_1^+ \boldsymbol{p}_1\, \bar{n}_1^- \boldsymbol{p}_1 \ldots\right\rangle$ of the number operators for particles and antiparticles

$$\begin{aligned} \hat{n}(\boldsymbol{p}, s) &= \hat{b}^\dagger(\boldsymbol{p}, s) \hat{b}(\boldsymbol{p}, s) \\ \hat{\bar{n}}(\boldsymbol{p}, s) &= \hat{d}^\dagger(\boldsymbol{p}, s) \hat{d}(\boldsymbol{p}, s). \end{aligned}$$

Here, the state $\left|n_1^+ \boldsymbol{p}_1\, n_1^- \boldsymbol{p}_1 \ldots \bar{n}_1^+ \boldsymbol{p}_1\, \bar{n}_1^- \boldsymbol{p}_1 \ldots\right\rangle$ represents a state with n_1^+ particles of momentum $\boldsymbol{p}_1$ spin $\frac{1}{2}$, n_1^- particles of momentum $\boldsymbol{p}_1$ and spin $-\frac{1}{2}$, $\bar{n}_1^+$ antiparticles of momentum $\boldsymbol{p}_1$ and spin $\frac{1}{2}$, $\bar{n}_1^-$ antiparticles of momentum $\boldsymbol{p}_1$ and spin $-\frac{1}{2}$, etcetera.
Moreover, since we have quantized the field following Fermi rules, $n_i^\pm$ can be only 0 or 1, in accordance with Pauli exclusion principle.

We can also define the momentum operator

$$\hat{\boldsymbol{P}} = -\int d^3x\, \boldsymbol{\nabla} \hat{\psi} \hat{\pi},$$

such that

$$\hat{\boldsymbol{P}} \left|n_1^+ \boldsymbol{p}_1\, n_1^- \boldsymbol{p}_1 \ldots \bar{n}_1^+ \boldsymbol{p}_1\, \bar{n}_1^- \boldsymbol{p}_1 \ldots\right\rangle = \underbrace{\sum_i \left[(n_i^+ + n_i^-) \boldsymbol{p_i} + (\bar{n}_i^+ + \bar{n}_i^-) \boldsymbol{p_i} \right]}_{=P} \left|n_1^+ \boldsymbol{p}_1\, n_1^- \boldsymbol{p}_1 \ldots \bar{n}_1^+ \boldsymbol{p}_1\, \bar{n}_1^- \boldsymbol{p}_1 \ldots\right\rangle;$$

the charge operator

$$\hat{Q} = e \int d^3x\, \hat{\psi}^\dagger \hat{\psi}$$

such that

$$\hat{Q} \left|n_1^+ \boldsymbol{p}_1\, n_1^- \boldsymbol{p}_1 \ldots \bar{n}_1^+ \boldsymbol{p}_1\, \bar{n}_1^- \boldsymbol{p}_1 \ldots\right\rangle = \underbrace{e \sum_i \left[(n_i^+ + n_i^-) - (\bar{n}_i^+ + \bar{n}_i^-) \right]}_{=Q} \left|n_1^+ \boldsymbol{p}_1\, n_1^- \boldsymbol{p}_1 \ldots \bar{n}_1^+ \boldsymbol{p}_1\, \bar{n}_1^- \boldsymbol{p}_1 \ldots\right\rangle;$$

and the spin operator along the direction of motion

$$\hat{S} = \frac{1}{2} \int d^3x\, \hat{\psi}^\dagger \boldsymbol{\Sigma} \hat{\psi}$$

such that

$$\hat{S}\,|n_1^+\boldsymbol{p}_1\,n_1^-\boldsymbol{p}_1\ldots\bar{n}_1^+\boldsymbol{p}_1\,\bar{n}_1^-\boldsymbol{p}_1\ldots\rangle = \underbrace{\frac{1}{2}\sum_i\left[(n_i^+ + \bar{n}_i^+) - (n_i^- + \bar{n}_i^-)\right]}_{=S}\,|n_1^+\boldsymbol{p}_1\,n_1^-\boldsymbol{p}_1\ldots\bar{n}_1^+\boldsymbol{p}_1\,\bar{n}_1^-\boldsymbol{p}_1\ldots\rangle\,.$$

d) The spin-statistics theorem states that, under the assumptions

- Lorentz invariance,
- microcausality
- positive-definite energy,

every bosonic particle has integer spin, while every fermionic particle has semi-integer spin. In our case, quantizing the Dirac field by following the Bose rules contradicts the positivity of the Hamiltonian. In fact, all the calculations done for the Hamiltonian in point (c) are still valid, until

$$\hat{H} = \sum_{r=1,2}\int d^3p\,\omega_p\,\hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r) - \sum_{r=3,4}\int d^3p\,\omega_p\,\hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r).$$

With the commutation relations for the operators with $r = 3, 4$ we get

$$\hat{H} = \sum_{r=1,2}\int d^3p\,\omega_p\,\hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r) - \sum_{r=3,4}\int d^3p\,\omega_p\,\hat{a}(\boldsymbol{p},r)\hat{a}^\dagger(\boldsymbol{p},r) + E_0$$

(where E_0 is the zero-point energy). We can normalize the Hamiltonian with

$$\hat{H} \longrightarrow \hat{H} = \sum_{r=1,2}\int d^3p\,\omega_p\,\hat{a}^\dagger(\boldsymbol{p},r)\hat{a}(\boldsymbol{p},r) - \sum_{r=3,4}\int d^3p\,\omega_p\,\hat{a}(\boldsymbol{p},r)\hat{a}^\dagger(\boldsymbol{p},r),$$

but it does not become positive-definite. Thus, positivity of the Hamiltonian is violated, which means that the Dirac field, which describes $\frac{1}{2}$-spin particles, cannot describe bosons.

11 Properties of the quantized Dirac field

a) Discuss the gauge symmetry of the Dirac Lagrangian and derive the conserved charge Q it leads to. Write it in terms of the field $\psi(x)$ and its complex conjugate. From that, derive the quantized expression of $\hat{Q}$ in terms of the operators $\hat{b}(\boldsymbol{p},s)$, $\hat{d}(\boldsymbol{p},s)$ and their adjoints. In the light of this result, comment on the physical meaning of these operators. Discuss the meaning of the expression thus derived, and why one has to introduce the normal ordering prescription (explain what it consists of).

b) Starting from the quantized momentum $\hat{\boldsymbol{P}}$ for the Dirac field expressed in terms of the fields $\hat{\psi}(x)$ and its conjugate, and using the plane wave decomposition, derive the corresponding expression in terms of $\hat{b}(\boldsymbol{p},s)$, $\hat{d}(\boldsymbol{p},s)$ and their adjoints.

c) Consider the *Feynman* propagator $i\Delta_{F\alpha\beta}(x-y) = \langle 0|T(\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y))|0\rangle$, and prove that it can be written as follows:

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)}\,\frac{\gamma\cdot p + m}{p^2 - m^2 + i\epsilon}.$$

d) Show that the anti-commutator $\{\hat{\psi}_\alpha(x), \hat{\bar{\psi}}_\beta(y)\}$ takes the form:

$$\{\hat{\psi}_\alpha(x), \hat{\bar{\psi}}_\beta(y)\} = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_p}\left((\gamma\cdot p + m)_{\alpha\beta}e^{-ip\cdot(x-y)} - (-\gamma\cdot p + m)_{\alpha\beta}e^{+ip\cdot(x-y)}\right),$$

and that it vanishes for space-like separated distances.

e) State what the *microcausality* condition means, and prove it for the Dirac field.

---

a) The Dirac Lagrangian density

$$\mathcal{L}(x) = \bar{\psi}(i\,\gamma^\mu \partial_\mu - m)\psi,$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, is invariant under phase rotation

$$\psi \longrightarrow e^{i\alpha}\psi$$
$$\psi^\dagger \longrightarrow e^{-i\alpha}\psi^\dagger.$$

Since the infinitesimal generators are $\delta\psi = i\alpha\psi$ and $\delta\psi^\dagger = -i\alpha\psi^\dagger$, from Noether theorem we derive the conserved quantity

$$Q \propto \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta\psi + \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\dagger} \delta\psi^\dagger \right) = -\alpha \int d^3x\, \psi^\dagger \psi.$$

We can set

$$Q = e \int d^3x\, \psi^\dagger \psi.$$

The quantized version of Q is

$$\hat{Q} = e \int d^3x\, \hat{\psi}^\dagger \hat{\psi}.$$

Using the plane wave expansion for $\hat{\psi}$

$$\hat{\psi}(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}} \sqrt{\frac{m}{\omega_p}} \left(\hat{b}(\boldsymbol{p}, s) u(\boldsymbol{p}, s) e^{-ipx} + \hat{d}^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s) e^{ipx} \right).$$

we obtain

$$\begin{aligned}
\hat{Q} &= e \int d^3x\, d^3p\, d^3p' \sum_{s,s'} \frac{1}{(2\pi)^3} \frac{m}{\sqrt{\omega_p \omega_{p'}}} \Big(\hat{d}(\boldsymbol{p}, s) v^\dagger(\boldsymbol{p}, s) e^{-ipx} \\
&\quad + \hat{b}^\dagger(\boldsymbol{p}, s) u^\dagger(\boldsymbol{p}, s) e^{ipx} \Big) \Big(\hat{b}(\boldsymbol{p}', s') u(\boldsymbol{p}', s') e^{-ip'x} + \hat{d}^\dagger(\boldsymbol{p}', s') v(\boldsymbol{p}', s') e^{ip'x} \Big) \\
&= e \int d^3x\, d^3p\, d^3p' \sum_{s,s'} \frac{1}{(2\pi)^3} \frac{m}{\sqrt{\omega_p \omega_{p'}}} \Big(\hat{d}(\boldsymbol{p}, s) \hat{b}(\boldsymbol{p}', s') v^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}', s') e^{-i(p+p')x} \\
&\quad + \hat{d}(\boldsymbol{p}, s) \hat{d}^\dagger(\boldsymbol{p}', s') v^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}', s') e^{-i(p-p')x} + \hat{b}^\dagger(\boldsymbol{p}, s) \hat{b}(\boldsymbol{p}', s') u^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}', s') e^{i(p-p')x} \\
&\quad + \hat{b}^\dagger(\boldsymbol{p}, s) \hat{d}^\dagger(\boldsymbol{p}', s') u^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}', s') e^{i(p+p')x} \Big) \\
&= e \int d^3p\, d^3p' \sum_{s,s'} \frac{m}{\sqrt{\omega_p \omega_{p'}}} \Big(\big(\hat{d}(\boldsymbol{p}, s) \hat{b}(\boldsymbol{p}', s') v^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}', s') e^{-i(\omega_p + \omega_{p'})t} \\
&\quad + \hat{b}^\dagger(\boldsymbol{p}, s) \hat{d}^\dagger(\boldsymbol{p}', s') u^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}', s') e^{i(\omega_p + \omega_{p'})t} \big) \delta^{(3)}(\boldsymbol{p} + \boldsymbol{p}') \\
&\quad + \big(\hat{d}(\boldsymbol{p}, s) \hat{d}^\dagger(\boldsymbol{p}', s') v^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}', s') e^{-i(\omega_p - \omega_{p'})t} \\
&\quad + \hat{b}^\dagger(\boldsymbol{p}, s) \hat{b}(\boldsymbol{p}', s') u^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}', s') e^{i(\omega_p - \omega_{p'})t} \big) \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \Big) \\
&= e \int d^3p \sum_{s,s'} \frac{m}{\omega_p} \Big(\hat{d}(\boldsymbol{p}, s) \hat{b}(-\boldsymbol{p}, s') v^\dagger(\boldsymbol{p}, s) u(-\boldsymbol{p}, s') e^{-i2\omega_p t} + \hat{b}^\dagger(\boldsymbol{p}, s) \hat{d}^\dagger(-\boldsymbol{p}, s') u^\dagger(\boldsymbol{p}, s) v(-\boldsymbol{p}, s') e^{i2\omega_p t} \\
&\quad + \hat{d}(\boldsymbol{p}, s) \hat{d}^\dagger(\boldsymbol{p}, s') v^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s') + \hat{b}^\dagger(\boldsymbol{p}, s) \hat{b}(\boldsymbol{p}, s') u^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}, s') \Big).
\end{aligned}$$

Now we can use the orthogonality relations

$$u^\dagger(\boldsymbol{p}, s) u(\boldsymbol{p}, s') = v^\dagger(\boldsymbol{p}, s) v(\boldsymbol{p}, s') = \frac{\omega_p}{m} \delta_{ss'}$$
$$u^\dagger(\boldsymbol{p}, s) v(-\boldsymbol{p}, s') = v^\dagger(\boldsymbol{p}, s) u(-\boldsymbol{p}, s') = 0,$$

so that

$$\hat{Q} = e\int d^3p \sum_{s,s'} \left(\hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s') + \hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p},s')\right)\delta_{ss'}$$
$$= e\sum_s \int d^3p \left(\hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s) + \hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p},s)\right).$$

We can see that, applying the anticommutation relation for $\hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p},s)$, we obtain

$$\hat{Q} = e\sum_s \int d^3p \left(\hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s) - \hat{d}^\dagger(\boldsymbol{p},s)\hat{d}(\boldsymbol{p},s)\right) + e\int d^3p\, \delta^{(3)}(\boldsymbol{0})$$

which has a divergent expectation value on the vacuum:

$$\langle 0|\hat{Q}|0\rangle = e\int d^3p\, \delta^{(3)}(\boldsymbol{0}).$$

The problem can be solved by introducing the normal ordering. Since the difficulty is originated by the terms $\hat{d}_{\boldsymbol{p}}\hat{d}^\dagger_{\boldsymbol{p}}$, we can separate for every operator the contribution given by the positive frequencies $e^{-i\omega_p t}$ and the negative frequencies $e^{i\omega_p t}$

$$\hat{\alpha}(x) = \hat{\alpha}^{(+)}(x) + \hat{\alpha}^{(-)}(x), \qquad \hat{\beta}(x) = \hat{\beta}^{(+)}(x) + \hat{\beta}^{(-)}(x)$$

and define the normal ordered product to be

$$:\hat{\alpha}\hat{\beta}: = \hat{\alpha}^{(+)}\hat{\beta}^{(+)} + \hat{\alpha}^{(-)}\hat{\beta}^{(+)} \pm \hat{\beta}^{(-)}\hat{\alpha}^{(+)} + \hat{\alpha}^{(-)}\hat{\beta}^{(-)},$$

where the sign is $+$ if the field are bosonic and $-$ if the fields are fermionic. Note that the negative frequencies contributions are moved to the left, with a minus sign when interchanged. With this prescription, the charge becomes

$$:\hat{Q}: = e\int d^3x :\hat{\psi}^\dagger\hat{\psi}: = e\sum_s \int d^3p \left(\hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s) - \hat{d}^\dagger(\boldsymbol{p},s)\hat{d}(\boldsymbol{p},s)\right)$$
$$= e\sum_s \int d^3p \Big(\hat{n}(\boldsymbol{p},s) - \hat{\bar{n}}(\boldsymbol{p},s)\Big)$$

that is the previous one with the divergence removed. Here we have introduced the number operators

$$\hat{n}(\boldsymbol{p},s) = \hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s)$$
$$\hat{\bar{n}}(\boldsymbol{p},s) = \hat{d}^\dagger(\boldsymbol{p},s)\hat{d}(\boldsymbol{p},s),$$

which generates the fermionic Fock space in terms of eigenstates. Then, we can interpret the operator as a charge operator, which distinguishes between particles and antiparticles. For example, if $|1^+\boldsymbol{p}\rangle = \hat{b}^\dagger(\boldsymbol{p},+s)\,|0\rangle$ is a particle of momentum $\boldsymbol{p}$ and spin $+\frac{1}{2}$, then

$$:\hat{Q}:\; |1^+\boldsymbol{p}\rangle = e,$$

since $\hat{n}(\boldsymbol{p},+s)\,|1^+\boldsymbol{p}\rangle = 1$ and 0 otherwise. Analogously, if $|\bar{1}^+\boldsymbol{p}\rangle = \hat{d}^\dagger(\boldsymbol{p},+s)\,|0\rangle$ is an antiparticle of momentum $\boldsymbol{p}$ and spin $+\frac{1}{2}$, then

$$:\hat{Q}:\; |\bar{1}^+\boldsymbol{p}\rangle = -e.$$

In general, for a state $|n_1^+\boldsymbol{p}_1\, n_1^-\boldsymbol{p}_1 \ldots \bar{n}_1^+\boldsymbol{p}_1\, \bar{n}_1^-\boldsymbol{p}_1 \ldots\rangle$, we have

$$:\hat{Q}:\; |n_1^+\boldsymbol{p}_1\, n_1^-\boldsymbol{p}_1 \ldots \bar{n}_1^+\boldsymbol{p}_1\, \bar{n}_1^-\boldsymbol{p}_1 \ldots\rangle = \underbrace{e\sum_i \left[(n_i^+ + n_i^-) - (\bar{n}_i^+ + \bar{n}_i^-)\right]}_{=Q} |n_1^+\boldsymbol{p}_1\, n_1^-\boldsymbol{p}_1 \ldots \bar{n}_1^+\boldsymbol{p}_1\, \bar{n}_1^-\boldsymbol{p}_1 \ldots\rangle$$

and Q represents the total charge of the state.

b) The momentum operator, which can be derived from the space-translation invariance of the Dirac Lagrangian density via Noether's theorem, is

$$\hat{\boldsymbol{P}} = -i \int d^3x\, \hat{\psi}^\dagger \boldsymbol{\nabla} \hat{\psi}.$$

Note that with this expression $\hat{\boldsymbol{P}}$ is not hermitian, although this fact will not affect the following calculations. With the wave expansion, we find

$$\begin{aligned}
\hat{\boldsymbol{P}} = -i \int d^3x\, d^3p\, d^3p' \sum_{s,s'} \frac{1}{(2\pi)^3} \frac{m}{\sqrt{\omega_p \omega_{p'}}} i\boldsymbol{p}' \Big(& \hat{d}(\boldsymbol{p},s) v^\dagger(\boldsymbol{p},s) e^{-ipx} \\
&+ \hat{b}^\dagger(\boldsymbol{p},s) u^\dagger(\boldsymbol{p},s) e^{ipx}\Big) \Big(\hat{b}(\boldsymbol{p}',s') u(\boldsymbol{p}',s') e^{-ip'x} - \hat{d}^\dagger(\boldsymbol{p}',s') v(\boldsymbol{p}',s') e^{ip'x}\Big) \\
= \int d^3x\, d^3p\, d^3p' \sum_{s,s'} \frac{1}{(2\pi)^3} \frac{m}{\sqrt{\omega_p \omega_{p'}}} \boldsymbol{p}' \Big(& \hat{d}(\boldsymbol{p},s)\hat{b}(\boldsymbol{p}',s') v^\dagger(\boldsymbol{p},s) u(\boldsymbol{p}',s') e^{-i(p+p')x} \\
&- \hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p}',s') v^\dagger(\boldsymbol{p},s) v(\boldsymbol{p}',s') e^{-i(p-p')x} + \hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p}',s') u^\dagger(\boldsymbol{p},s) u(\boldsymbol{p}',s') e^{i(p-p')x} \\
&- \hat{b}^\dagger(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p}',s') u^\dagger(\boldsymbol{p},s) v(\boldsymbol{p}',s') e^{i(p+p')x}\Big) \\
= \int d^3p\, d^3p' \sum_{s,s'} \frac{m}{\sqrt{\omega_p \omega_{p'}}} \boldsymbol{p}' \Big(\Big(& \hat{d}(\boldsymbol{p},s)\hat{b}(\boldsymbol{p}',s') v^\dagger(\boldsymbol{p},s) u(\boldsymbol{p}',s') e^{-i(\omega_p+\omega_{p'})t} \\
&- \hat{b}^\dagger(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p}',s') u^\dagger(\boldsymbol{p},s) v(\boldsymbol{p}',s') e^{i(\omega_p+\omega_{p'})t}\Big) \delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}') \\
&+ \Big(-\hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p}',s') v^\dagger(\boldsymbol{p},s) v(\boldsymbol{p}',s') e^{-i(\omega_p-\omega_{p'})t} \\
&+ \hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p}',s') u^\dagger(\boldsymbol{p},s) u(\boldsymbol{p}',s') e^{i(\omega_p-\omega_{p'})t}\Big) \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')\Big)
\end{aligned}$$

$$\begin{aligned}
= \int d^3p \sum_{s,s'} \frac{m}{\omega_p} \boldsymbol{p} \Big(& -\hat{d}(\boldsymbol{p},s)\hat{b}(-\boldsymbol{p},s') v^\dagger(\boldsymbol{p},s) u(-\boldsymbol{p},s') e^{-i2\omega_p t} + \hat{b}^\dagger(\boldsymbol{p},s)\hat{d}^\dagger(-\boldsymbol{p},s') u^\dagger(\boldsymbol{p},s) v(-\boldsymbol{p},s') e^{i2\omega_p t} \\
&- \hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p},s') v^\dagger(\boldsymbol{p},s) v(\boldsymbol{p},s') + \hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s') u^\dagger(\boldsymbol{p},s) u(\boldsymbol{p},s')\Big).
\end{aligned}$$

With the othogonal relations, we obtain

$$\begin{aligned}
\hat{\boldsymbol{P}} &= \int d^3p \sum_{s,s'} \boldsymbol{p} \left(\hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s') - \hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p},s')\right) \delta_{ss'} \\
&= \sum_s \int d^3p\, \boldsymbol{p} \left(\hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s) - \hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p},s)\right).
\end{aligned}$$

We can see that, applying the anticommutation relation for $\hat{d}(\boldsymbol{p},s)\hat{d}^\dagger(\boldsymbol{p},s)$, we obtain

$$\begin{aligned}
\hat{\boldsymbol{P}} &= \sum_s \int d^3p\, \boldsymbol{p} \left(\hat{b}^\dagger(\boldsymbol{p},s)\hat{b}(\boldsymbol{p},s) + \hat{d}^\dagger(\boldsymbol{p},s)\hat{d}(\boldsymbol{p},s)\right) - 2\int d^3p\, \boldsymbol{p}\, \delta^{(3)}(\boldsymbol{0}) \\
&= \sum_s \int d^3p\, \boldsymbol{p} \left(\hat{n}(\boldsymbol{p},s) + \hat{\bar{n}}(\boldsymbol{p},s)\right).
\end{aligned}$$

Note that, contrarily to the charge operator, here the zero-point momentum vanishes due to the symmetry of the integral. This fact is a consequence of the isotropy of space. Thus, normal ordering is not necessary, though its application does not affect the result.

c) Let us consider the Feynman propagator

$$\Delta_{F\alpha\beta}(x-y) = -i \langle 0| T\left(\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y)\right) |0\rangle,$$

where $T(\,\cdot\,)$ is the time-ordered product[5]

$$T\left(\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y)\right) = \Theta(x^0-y^0)\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y) - \Theta(y^0-x^0)\hat{\bar{\psi}}_\beta(y)\hat{\psi}_\alpha(x).$$

Expanding the fields in terms of plane waves, we obtain

$$\begin{aligned} i\Delta_{F\alpha\beta}(x-y) = \int d^3p\, d^3p' \sum_{s,s'} \frac{1}{(2\pi)^3} \frac{m}{\sqrt{\omega_p\omega_{p'}}} \Big(&\Theta(x^0-y^0)\langle 0| \big(\hat{b}(\boldsymbol{p},s)u_\alpha(\boldsymbol{p},s)e^{-ipx} + \\ &+ \hat{d}^\dagger(\boldsymbol{p},s)v_\alpha(\boldsymbol{p},s)e^{ipx}\big) \big(\hat{d}(\boldsymbol{p}',s')\bar{v}_\beta(\boldsymbol{p}',s')e^{-ip'y} + \hat{b}^\dagger(\boldsymbol{p}',s')\bar{u}_\beta(\boldsymbol{p}',s')e^{ip'y}\big)|0\rangle + \\ &- \Theta(y^0-x^0)\langle 0| \big(\hat{d}(\boldsymbol{p}',s')\bar{v}_\beta(\boldsymbol{p}',s')e^{-ip'y} + \hat{b}^\dagger(\boldsymbol{p}',s')\bar{u}_\beta(\boldsymbol{p}',s')e^{ip'y}\big)\times \\ &\times \big(\hat{b}(\boldsymbol{p},s)u_\alpha(\boldsymbol{p},s)e^{-ipx} + \hat{d}^\dagger(\boldsymbol{p},s)v_\alpha(\boldsymbol{p},s)e^{ipx}\big)|0\rangle\Big) \\ = \int d^3p\, d^3p' \sum_{s,s'} \frac{1}{(2\pi)^3} \frac{m}{\sqrt{\omega_p\omega_{p'}}} \Big(&\Theta(x^0-y^0)\,\langle 0|\hat{b}(\boldsymbol{p},s)\hat{b}^\dagger(\boldsymbol{p}',s')|0\rangle\, u_\alpha(\boldsymbol{p},s)\bar{u}_\beta(\boldsymbol{p}',s')\times \\ &\times e^{-ipx}e^{ip'y} - \Theta(y^0-x^0)\,\langle 0|\hat{d}(\boldsymbol{p}',s')\hat{d}^\dagger(\boldsymbol{p},s)|0\rangle\, \bar{v}_\beta(\boldsymbol{p}',s')v_\alpha(\boldsymbol{p},s)e^{-ip'y}e^{ipx}\Big). \end{aligned}$$

since the vacuum is destroyed by the annihilator operators

$$\hat{d}(\boldsymbol{p}',s')\,|0\rangle = \hat{b}(\boldsymbol{p},s)\,|0\rangle = 0$$
$$\langle 0|\,\hat{d}^\dagger(\boldsymbol{p},s) = \langle 0|\,\hat{b}^\dagger(\boldsymbol{p}',s') = 0.$$

On the other hand,

$$\langle 0|\hat{b}(\boldsymbol{p},s)\hat{b}^\dagger(\boldsymbol{p}',s')|0\rangle = \langle 0|\hat{d}(\boldsymbol{p}',s')\hat{d}^\dagger(\boldsymbol{p},s)|0\rangle = \delta_{ss'}\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'),$$

so that

$$\begin{aligned} i\Delta_{F\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{m}{\omega_p} \Big(&\Theta(x^0-y^0)u_\alpha(\boldsymbol{p},s)\bar{u}_\beta(\boldsymbol{p},s)e^{-ip(x-y)} + \\ &- \Theta(y^0-x^0)\bar{v}_\beta(\boldsymbol{p},s)v_\alpha(\boldsymbol{p},s)e^{ip(x-y)}\Big). \end{aligned}$$

We now use the completeness relations

$$\sum_s u_\alpha(\boldsymbol{p},s)\bar{u}_\beta(\boldsymbol{p},s) = \frac{\not{p}_{\alpha\beta}+m}{2m}$$
$$\sum_s v_\alpha(\boldsymbol{p},s)\bar{v}_\beta(\boldsymbol{p},s) = \frac{\not{p}_{\alpha\beta}-m}{2m}$$

to obtain

$$\begin{aligned} i\Delta_{F\alpha\beta}(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{m}{\omega_p}\Big(\Theta(x^0-y^0)\frac{\not{p}_{\alpha\beta}+m}{2m}e^{-ip(x-y)} - \Theta(y^0-x^0)\frac{\not{p}_{\alpha\beta}-m}{2m}e^{ip(x-y)}\Big) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{i\not{\partial}_{\alpha\beta}+m}{2\omega_p} \Big(\Theta(x^0-y^0)e^{-i\omega_p(x^0-y^0)} + \Theta(y^0-x^0)e^{i\omega_p(x^0-y^0)}\Big)\, e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \end{aligned}$$

We can rewrite the integral as a four-dimensional integral (in d^4p), by looking at a term inside the integral as a residue. In fact, setting

$$f(p^0) = -\frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)}$$

we have two cases. If $y^0 > x^0$ we can integrate f along the contour Γ^+ in figure and, using the Residue theorem,

$$\begin{aligned} \frac{1}{2\pi i}\int_{\Gamma^+} dp^0\, f(p^0) &= \mathrm{Res}(-\omega_p+i\epsilon) \\ &= \lim_{p^0\to-\omega_p+i\epsilon}(p^0+\omega_p-i\epsilon)f(p^0) \\ &= \frac{e^{i\omega_p(x^0-y^0)+\epsilon(x^0-y^0)}}{2\omega_p-2i\epsilon} \\ &= \frac{e^{i\omega_p(x^0-y^0)}}{2\omega_p}. \end{aligned}$$

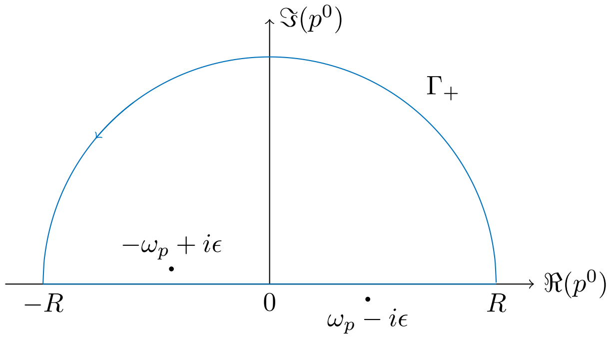
[5]The minus sign would be substituted by a plus sign in case of Bose particles.

On the other hand, the contribution along the semicircle is zero, since

$$\lim_{|p^0|\to\infty} p^0 f(p^0) = 0.$$

Thus,

$$-\frac{1}{2\pi i}\int dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)} = \Theta(y^0-x^0)\frac{e^{i\omega_p(x^0-y^0)}}{2\omega_p}.$$



With analogous calculations for $x^0 > y^0$, by integrating along a contour Γ^- in the lower half plane, we find

$$-\frac{1}{2\pi i}\int dp^0 \frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)} = \Theta(x^0-y^0)\frac{e^{-i\omega_p(x^0-y^0)}}{2\omega_p}.$$

On the other hand,

$$\begin{aligned} f(p^0) &= -\frac{e^{-ip^0(x^0-y^0)}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p-i\epsilon)} \\ &= -\frac{e^{-ip^0(x^0-y^0)}}{(p^0)^2-\omega_p^2+2i\omega_p\epsilon+\epsilon^2} \\ &= -\frac{e^{-ip^0(x^0-y^0)}}{p^2-m^2+2i\omega_p\epsilon} \end{aligned}$$

Writing ϵ instead of $2\omega_p\epsilon$, we have

$$\begin{aligned} \Delta_{F\alpha\beta}(x-y) &= -i\int \frac{d^3p}{(2\pi)^3}(i\partial\!\!\!/_{\alpha\beta}+m)\frac{1}{2\pi i}\int dp^0 \frac{-e^{-ip^0(x^0-y^0)}}{p^2-m^2+i\epsilon} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{p\!\!\!/_{\alpha\beta}+m}{p^2-m^2+i\epsilon}e^{-ip(x-y)}. \end{aligned}$$

Without reference to the components,

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{p\!\!\!/+m}{p^2-m^2+i\epsilon}e^{-ip(x-y)}.$$

With this expression, it can be shown that the Feynman propagator is a Green function for the Dirac equation with a Dirac delta as a source term:

$$\begin{aligned} (i\partial\!\!\!/_x - m)\Delta_F(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{p\!\!\!/+m}{p^2-m^2+i\epsilon}(-p\!\!\!/+m)e^{-ip(x-y)} \\ &= -\delta^{(4)}(x-y). \end{aligned}$$

d) The anticommutator will be

$$
\begin{aligned}
\{\hat{\psi}_\alpha(x), \hat{\bar{\psi}}_\beta(y)\} &= \int d^3p\, d^3p' \frac{1}{(2\pi)^3} \sum_{s,s'} \frac{m}{\sqrt{\omega_p \omega_{p'}}} \{\hat{b}(\boldsymbol{p}, s) u_\alpha(\boldsymbol{p}, s) e^{-ipx} + \hat{d}^\dagger(\boldsymbol{p}, s) v_\alpha(\boldsymbol{p}, s) e^{ipx} + \\
&\quad + \hat{d}(\boldsymbol{p'}, s') \bar{v}_\beta(\boldsymbol{p'}, s') e^{-ip'y} + \hat{b}^\dagger(\boldsymbol{p'}, s') \bar{u}_\beta(\boldsymbol{p'}, s') e^{ip'y}\} \\
&= \int d^3p\, d^3p' \frac{1}{(2\pi)^3} \sum_{s,s'} \frac{m}{\sqrt{\omega_p \omega_{p'}}} \Big(\{\hat{b}(\boldsymbol{p}, s), \hat{d}(\boldsymbol{p'}, s')\} u_\alpha(\boldsymbol{p}, s) \bar{v}_\beta(\boldsymbol{p'}, s') e^{-ipx} e^{-ip'y} + \\
&\quad + \{\hat{b}(\boldsymbol{p}, s), \hat{b}^\dagger(\boldsymbol{p'}, s')\} u_\alpha(\boldsymbol{p}, s) \bar{u}_\beta(\boldsymbol{p'}, s') e^{-ipx} e^{ip'y} + \\
&\quad + \{\hat{d}^\dagger(\boldsymbol{p}, s), \hat{d}(\boldsymbol{p'}, s')\} v_\alpha(\boldsymbol{p}, s) \bar{v}_\beta(\boldsymbol{p'}, s') e^{ipx} e^{-ip'y} + \\
&\quad + \{\hat{d}^\dagger(\boldsymbol{p}, s), \hat{b}^\dagger(\boldsymbol{p'}, s')\} v_\alpha(\boldsymbol{p}, s) \bar{u}_\beta(\boldsymbol{p'}, s') e^{ipx} e^{ip'y}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_s \frac{m}{\omega_p} \Big(u_\alpha(\boldsymbol{p}, s) \bar{u}_\beta(\boldsymbol{p}, s) e^{-ip(x-y)} + v_\alpha(\boldsymbol{p}, s) \bar{v}_\beta(\boldsymbol{p}, s) e^{ip(x-y)}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{m}{\omega_p} \Big(\frac{\not{p}_{\alpha\beta} + m}{2m} e^{-ip(x-y)} + \frac{\not{p}_{\alpha\beta} - m}{2m} e^{ip(x-y)}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \Big(\big(\not{p}_{\alpha\beta} + m\big) e^{-ip(x-y)} + \big(\not{p}_{\alpha\beta} - m\big) e^{ip(x-y)}\Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \big(i\not{\partial}_{\alpha\beta} + m\big) \Big(e^{-ip(x-y)} - e^{ip(x-y)}\Big) \\
&= \big(i\not{\partial}_{\alpha\beta} + m\big) i\Delta(x-y),
\end{aligned}
$$

which is zero for space-like separated events, thanks to a property of the Pauli-Jordan function.

e) The microcausality condition states that measurements of an observable $\hat{O}$ in points with a space-like separation do not interfere each other, *i.e.* if $(x-y)^2 < 0$, then $[\hat{O}(x), \hat{O}(y)] = 0$. For the Dirac field, since every observable has the form

$$\hat{O}(x) = \hat{\bar{\psi}}_\alpha(x) O_{\alpha\beta}(x) \hat{\psi}_\beta(x),$$

we obtain

$$
\begin{aligned}
[\hat{O}(x), \hat{O}(y)] &= O_{\alpha\beta}(x) O_{\gamma\delta}(y) [\hat{\bar{\psi}}_\alpha(x) \hat{\psi}_\beta(x), \hat{\bar{\psi}}_\gamma(y) \hat{\psi}_\delta(y)] \\
&= O_{\alpha\beta}(x) O_{\gamma\delta}(y) \Big([\hat{\bar{\psi}}_\alpha(x), \hat{\bar{\psi}}_\gamma(y) \hat{\psi}_\delta(y)] \hat{\psi}_\beta(x) + \hat{\bar{\psi}}_\alpha(x) [\hat{\psi}_\beta(x), \hat{\bar{\psi}}_\gamma(y) \hat{\psi}_\delta(y)]\Big) \\
&= O_{\alpha\beta}(x) O_{\gamma\delta}(y) \Big(\{\hat{\bar{\psi}}_\alpha(x), \hat{\bar{\psi}}_\gamma(y)\} \hat{\psi}_\delta(y) \hat{\psi}_\beta(x) - \hat{\bar{\psi}}_\gamma(y) \{\hat{\bar{\psi}}_\alpha(x), \hat{\psi}_\delta(y)\} \hat{\psi}_\beta(x) \\
&\qquad + \hat{\bar{\psi}}_\alpha(x) \{\hat{\psi}_\beta(x), \hat{\bar{\psi}}_\gamma(y)\} \hat{\psi}_\delta(y) - \hat{\bar{\psi}}_\alpha(x) \hat{\bar{\psi}}_\gamma(y) \{\hat{\psi}_\beta(x), \hat{\psi}_\delta(y)\}\Big).
\end{aligned}
$$

Using the fact that

$$\{\hat{\psi}_\lambda(x), \hat{\psi}_\sigma(y)\} = \{\hat{\bar{\psi}}_\lambda(x), \hat{\bar{\psi}}_\sigma(y)\} = 0,$$

we obtain

$$
\begin{aligned}
[\hat{O}(x), \hat{O}(y)] &= O_{\alpha\beta}(x) O_{\gamma\delta}(y) \Big(\hat{\bar{\psi}}_\alpha(x) \{\hat{\psi}_\beta(x), \hat{\bar{\psi}}_\gamma(y)\} \hat{\psi}_\delta(y) - \hat{\bar{\psi}}_\gamma(y) \{\hat{\bar{\psi}}_\alpha(x), \hat{\psi}_\delta(y)\} \hat{\psi}_\beta(x)\Big) \\
&= O_{\alpha\beta}(x) O_{\gamma\delta}(y) \Big(\hat{\bar{\psi}}_\alpha(x) \hat{\psi}_\delta(y) \big(i\not{\partial}_{\beta\gamma} + m\big) i\Delta(x-y) - \hat{\bar{\psi}}_\gamma(y) \hat{\psi}_\beta(x) \big(i\not{\partial}_{\delta\alpha} + m\big) i\Delta(y-x)\Big) \\
&= O_{\alpha\beta}(x) O_{\gamma\delta}(y) \Big(\hat{\bar{\psi}}_\alpha(x) \hat{\psi}_\delta(y) \big(i\not{\partial}_{\beta\gamma} + m\big) + \hat{\bar{\psi}}_\gamma(y) \hat{\psi}_\beta(x) \big(i\not{\partial}_{\delta\alpha} + m\big)\Big) i\Delta(x-y),
\end{aligned}
$$

which is zero for space-like separated events, as follows from point (d). Thus, the microcausality condition is satisfied for the Dirac field.

12 The electromagnetic field

a) Write down Maxwell's equations for $\boldsymbol{E}$ and $\boldsymbol{B}$. Introduce the field strength tensor $F_{\mu\nu}$ and its dual, and re-write Maxwell's equations in an explicitly covariant form. Introduce the 4-current j_μ and show that it is conserved. Introduce the 4-potential A_μ and re-write Maxwell's equations in terms of A_μ.

b) Discuss the gauge freedom of the theory, and introduce the Lorenz and Coulomb gauge. Write down the Lagrangian density for the e.m. field. Show the connection between gauge invariance and current conservation.

c) How many independent degrees of freedom does the free field A_μ have? Show it in the Lorenz gauge as well as in the Coulomb gauge. Why is this a difficulty for canonical quantization? How it is solved? (Say it in words.)

d) Consider Maxwell's equations for A_μ in the Lorenz gauge, and the plane-wave solutions:

$$A_\mu(\boldsymbol{k}, \lambda, x) = N_k \epsilon_\mu(\boldsymbol{k}, \lambda) e^{ik\cdot x}.$$

Construct a basis for the polarization vectors $\epsilon_\mu(\boldsymbol{k}, \lambda)$. Discuss their physical meaning.

e) Prove the orthonormality and completeness relations for the polarization vectors $\epsilon_\mu(\boldsymbol{k}, \lambda)$.

---

a) The Maxwell equations for the electric field $\boldsymbol{E}$ and the magnetic one $\boldsymbol{B}$ are

$$\begin{aligned} &\boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \\ &\boldsymbol{\nabla} \times \boldsymbol{B} - \dot{\boldsymbol{E}} = \boldsymbol{j} \\ &\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \\ &\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \end{aligned}$$

where ρ is the charge distribution and $\boldsymbol{j}$ is the current. We can introduce the electromagnetic strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

and its dual

$$*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} = F^{\mu\nu}(\boldsymbol{E} \to \boldsymbol{B}, \boldsymbol{B} \to -\boldsymbol{E}).$$

With this notation, Maxwell's equations can be written as

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \qquad && \text{dynamic equation} \\ \partial_\mu {*F^{\mu\nu}} &= 0 && \text{constraint equation,} \end{aligned}$$

where $j^\nu = (\rho, \boldsymbol{j})$ is the 4-current. It can be shown that $F^{\mu\nu}$ and its dual are tensors, while the 4-current is a 4-vector. Hence, the Maxwell equations in terms of the field strength tensor are manifestly Lorentz invariant.

Taking the divergence of the 4-current, we immediately obtain the continuity equation:

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0,$$

since $\partial_\nu \partial_\mu$ is symmetric, while $F^{\mu\nu}$ is antisymmetric in the indices μ, ν.

We introduce the 4-potential A^μ, defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

In terms of the electric and magnetic fields,

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla} A^0 - \frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A}. \end{aligned}$$

In such a way, the constraint equation is already satisfied, as it can be easily checked. The dynamical equation becomes an inhomogeneous wave equation with a divergence term

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$$

or in terms of A^0 and $\boldsymbol{A}$

$$\frac{\partial^2}{\partial t^2}A^0 - \nabla^2 A^0 - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}A^0 + \boldsymbol{\nabla}\cdot\boldsymbol{A}\right) = -\nabla^2 A^0 - \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A} = \rho$$
$$\Box\boldsymbol{A} - \boldsymbol{\nabla}\left(\frac{\partial}{\partial t}A^0 + \boldsymbol{\nabla}\cdot\boldsymbol{A}\right) = \boldsymbol{j}.$$

b) From the definition of the 4-potential, it is clear that it is not uniquely defined. Every transformation

$$A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu\Lambda(x),$$

where Λ is a scalar function, leads to same electromagnetic strength tensor, *i.e.* to the same observable quantities. This fact can be seen by counting the d.o.f. of the different quantities here involved. The electromagnetic strength tensor has 6 different terms, but the constraint equation reduces them to 2. On the other hand, the 4-potential has 4 independent terms and the constraint equation is already satisfied, so that we are left with 2 more d.o.f.

This is the so-called gauge freedom and the above transformation is called a gauge transformation. Such a freedom can be used to fix a gauge. For example, we can introduce the Lorenz gauge by imposing

$$\partial_\mu A^\mu = 0,$$

which can be implemented from a potential A^μ via a gauge function satisfying the inhomogeneous wave equation

$$\Box\Lambda = -\partial_\mu A^\mu.$$

Another important gauge is the Coulomb gauge, which is obtained by imposing

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0.$$

It can be obtained from a potential A^μ via a gauge function satisfying the Poisson equation

$$\nabla^2\Lambda = -\boldsymbol{\nabla}\cdot\boldsymbol{A}.$$

The electromagnetic Lagrangian density is

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu,$$

which can be used to describe photons. Note that the electromagnetic strength tensor is gauge invariant, thus the free part of the Lagrangian also is. On the other hand, the interacting term $j_\mu A^\mu$ is not *a priori* gauge invariant. However, the action is invariant under gauge transformation provided that j_μ satisfies the continuity equation:

$$\begin{aligned} j_\mu A^\mu \longrightarrow j_\mu A'^\mu &= j_\mu A^\mu + j_\mu\partial^\mu\Lambda \\ &= j_\mu A^\mu + \partial^\mu(j_\mu\Lambda) - (\partial^\mu j_\mu)\Lambda. \end{aligned}$$

The surface term $\partial^\mu(j_\mu\Lambda)$ does not affect the action, so that the gauge invariance is implied from the continuity equation $\partial^\mu j_\mu = 0$.

c) As we already discussed, the 4-potential has 4 independent terms, since the constraint equation is already satisfied. Hence, we are left we 2 redundant dof.

The Lorenz gauge has the advantage that it is manifestly covariant, but it does not fix all gauge dof. The new 4-divergenceless potential leads to the same observable quantities if transformed as

$$A'^{\mu}(x) \longrightarrow A'^{\mu}(x) + \partial^{\mu}\Lambda'(x),$$

where Λ' satisfies the wave equation $\Box\Lambda' = 0$. Thus, we are left with one gauge dof.

On the other hand, the Coulomb gauge fixes all dof, but it has the disadvantage of not being manifestly Lorentz invariant. In this case, it can be easily shown which dof are fixed. In fact the 3-divergenceless condition becomes in momentum space

$$\boldsymbol{k}\cdot\boldsymbol{A}(t,\boldsymbol{k}) = 0,$$

that is, the longitudinal component of $\boldsymbol{A}$ is zero. Thus, we are left with $4-1=3$ dof. Furthermore, the second equation of motion expressed in terms of the potential is, in general

$$\nabla^2 A^0 = -\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{A}}{\partial t} - \rho = -\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A} - \rho,$$

which is a Poisson equation for A^0, whose solution can be expressed via the propagator $G(\boldsymbol{x}-\boldsymbol{x}') = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$ as

$$A^0(t,\boldsymbol{x}) = \int d^3x'\, G(\boldsymbol{x}-\boldsymbol{x}')\left(\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A}(t,\boldsymbol{x}') - \rho(t,\boldsymbol{x}')\right).$$

Thus, in the Coulomb gauge,

$$A^0(t,\boldsymbol{x}) = \int d^3x'\,\frac{1}{4\pi}\frac{\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

so that A^0 is no longer a dynamical variable, but it is determined by the source ρ. Hence, we have $4-2=2$ dof and the gauge freedom is completely fixed. Note that in this case the scalar potential is instantaneous, in the sense that a change in the charge distribution propagates instantaneously. However, this fact does not contradict relativity, since A^0 is not a physical quantity; the electromagnetic field is expressed in terms of the derivatives of A^{μ}, in such a way that relativity is not violated.

The redundancy in dof of the potential creates a difficulty in the application of canonical quantization. In order to quantize and impose the commutation relations, we have to compute the conjugate momenta. If we consider A^{μ} as dynamical variables, then

$$\pi_{\mu} = \frac{\partial\mathcal{L}}{\partial\dot{A}^{\mu}} = -F_{0\mu}.$$

In particular, $\pi_0 = 0$. Thus, we cannot impose the relation

$$[\hat{A}^0(t,\boldsymbol{x}), \hat{\pi}^0(t,\boldsymbol{x}')] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}').$$

This is a consequence of the fact that A^0 can be expressed in terms of the vector potential (and the charge distribution). The problem can be solved in several different ways. The two most popular are: either we quantize the theory in the Coulomb gauge, considering only the relevant dof (*i.e.* the transverse components of the vector potential), or we quantize the theory in the Lorenz gauge and use an appropriate prescription for dealing with the remaining dof.

d) The Maxwell equations in the Lorenz gauge are inhomogeneous wave equations fro the potential

$$\Box A^{\mu} = j^{\mu}.$$

Without sources, the equations in momentum space become $k^2A^{\mu}(k) = 0$, that is $k^2 = 0$. Let us consider now a plane wave expansion of the potential in terms of the free modes

$$A^{\mu}(\boldsymbol{k},\lambda;x) = N_k\epsilon^{\mu}(\boldsymbol{k},\lambda)e^{-ikx},$$

where $\epsilon^{\mu}(\boldsymbol{k},\lambda)$ for $\lambda = 0,1,2,3$ are called polarization vectors, which we now construct in such a way that they are orthogonal to each other and normalized. We can choose $\epsilon^{\mu}(\boldsymbol{k},0) = n$ to be a time-like unit vector. In a fixed (arbitrary) coordinate system, we set

$$n = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}.$$

Next we take $\epsilon^\mu(\boldsymbol{k},3)$ along $\boldsymbol{k}$ (thus space-like) a normalized

$$\epsilon^\mu(\boldsymbol{k},3) = \begin{pmatrix} 0 & \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \end{pmatrix} \qquad \text{longitudinal condition.}$$

In explicitly covariant form, we have to implement the projection of k onto the plane orthogonal to n:

$$\epsilon^\mu(\boldsymbol{k},3) = \frac{k-(nk)n}{\sqrt{(nk)^2-k^2}}.$$

Finally, we choose $\epsilon^\mu(\boldsymbol{k},\lambda)$ for $\lambda = 1,2$ to have no time-like component ($\epsilon^\mu(\boldsymbol{k},\lambda) = (0\ \boldsymbol{\epsilon}(\boldsymbol{k},\lambda))$) and satisfy

$$\begin{aligned} &\boldsymbol{k}\cdot\boldsymbol{\epsilon}(\boldsymbol{k},\lambda) = 0 && \text{transversality condition} \\ &\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\cdot\boldsymbol{\epsilon}(\boldsymbol{k},\sigma) = \delta_{\lambda\sigma} && \text{orthonormalization condition.} \end{aligned}$$

The above construction required the choice of a particular frame, but we can impose the polarization vectors to transform like vectors, to remove the dependence on the frame. The polarization vectors thus defined are called

$$\begin{aligned} &\epsilon(\boldsymbol{k},0) && \text{time-like polarization vector} \\ &\epsilon(\boldsymbol{k},1), \epsilon(\boldsymbol{k},2) && \text{transverse polarization vectors} \\ &\epsilon(\boldsymbol{k},3) && \text{longitudinal polarization vector.} \end{aligned}$$

They represent the polarization directions of the photons.

e) From the above construction, it is clear that if $k^2 = 0$, then

$$k\epsilon(\boldsymbol{k},0) = -k\epsilon(\boldsymbol{k},3) = kn.$$

The first relation follows by the definition of $\epsilon(\boldsymbol{k},0)$, while for the second one

$$-k\epsilon(\boldsymbol{k},3) = -\frac{k^2-(kn)(kn)}{\sqrt{(nk)^2-k^2}} = nk$$

if $k^2 = 0$. For the transverse polarization vectors

$$k\epsilon(\boldsymbol{k},1) = k\epsilon(\boldsymbol{k},2) = 0,$$

which follows from the definition.

Furthermore, we have the orthogonality relation

$$\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda') = \eta_{\lambda\lambda'}.$$

We can also prove the completeness relation

$$\sum_\lambda \eta_{\lambda\lambda}\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon_\nu(\boldsymbol{k},\lambda) = \eta_{\mu\nu}.$$

In fact, in the particular frame where we first defined the vectors, the relation can be written as

$$\begin{pmatrix}1\\ \boldsymbol{0}\end{pmatrix}_\mu \begin{pmatrix}1\\ \boldsymbol{0}\end{pmatrix}_\nu - \sum_{\lambda=1}^{3} \begin{pmatrix}0\\ \boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\end{pmatrix}_\mu \begin{pmatrix}0\\ \boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\end{pmatrix}_\nu.$$

For $\mu = \nu = 0$, the sum is $+1$. When we have a time index and a space index, the sum is 0. Last, when both are space indices, we find

$$\sum_{\lambda=1}^{3} \boldsymbol{\epsilon}_i(\boldsymbol{k},\lambda)\boldsymbol{\epsilon}_j(\boldsymbol{k},\lambda) = \delta_{ij}.$$

Thus, the completeness relation is proved.

13 Quantization of the electromagnetic field: Lorenz gauge

a) Write down the Lagrangian density of the free electromagnetic field in a covariant form (in terms of A^μ). Compute the conjugate momenta π^μ associated to A^μ and show that $\pi^0 = 0$. Add a gauge fixing term, such that the Lagrangian density does not change if the Lorenz gauge condition is satisfied. Compute again the conjugate momenta π^μ. Now $\pi^0 \neq 0$. Compute the Euler-Lagrange equations from the gauge fixed Lagrangian density.

b) Choose the Feynman gauge (gauge fixing parameter $\zeta = 1$). Perform an integration by parts, to further simplify the Lagrangian density. Compute once again the conjugate momenta π^μ. Compute the Hamiltonian density. What's unusual about its form?

c) Quantize the electromagnetic field as prescribed by canonical quantization. What's unusual about its form? Show that the Lorenz gauge condition is not satisfied for the quantized field. Expand $\hat{A}^\mu$ in plane waves. Do the same for the conjugate momenta. Write all factors explicitly. Reverse these relations and write the operators $\hat{a}_{\boldsymbol{k}\lambda}$ and $\hat{a}^\dagger_{\boldsymbol{k}\lambda}$ as a function of $\hat{A}^\mu$ and their conjugate momenta. Compute the commutation relations for the operators $\hat{a}_{\boldsymbol{k}\lambda}$ and $\hat{a}^\dagger_{\boldsymbol{k}\lambda}$ starting from those of $\hat{A}^\mu$ and their conjugate momenta.

d) Compute the normal-ordered Hamiltonian $\hat{H}$ and momentum $\hat{\boldsymbol{P}}$ as a function of $\hat{a}_{\boldsymbol{k}\lambda}$ and $\hat{a}^\dagger_{\boldsymbol{k}\lambda}$, starting from their definition in terms of $\hat{A}^\mu$ and their conjugate momenta. What's unusual about their form?

e) Explain the Gupta-Bleuler method and show how it solves all the problems encountered.

---

a) The Lagrangian density of the free electromagnetic field is

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right).$$

Considering A^μ as dynamical variables, the conjugate momenta are

$$\begin{aligned}
\pi_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} &= -\frac{1}{4}\left(\frac{\partial F_{\mu\nu}}{\partial \dot{A}^\lambda} F^{\mu\nu} + F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial \dot{A}^\lambda}\right) \\
&= -\frac{1}{2} F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial \dot{A}^\lambda} = -\frac{1}{2} F_{\mu\nu} \left(\delta^\mu_0\, \delta^\nu_\lambda - \delta^\nu_0\, \delta^\mu_\lambda\right) \\
&= -\frac{1}{2}\left(F_{0\lambda} - F_{\lambda 0}\right) = -F_{0\lambda}.
\end{aligned}$$

This fact shows that $\pi_0 = 0$, that is A^0 has no conjugate momentum. The problem can be solved by adding to the Lagrangian a gauge fixing term

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\zeta}{2} \left(\partial_\sigma A^\sigma\right)^2.$$

The new Lagrangian trivially reduced to the previous one in the Lorenz gauge. But for the time being, we will not impose the gauge condition.

With this addition, the conjugate momenta become

$$\pi_\lambda = -F_{0\lambda} - \zeta\, \eta_{0\lambda}\, \partial_\sigma A^\sigma$$

and in particular $\pi_0 = -\zeta\, \partial_\sigma A^\sigma \neq 0$, provided that $\zeta \neq 0$. The Euler-Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu \frac{\partial \mathcal{L}}{\partial \partial^\mu A^\nu} = 0.$$

We have

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \partial^\mu A^\nu} = -\frac{1}{4}\left(\frac{\partial F_{\rho\sigma}}{\partial \partial^\mu A^\nu} F^{\rho\sigma} + F_{\rho\sigma}\frac{\partial F^{\rho\sigma}}{\partial \partial^\mu A^\nu}\right) - \zeta(\partial_\sigma A^\sigma)\frac{\partial \partial_\sigma A^\sigma}{\partial \partial^\mu A^\nu}.$$

The first term in the last equation is

$$\begin{aligned}
-\frac{1}{4}\left(\frac{\partial F_{\rho\sigma}}{\partial \partial^\mu A^\nu} F^{\rho\sigma} + F_{\rho\sigma}\frac{\partial F^{\rho\sigma}}{\partial \partial^\mu A^\nu}\right) &= -\frac{1}{2}F_{\rho\sigma}\frac{\partial F^{\rho\sigma}}{\partial \partial^\mu A^\nu} = -\frac{1}{2}F_{\rho\sigma}\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu\right) \\
&= -\frac{1}{2}\left(F_{\mu\nu} - F_{\nu\mu}\right) = -F_{\mu\nu},
\end{aligned}$$

while the second term becomes

$$\frac{\partial \partial_\sigma A^\sigma}{\partial \partial^\mu A^\nu} = \eta_{\lambda\sigma}\frac{\partial \partial^\lambda A^\sigma}{\partial \partial^\mu A^\nu} = \eta_{\lambda\sigma}\delta^\lambda_\mu\delta^\sigma_\nu = \eta_{\mu\nu}.$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial \partial^\mu A^\nu} = -F_{\mu\nu} - \zeta\,\eta_{\mu\nu}(\partial_\sigma A^\sigma)$$

and the Euler-Lagrange equations are

$$\begin{aligned}
0 = -\partial^\mu\frac{\partial \mathcal{L}}{\partial \partial^\mu A^\nu} &= \partial^\mu F_{\mu\nu} + \zeta\,\eta_{\mu\nu}\partial^\mu(\partial_\sigma A^\sigma) \\
&= \partial^\mu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \zeta\,\partial_\nu(\partial_\sigma A^\sigma) \\
&= \Box A_\nu - \partial_\nu(\partial^\mu A_\mu) + \zeta\,\partial_\nu(\partial_\sigma A^\sigma) \\
&= \Box A_\nu + (\zeta - 1)\partial_\nu(\partial_\sigma A^\sigma).
\end{aligned}$$

We can see that, if $\zeta = 1$, the equations are the free Maxwell ones in the Lorenz gauge.

b) In the Feynman gauge $\zeta = 1$ the Lagrangian density becomes

$$\begin{aligned}
\mathcal{L}(x) &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\sigma A^\sigma\right)^2 \\
&= -\frac{1}{4}\left(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu - \partial_\nu A_\mu\partial^\mu A^\nu + \partial_\nu A_\mu\partial^\nu A^\mu\right) - \frac{1}{2}\partial_\mu A^\mu\partial_\nu A^\nu \\
&= -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu - \frac{1}{2}\partial_\mu A^\mu\partial_\nu A^\nu \\
&= -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu\left(A_\nu\partial^\nu A^\mu - A^\mu\partial_\nu A^\nu\right).
\end{aligned}$$

In the last step the terms $A_\nu\partial_\mu\partial^\nu A^\mu$ and $-A^\mu\partial_\mu\partial_\nu A^\nu$ cancel each other. The 4-divergence term does not affect the variation of the action. Thus, we can take the Lagrangian density as

$$\mathcal{L}(x) = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu.$$

The conjugate momenta become

$$\begin{aligned}
\pi_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} &= -\frac{1}{2}\left(\frac{\partial \partial_\mu A_\nu}{\partial \dot{A}^\lambda}\partial^\mu A^\nu + \partial_\mu A_\nu\frac{\partial \partial^\mu A^\nu}{\partial \dot{A}^\lambda}\right) \\
&= -\partial_\mu A_\nu\frac{\partial \partial^\mu A^\nu}{\partial \dot{A}^\lambda} = -\partial_\mu A_\nu\,\delta^\mu_0\,\delta^\nu_\lambda = -\partial_0 A_\lambda.
\end{aligned}$$

Thus, the Hamiltonian density will be

$$\begin{aligned}
\mathcal{H}(x) = \pi_\mu\dot{A}^\mu - \mathcal{L}(x) &= -\pi_\mu\pi^\mu + \frac{1}{2}\pi_\mu\pi^\mu + \frac{1}{2}\partial_i A_\nu\partial^i A^\nu \\
&= -\frac{1}{2}\pi_\mu\pi^\mu + \frac{1}{2}\boldsymbol{\nabla}A_\nu\cdot\boldsymbol{\nabla}A^\nu \\
&= \frac{1}{2}\sum_{k=1}^{3}\Big((\pi^k)^2 + (\boldsymbol{\nabla}A^k)^2\Big) - \frac{1}{2}\Big((\pi^0)^2 + (\boldsymbol{\nabla}A^0)^2\Big),
\end{aligned}$$

which is not positive definite due to presence of the temporal component. This will create a problem.

c) Let us quantize the electromagnetic field

$$A^\mu, \pi^\mu \longrightarrow \hat{A}^\mu, \hat{\pi}^\mu$$

$$\{\,\cdot\,,\,\cdot\,\}_{\mathrm{PB}} \longrightarrow -i\,[\,\cdot\,,\,\cdot\,].$$

The fundamental equal-time commutation relation becomes

$$[\hat{A}^\mu(t,\boldsymbol{x}), \hat{\pi}^\nu(t,\boldsymbol{x}')] = i\,\eta^{\mu\nu}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$$
$$[\hat{A}^\mu(t,\boldsymbol{x}), \hat{A}^\nu(t,\boldsymbol{x}')] = [\hat{\pi}^\mu(t,\boldsymbol{x}), \hat{\pi}^\nu(t,\boldsymbol{x}')] = 0.$$

In particular, we find the "wrong signed" commutation relation

$$[\hat{A}^0(t,\boldsymbol{x}), \dot{\hat{A}}^0(t,\boldsymbol{x}')] = -i\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}').$$

This is another problem to deal with. We can also see now that the Lorenz gauge condition cannot be implemented for the field operator $\hat{A}^\mu$. In fact,

$$\begin{aligned}
[\partial_\mu \hat{A}^\mu(t,\boldsymbol{x}), \hat{A}^\nu(t,\boldsymbol{x}')] &= [\dot{\hat{A}}^0(t,\boldsymbol{x}) + \boldsymbol{\nabla}_x \hat{\boldsymbol{A}}(t,\boldsymbol{x}), \hat{A}^\nu(t,\boldsymbol{x}')] \\
&= [\dot{\hat{A}}^0(t,\boldsymbol{x}), \hat{A}^\nu(t,\boldsymbol{x}')] + \boldsymbol{\nabla}_x[\hat{\boldsymbol{A}}(t,\boldsymbol{x}), \hat{A}^\nu(t,\boldsymbol{x}')] \\
&= [\hat{A}^\nu(t,\boldsymbol{x}), \hat{\pi}^0(t,\boldsymbol{x}')] \\
&= i\,\eta^{\nu 0}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}') \neq 0.
\end{aligned}$$

Thus, canonical quantization and the Lorenz gauge are not compatible.

A basis of plane wave solutions to the equation $\Box A^\mu = 0$ is given by

$$A^\mu(\boldsymbol{k},\lambda;x) = N_k\,\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx},$$

where $\epsilon^\mu(\boldsymbol{k},\lambda)$ for $\lambda = 0,1,2,3$ are the polarization vectors. They can be assumed to be real. In addition, the normalization factor can be chosen as

$$N_k = \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}$$

as in the Klein-Gordon (KG) case. Here $\omega_k = |\boldsymbol{k}|$, in accordance with the massless dispersion relation. Expanding the fields, we find

$$\hat{A}^\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}} \sum_\lambda \Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx} + \hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big),$$

while for the conjugate momenta we find

$$\begin{aligned}
\hat{\pi}^\mu(x) = -\partial_0 \hat{A}^\mu(x) &= i\int \frac{d^3k}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}} \sum_\lambda \omega_k \Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx} - \hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big) \\
&= i\int \frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{\omega_k}{2}} \sum_\lambda \Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx} - \hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big).
\end{aligned}$$

In order to obtain the commutation relation between the operators $\hat{a}(\boldsymbol{k},\lambda)$ and their adjoints it is useful to isolate the operators using some Fourier transforms

$$\sum_\lambda \Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-i\omega_k t} + \hat{a}^\dagger(-\boldsymbol{k},\lambda)\epsilon^\mu(-\boldsymbol{k},\lambda)e^{i\omega_k t}\Big) = \sqrt{(2\pi)^3}\,\sqrt{2\omega_k}\int \frac{d^3x}{(2\pi)^3}\,\hat{A}^\mu(x)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\sum_\lambda \Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-i\omega_k t} - \hat{a}^\dagger(-\boldsymbol{k},\lambda)\epsilon^\mu(-\boldsymbol{k},\lambda)e^{i\omega_k t}\Big) = -i\sqrt{(2\pi)^3}\,\sqrt{\frac{2}{\omega_k}}\int \frac{d^3x}{(2\pi)^3}\,\hat{\pi}^\mu(x)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

here we have replaced $\boldsymbol{k} \to -\boldsymbol{k}$ in the $\hat{a}^\dagger_{\boldsymbol{p}}$ terms of the $\hat{A}^\mu(x)$ and $\hat{\pi}^\mu(x)$ integrals and used the fact that $\omega_k = \omega_{-k}$. So that

$$\sum_\lambda \hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda) = \frac{1}{2}\sqrt{(2\pi)^3}\, e^{i\omega_k t} \int \frac{d^3x}{(2\pi)^3}\Big(\sqrt{2\omega_k}\,\hat{A}^\mu(x) - i\sqrt{\frac{2}{\omega_k}}\,\hat{\pi}^\mu(x)\Big)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$\sum_\lambda \hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda) = \frac{1}{2}\sqrt{(2\pi)^3}\, e^{-i\omega_k t} \int \frac{d^3x}{(2\pi)^3}\Big(\sqrt{2\omega_k}\,\hat{A}^\mu(x) + i\sqrt{\frac{2}{\omega_k}}\,\hat{\pi}^\mu(x)\Big)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

We can now compute the commutator $[\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k'},\lambda')]$

$$\begin{aligned}\sum_{\lambda,\lambda'} [\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k'},\lambda')]\,\epsilon^\mu(\boldsymbol{k},\lambda)\,\epsilon^\nu(\boldsymbol{k'},\lambda') = \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3x \int d^3y\, \times \\
\times\Big(\sqrt{\omega_k}\sqrt{\omega_{k'}}\,[\hat{A}^\mu(x),\hat{A}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{-i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})} + \\
+ i\sqrt{\frac{\omega_k}{\omega_{k'}}}\,[\hat{A}^\mu(x),\hat{\pi}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{-i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})} + \\
- i\sqrt{\frac{\omega_{k'}}{\omega_k}}\,[\hat{\pi}^\mu(x),\hat{A}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{-i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})} + \\
+ \frac{1}{\sqrt{\omega_k}\sqrt{\omega_{k'}}}\,[\hat{\pi}^\mu(x),\hat{\pi}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{-i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})}\Big),\end{aligned}$$

using the equal-time commutation relations we find

$$\begin{aligned}\sum_{\lambda,\lambda'} [\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k'},\lambda')]\,\epsilon^\mu(\boldsymbol{k},\lambda)\,\epsilon^\nu(\boldsymbol{k'},\lambda') &= -\frac{1}{2}\frac{1}{(2\pi)^3}\,\eta^{\mu\nu}\int d^3x\,\Big(\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}}\,\Big)\, e^{i(\omega_k-\omega_{k'})t}\, e^{i(\mathbf{k}-\mathbf{k'})\cdot\mathbf{x}} \\
&= -\frac{1}{2}\,\eta^{\mu\nu}\Big(\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}}\,\Big)\, e^{i(\omega_k-\omega_{k'})t}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'}) \\
&= -\,\eta^{\mu\nu}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'}).\end{aligned}$$

(The equalities have to be intended as such if integrated over momentum space.)
We can now multiply both sides by $\epsilon_\mu(\boldsymbol{k},\bar{\lambda})\,\epsilon_\nu(\boldsymbol{k'},\bar{\lambda}')$ and use the relation $\epsilon_\mu(\boldsymbol{k},\lambda)\,\epsilon^\mu(\boldsymbol{k},\lambda') = \eta_{\lambda\lambda'}$. So that

$$\sum_{\lambda,\lambda'} [\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k'},\lambda')]\,\eta_{\lambda\bar{\lambda}}\,\eta_{\lambda'\bar{\lambda}'} = -\,\eta_{\bar{\lambda}\bar{\lambda}'}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'})$$

The last equality can be written as

$$[\hat{a}(\boldsymbol{k},\bar{\lambda}), \hat{a}^\dagger(\boldsymbol{k'},\bar{\lambda}')]\,\eta_{\bar{\lambda}\bar{\lambda}}\,\eta_{\bar{\lambda}'\bar{\lambda}'} = -\,\eta_{\bar{\lambda}\bar{\lambda}'}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'})$$

The last equality tells us that if $\bar{\lambda} = \bar{\lambda}'$ we have $[\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k'},\lambda')] = -\eta_{\lambda\lambda'}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'})$ otherwise $[\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k'},\lambda')] = 0$.

With analogous calculation we can compute *i.e.* $[\hat{a}(\boldsymbol{k},\lambda), \hat{a}(\boldsymbol{k'},\lambda')]$

$$\begin{aligned}\sum_{\lambda,\lambda'} [\hat{a}(\boldsymbol{k},\lambda), \hat{a}(\boldsymbol{k'},\lambda')]\,\epsilon^\mu(\boldsymbol{k},\lambda)\,\epsilon^\nu(\boldsymbol{k'},\lambda') = \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3x \int d^3y\, \times \\
\times\Big(\sqrt{\omega_k}\sqrt{\omega_{k'}}\,[\hat{A}^\mu(x),\hat{A}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})} + \\
- i\sqrt{\frac{\omega_k}{\omega_{k'}}}\,[\hat{A}^\mu(x),\hat{\pi}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})} + \\
- i\sqrt{\frac{\omega_{k'}}{\omega_k}}\,[\hat{\pi}^\mu(x),\hat{A}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})} + \\
- \frac{1}{\sqrt{\omega_k}\sqrt{\omega_{k'}}}\,[\hat{\pi}^\mu(x),\hat{\pi}^\nu(y)]\, e^{i(\omega_k t+\boldsymbol{k}\cdot\boldsymbol{x})}\, e^{i(\omega_{k'}t+\boldsymbol{k'}\cdot\boldsymbol{y})}\Big),\end{aligned}$$

using the equal-time commutation relations we find

$$\begin{aligned}\sum_{\lambda,\lambda'}[\hat{a}(\boldsymbol{k},\lambda),\hat{a}^\dagger(\boldsymbol{k'},\lambda')]\,\epsilon^\mu(\boldsymbol{k},\lambda)\,\epsilon^\nu(\boldsymbol{k'},\lambda') &= \frac{1}{2}\frac{1}{(2\pi)^3}\,\eta^{\mu\nu}\int d^3x\left(\sqrt{\frac{\omega_k}{\omega_{k'}}}-\sqrt{\frac{\omega_{k'}}{\omega_k}}\right)e^{i(\omega_k+\omega_{k'})t}\,e^{i(\boldsymbol{k}+\boldsymbol{k'})}\\ &= \frac{1}{2}\,\eta^{\mu\nu}\left(\sqrt{\frac{\omega_k}{\omega_{k'}}}-\sqrt{\frac{\omega_{k'}}{\omega_k}}\right)e^{i(\omega_k+\omega_{k'})t}\,\delta^{(3)}(\boldsymbol{k}+\boldsymbol{k'})\\ &= 0\end{aligned}$$

For symmetry $[\hat{a}^\dagger(\boldsymbol{k},\lambda),\hat{a}^\dagger(\boldsymbol{k'},\lambda')]=0$

We have finally found the usual harmonic oscillator algebra

$$\begin{aligned}&[\hat{a}(\boldsymbol{k},\lambda),\hat{a}^\dagger(\boldsymbol{k'},\lambda')] = -\eta_{\lambda\lambda'}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'})\\ &[\hat{a}(\boldsymbol{k},\lambda),\hat{a}(\boldsymbol{k'},\lambda')] = [\hat{a}^\dagger(\boldsymbol{k},\lambda),\hat{a}^\dagger(\boldsymbol{k'},\lambda')] = 0.\end{aligned}$$

d) Using the relations

$$\begin{aligned}\hat{A}^\mu(x) &= \int\frac{d^3k}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}\sum_\lambda\Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx}+\hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big),\\ \hat{\pi}^\mu(x) &= -\partial_0\hat{A}^\mu(x) = i\int\frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{\omega_k}{2}}\sum_\lambda\Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx}-\hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big),\\ \boldsymbol{\nabla}\hat{A}^\mu(x) &= i\int\frac{d^3k}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}\boldsymbol{k}\sum_\lambda\Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx}-\hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big),\end{aligned}$$

the normal ordered Hamiltonian will be

$$\begin{aligned}\hat{H} &= -\frac{1}{2}\int d^3x\,\boldsymbol{:}\Big(\hat{\pi}_\mu\hat{\pi}^\mu+\boldsymbol{\nabla}A_\mu\cdot\boldsymbol{\nabla}A^\mu\Big)\boldsymbol{:}\\ &= -\frac{1}{2}\int\frac{d^3x}{(2\pi)^3}\frac{d^3k}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\sum_{\lambda,\lambda'}\boldsymbol{:}\Big(-\omega_k\omega_{k'}\Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon_\mu(\boldsymbol{k},\lambda)e^{-ikx}-\hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon_\mu(\boldsymbol{k},\lambda)e^{ikx}\Big)\times\\ &\qquad\times\Big(\hat{a}(\boldsymbol{k'},\lambda')\epsilon^\mu(\boldsymbol{k'},\lambda')e^{-ik'x}-\hat{a}^\dagger(\boldsymbol{k'},\lambda')\epsilon^\mu(\boldsymbol{k'},\lambda')e^{ik'x}\Big)+\\ &\qquad-\boldsymbol{k}\cdot\boldsymbol{k'}\Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon_\mu(\boldsymbol{k},\lambda)e^{-ikx}-\hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon_\mu(\boldsymbol{k},\lambda)e^{ikx}\Big)\times\\ &\qquad\times\Big(\hat{a}(\boldsymbol{k'},\lambda')\epsilon^\mu(\boldsymbol{k'},\lambda')e^{-ik'x}-\hat{a}^\dagger(\boldsymbol{k'},\lambda')\epsilon^\mu(\boldsymbol{k'},\lambda')e^{ik'x}\Big)\Big)\boldsymbol{:}\\ &= \frac{1}{2}\int\frac{d^3x}{(2\pi)^3}\frac{d^3k}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\sum_{\lambda,\lambda'}(\omega_k\omega_{k'}+\boldsymbol{k}\cdot\boldsymbol{k'})\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k'},\lambda')\times\\ &\qquad\times\boldsymbol{:}\Big(\hat{a}(\boldsymbol{k},\lambda)e^{-ikx}-\hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}\Big)\Big(\hat{a}(\boldsymbol{k'},\lambda')e^{-ik'x}-\hat{a}^\dagger(\boldsymbol{k'},\lambda')e^{ik'x}\Big)\boldsymbol{:}\,.\end{aligned}$$

The normal ordered term (integrated over space) becomes

$$\begin{aligned}&\int\frac{d^3x}{(2\pi)^3}\boldsymbol{:}\Big(\hat{a}(\boldsymbol{k},\lambda)e^{-ikx}-\hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}\Big)\Big(\hat{a}(\boldsymbol{k'},\lambda')e^{-ik'x}-\hat{a}^\dagger(\boldsymbol{k'},\lambda')e^{ik'x}\Big)\boldsymbol{:}=\\ &\qquad= \int\frac{d^3x}{(2\pi)^3}\Big(\hat{a}(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k'},\lambda')e^{-i(k+k')x}-\hat{a}^\dagger(\boldsymbol{k'},\lambda')\hat{a}(\boldsymbol{k},\lambda)e^{-i(k-k')x}+\\ &\qquad\quad-\hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k'},\lambda')e^{i(k-k')x}+\hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}^\dagger(\boldsymbol{k'},\lambda')e^{i(k+k')x}\Big)\\ &\qquad= \Big(\hat{a}(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k'},\lambda')e^{-i(\omega_k+\omega_{k'})t}+\hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}^\dagger(\boldsymbol{k'},\lambda')e^{i(\omega_k+\omega_{k'})t}\Big)\delta^{(3)}(\boldsymbol{k}+\boldsymbol{k'})+\\ &\qquad\quad-\Big(\hat{a}^\dagger(\boldsymbol{k'},\lambda')\hat{a}(\boldsymbol{k},\lambda)e^{-i(\omega_k-\omega_{k'})t}+\hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k'},\lambda')e^{i(\omega_k-\omega_{k'})t}\Big)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k'}).\end{aligned}$$

Thus, with an integration over $\boldsymbol{k}'$ in $\hat{H}$, we find

$$\hat{H} = \frac{1}{2}\int \frac{d^3k}{2\omega_k}\sum_{\lambda,\lambda'}(\omega_k^2 - |\boldsymbol{k}|^2)\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(-\boldsymbol{k},\lambda')\Big(\Big(\hat{a}(\boldsymbol{k},\lambda)\hat{a}(-\boldsymbol{k},\lambda')e^{-2i\omega_k t} + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}^\dagger(-\boldsymbol{k},\lambda')e^{2i\omega_k t}\Big) + \\ - (\omega_k^2 + |\boldsymbol{k}|^2)\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda')\Big(\hat{a}^\dagger(\boldsymbol{k},\lambda')\hat{a}(\boldsymbol{k},\lambda) + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda')\Big)\Big).$$

With the massless dispersion relation $\omega_k^2 = |\boldsymbol{k}|^2$ and then the orthogonality relation $\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda') = \eta_{\lambda\lambda'}$, we obtain

$$\begin{aligned}
\hat{H} &= -\frac{1}{2}\int d^3k\,\omega_k \sum_{\lambda,\lambda'}\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda')\Big(\hat{a}^\dagger(\boldsymbol{k},\lambda')\hat{a}(\boldsymbol{k},\lambda) + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda')\Big) \\
&= -\frac{1}{2}\int d^3k\,\omega_k \sum_{\lambda,\lambda'}\eta_{\lambda\lambda'}\Big(\hat{a}^\dagger(\boldsymbol{k},\lambda')\hat{a}(\boldsymbol{k},\lambda) + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda')\Big) \\
&= -\int d^3k\,\omega_k \sum_{\lambda}\eta_{\lambda\lambda}\hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda).
\end{aligned}$$

Defining the number operator $\hat{n}(\boldsymbol{k},\lambda) = \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda)$, we eventually arrive at

$$\hat{H} = -\sum_\lambda \eta_{\lambda\lambda}\int d^3k\,\omega_k\,\hat{n}(\boldsymbol{k},\lambda).$$

From Noether theorem, we find the expression for the momentum

$$\begin{aligned}
\hat{\boldsymbol{P}} &= \int d^3x :\dot{\hat{A}}_\mu(x)\boldsymbol{\nabla}\hat{A}^\mu(x): \\
&= \int \frac{d^3x}{(2\pi)^3}\frac{d^3k}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\sum_{\lambda,\lambda'}\omega_k\boldsymbol{k}' :\Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon_\mu(\boldsymbol{k},\lambda)e^{-ikx} - \hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon_\mu(\boldsymbol{k},\lambda)e^{ikx}\Big)\times \\
&\qquad \times\Big(\hat{a}(\boldsymbol{k}',\lambda')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-ik'x} - \hat{a}^\dagger(\boldsymbol{k}',\lambda')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{ik'x}\Big): \\
&= \int \frac{d^3x}{(2\pi)^3}\frac{d^3k}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\sum_{\lambda,\lambda'}\omega_k\boldsymbol{k}'\,\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k}',\lambda') :\Big(\hat{a}(\boldsymbol{k},\lambda)e^{-ikx} - \hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}\Big)\times \\
&\qquad \times\Big(\hat{a}(\boldsymbol{k}',\lambda')e^{-ik'x} - \hat{a}^\dagger(\boldsymbol{k}',\lambda')e^{ik'x}\Big):.
\end{aligned}$$

Using the previous result, we find

$$\begin{aligned}
\hat{\boldsymbol{P}} &= \int \frac{d^3k}{\sqrt{2\omega_k}}\frac{d^3k'}{\sqrt{2\omega_{k'}}}\sum_{\lambda,\lambda'}\omega_k\boldsymbol{k}'\,\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k}',\lambda')\times \\
&\qquad \times\Big(\Big(\hat{a}(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k}',\lambda')e^{-i(\omega_k+\omega_{k'})t} + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}^\dagger(\boldsymbol{k}',\lambda')e^{i(\omega_k+\omega_{k'})t}\Big)\delta^{(3)}(\boldsymbol{k}+\boldsymbol{k}') + \\
&\qquad - \Big(\hat{a}^\dagger(\boldsymbol{k}',\lambda')\hat{a}(\boldsymbol{k},\lambda)e^{-i(\omega_k-\omega_{k'})t} + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k}',\lambda')e^{i(\omega_k-\omega_{k'})t}\Big)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\Big) \\
&= \int \frac{d^3k}{2\omega_k}\sum_{\lambda,\lambda'}\Big(-\omega_k\boldsymbol{k}\,\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(-\boldsymbol{k},\lambda')\Big(\hat{a}(\boldsymbol{k},\lambda)\hat{a}(-\boldsymbol{k},\lambda')e^{-2i\omega_k t} + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}^\dagger(-\boldsymbol{k},\lambda')e^{2i\omega_k t}\Big) + \\
&\qquad - \omega_k\boldsymbol{k}\,\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k}',\lambda')\Big(\hat{a}^\dagger(\boldsymbol{k},\lambda')\hat{a}(\boldsymbol{k},\lambda) + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda')\Big)\Big).
\end{aligned}$$

The terms in the first line are equal to zero due to symmetry reasons, since the integrand is an odd function of $\boldsymbol{k}$. This is a consequence of the fact that $[\hat{a}(\boldsymbol{k},\lambda),\hat{a}(-\boldsymbol{k},\lambda')] = [\hat{a}^\dagger(\boldsymbol{k},\lambda),\hat{a}^\dagger(-\boldsymbol{k},\lambda')] = 0.$

Using the orthogonality relation $\epsilon_\mu(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda') = \eta_{\lambda\lambda'}$, we obtain

$$\begin{aligned}
\hat{\boldsymbol{P}} &= -\int \frac{d^3k}{2\omega_k} \sum_{\lambda,\lambda'} \omega_k \boldsymbol{k}\, \eta_{\lambda\lambda'} \Big(\hat{a}^\dagger(\boldsymbol{k},\lambda')\hat{a}(\boldsymbol{k},\lambda) + \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda') \Big) \\
&= -\int d^3k \sum_\lambda \boldsymbol{k}\, \eta_{\lambda\lambda} \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda) \\
&= -\sum_\lambda \eta_{\lambda\lambda} \int d^3k\, \boldsymbol{k}\, \hat{n}(\boldsymbol{k},\lambda).
\end{aligned}$$

In both cases, we have the expected operators, except for the time-like photons which carry the wrong sign:

$$\begin{aligned}
\hat{H} &= -\int d^3k\, \omega_k\, \hat{n}(\boldsymbol{k},0) + \sum_{\lambda=1}^{3} \int d^3k\, \omega_k\, \hat{n}(\boldsymbol{k},\lambda) \\
\hat{\boldsymbol{P}} &= -\int d^3k\, \boldsymbol{k}\, \hat{n}(\boldsymbol{k},0) + \sum_{\lambda=1}^{3} \int d^3k\, \boldsymbol{k}\, \hat{n}(\boldsymbol{k},\lambda).
\end{aligned}$$

e) We have found quantities with wrong signs in the commutation relation, the Hamiltonian and the momentum. This fact has undesired consequences, like imaginary norms for the time-like photons in the Fock space:

$$\begin{aligned}
\langle 1_{\boldsymbol{k},0} | 1_{\boldsymbol{k},0} \rangle &= \langle 0 | \hat{a}(\boldsymbol{k},0)\hat{a}^\dagger(\boldsymbol{k},0) | 0 \rangle \\
&= \langle 0 | \hat{a}^\dagger(\boldsymbol{k},0)\hat{a}(\boldsymbol{k},0) | 0 \rangle - \delta^{(3)}(\boldsymbol{0})\, \langle 0|0 \rangle \\
&= -\delta^{(3)}(\boldsymbol{0}).
\end{aligned}$$

These problems are connected to the Lorenz gauge, which has not been implemented. As we saw in point (c), we cannot impose $\partial^\mu \hat{A}_\mu$. The Gupta-Bleuer method gets rid of this problem by restricting the Hilbert space to states $|\phi\rangle$ such that

$$\langle \phi | \partial^\mu \hat{A}_\mu | \phi \rangle = 0.$$

We will implement the stronger condition

$$\partial^\mu \hat{A}_\mu^{(+)} |\phi\rangle = 0,$$

so that, with the adjoint one $\langle\phi|\, \partial^\mu \hat{A}_\mu^{(-)} = 0$, we obtain the first constraint

$$\langle \phi | \partial^\mu \hat{A}_\mu | \phi \rangle = \langle \phi | \partial^\mu \hat{A}_\mu^{(+)} | \phi \rangle + \langle \phi | \partial^\mu \hat{A}_\mu^{(-)} | \phi \rangle = 0.$$

Expanding the field $\partial^\mu \hat{A}_\mu^{(+)}$, the condition becomes

$$-i \sum_\lambda \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\, k^\mu \epsilon_\mu(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda) e^{-ikx} |\phi\rangle = 0.$$

With the orthogonality relations

$$\begin{aligned}
k^\mu \epsilon_\mu(\boldsymbol{k},0) &= -k^\mu \epsilon_\mu(\boldsymbol{k},3) \\
k^\mu \epsilon_\mu(\boldsymbol{k},1) &= k^\mu \epsilon_\mu(\boldsymbol{k},2) = 0,
\end{aligned}$$

we find

$$-i \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\, k^\mu \epsilon_\mu(\boldsymbol{k},0) e^{-ikx} \Big(\hat{a}(\boldsymbol{k},0) - \hat{a}(\boldsymbol{k},3) \Big) |\phi\rangle = 0.$$

A sufficient condition is that

$$\Big(\hat{a}(\boldsymbol{k},0) - \hat{a}(\boldsymbol{k},3) \Big) |\phi\rangle = 0.$$

As a consequence, the expectation values of the time-like and longitudinal number operators are equal

$$\langle\phi|\hat{n}(\boldsymbol{k},0)|\phi\rangle = \langle\phi|\hat{n}(\boldsymbol{k},3)|\phi\rangle$$

and thus all the expectation values of observables previously computed have the correct expressions and dof:

$$\begin{aligned}
\langle\phi|\hat{H}|\phi\rangle &= \sum_{\lambda=1}^{2}\int d^3k\,\omega_k\;\langle\phi|\hat{n}(\boldsymbol{k},\lambda)|\phi\rangle \\
\langle\phi|\hat{\boldsymbol{P}}|\phi\rangle &= \sum_{\lambda=1}^{2}\int d^3k\,\boldsymbol{k}\;\langle\phi|\hat{n}(\boldsymbol{k},\lambda)|\phi\rangle\,.
\end{aligned}$$

14 The quantized electromagnetic field

a) Consider the quantized electromagnetic potential $\hat{A}^\mu(x)$. Show that the Feynman propagator $D_F^{\mu\nu}(x-y) = -i\langle 0|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|0\rangle$ can be written in the form:

$$D_F^{\mu\nu}(x-y) = \int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{-\eta^{\mu\nu}}{k^2+i\epsilon}.$$

b) The photon propagator could take a simple form only because the summation over the polarization states was not restricted to the transverse polarizations only. Show that the propagator can be split as follows:

$$D_F^{\mu\nu}(k) = \frac{-\eta^{\mu\nu}}{k^2+i\epsilon} = D^{\mu\nu}_{F\,(\text{transverse})}(k) + D^{\mu\nu}_{F(\text{Coulomb})}(k) + D^{\mu\nu}_{F(\text{residual})}(k)$$

and explain the meaning and role of the three terms.

c) Compute the commutation relations $[\hat{A}^\mu(x),\hat{A}^\nu(y)]$ and show where they are non-vanishing. Comment on the meaning of the result.

d) Compute the commutation relations among the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields and show that microcausality is satisfied.

e) Briefly review the quantization of the electromagnetic field in the Coulomb gauge.

---

a) The Feynman propagator is defined as

$$\begin{aligned}
iD_F^{\mu\nu}(x-y) &= \langle 0|T\Big(\hat{A}^\mu(x)\hat{A}^\nu(y)\Big)|0\rangle \\
&= \Theta(x^0-y^0)\,\langle 0|\hat{A}^\mu(x)\hat{A}^\nu(y)|0\rangle + \Theta(y^0-x^0)\,\langle 0|\hat{A}^\nu(y)\hat{A}^\mu(x)|0\rangle\,.
\end{aligned}$$

The first term, when expanding the field into plane waves, becomes

$$\begin{aligned}
\langle 0|\hat{A}^\mu(x)\hat{A}^\nu(y)|0\rangle &= \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}}\sum_{\lambda,\lambda'}\times \\
&\quad\times\langle 0|\Big(\epsilon^\mu(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda)e^{-ikx} + \epsilon^\mu(\boldsymbol{k},\lambda)\hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}\Big)\times \\
&\quad\times\Big(\epsilon^\nu(\boldsymbol{k}',\lambda')\hat{a}(\boldsymbol{k}',\lambda')e^{-ik'y} + \epsilon^\nu(\boldsymbol{k}',\lambda')\hat{a}^\dagger(\boldsymbol{k}',\lambda')e^{ik'y}\Big)|0\rangle \\
&= \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}}\sum_{\lambda,\lambda'}\epsilon^\mu(\boldsymbol{k},\lambda)\epsilon^\nu(\boldsymbol{k}',\lambda')\,\langle 0|\hat{a}(\boldsymbol{k},\lambda)\hat{a}^\dagger(\boldsymbol{k}',\lambda')|0\rangle\,e^{-i(kx-k'y)} \\
&= -\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}}\sum_{\lambda,\lambda'}\epsilon^\mu(\boldsymbol{k},\lambda)\epsilon^\nu(\boldsymbol{k}',\lambda')\,\eta_{\lambda\lambda'}\,\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,e^{-i(kx-k'y)} \\
&= -\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\sum_\lambda\eta_{\lambda\lambda}\epsilon^\mu(\boldsymbol{k},\lambda)\epsilon^\nu(\boldsymbol{k},\lambda)\,e^{-ik(x-y)} \\
&= -\int\frac{d^3k}{(2\pi)^3}\frac{\eta^{\mu\nu}}{2\omega_k}\,e^{-ik(x-y)}
\end{aligned}$$

The second term can be treated similarly. We then have

$$iD_F^{\mu\nu}(x-y) = -\int \frac{d^3k}{(2\pi)^3}\frac{\eta^{\mu\nu}}{2\omega_k}\left(\Theta(x^0-y^0)e^{-ik(x-y)} + \Theta(y^0-x^0)e^{ik(x-y)}\right).$$

With the change of variable $\boldsymbol{k} \to -\boldsymbol{k}$ in the second integral, we obtain

$$iD_F^{\mu\nu}(x-y) = -\int \frac{d^3k}{(2\pi)^3}\frac{\eta^{\mu\nu}}{2\omega_k}\left(\Theta(x^0-y^0)e^{-i\omega_k(x^0-y^0)} + \Theta(y^0-x^0)e^{i\omega_k(x^0-y^0)}\right)e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}.$$

We can rewrite the integral as a four-dimensional integral (in d^4k), by looking at the term inside the integral as a residue. In fact, setting

$$f(k^0) = -\frac{e^{-ik^0(x^0-y^0)}}{(k^0-\omega_k+i\epsilon)(k^0+\omega_k-i\epsilon)}$$

we have two cases. If $y^0 > x^0$ we can integrate f along the contour Γ^+ in figure and, using the Residue theorem,

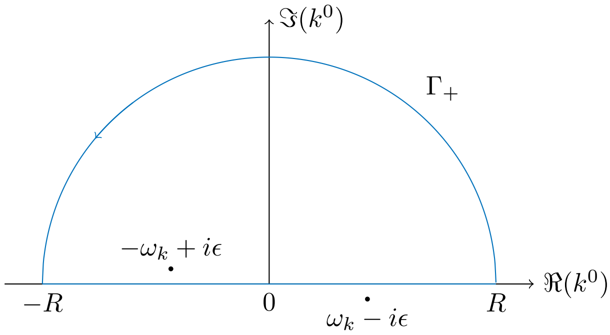
$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma^+} dk^0\, f(k^0) &= \text{Res}(-\omega_k+i\epsilon)\\ &= \lim_{k^0\to-\omega_k+i\epsilon}(k^0+\omega_k-i\epsilon)f(k^0)\\ &= \frac{e^{i\omega_k(x^0-y^0)+\epsilon(x^0-y^0)}}{2\omega_k-2i\epsilon}\\ &= \frac{e^{i\omega_k(x^0-y^0)}}{2\omega_k}.\end{aligned}$$

On the other hand, the contribution along the semicircle is zero, since

$$\lim_{|k^0|\to\infty} k^0\, f(k^0) = 0.$$

Thus,

$$-\frac{1}{2\pi i}\int dk^0\, \frac{e^{-ik^0(x^0-y^0)}}{(k^0-\omega_k+i\epsilon)(k^0+\omega_k-i\epsilon)} = \Theta(y^0-x^0)\frac{e^{i\omega_k(x^0-y^0)}}{2\omega_k}.$$



With analogous calculations for $x^0 > y^0$, by integrating along a contour Γ^- in the lower half plane, we find

$$-\frac{1}{2\pi i}\int dk^0\, \frac{e^{-ik^0(x^0-y^0)}}{(k^0-\omega_k+i\epsilon)(k^0+\omega_k-i\epsilon)} = \Theta(x^0-y^0)\frac{e^{-i\omega_k(x^0-y^0)}}{2\omega_k}.$$

On the other hand,

$$\begin{aligned}f(k^0) &= -\frac{e^{-ik^0(x^0-y^0)}}{(k^0-\omega_k+i\epsilon)(k^0+\omega_k-i\epsilon)}\\ &= -\frac{e^{-ik^0(x^0-y^0)}}{(k^0)^2-\omega_k^2+2i\omega_k\epsilon+\epsilon^2}\\ &= -\frac{e^{-ik^0(x^0-y^0)}}{k^2+2i\omega_k\epsilon}\end{aligned}$$

Writing ϵ instead of $2\omega_k\epsilon$, we have

$$\begin{aligned} D_F^{\mu\nu}(x-y) &= i\int \frac{d^3k}{(2\pi)^3}\frac{\eta^{\mu\nu}}{2\pi i}\int dk^0\, \frac{-e^{-ik^0(x^0-y^0)}}{k^2+i\epsilon} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{-\eta^{\mu\nu}}{k^2+i\epsilon} e^{-ik(x-y)}. \end{aligned}$$

With this expression, it can be easily seen that the Feynman propagator is a Green function for the wave equation with a Dirac delta as a source term:

$$\begin{aligned} \Box_x D_F^{\mu\nu}(x-y) &= \int \frac{d^4k}{(2\pi)^4}\frac{-\eta^{\mu\nu}}{k^2+i\epsilon}(-k^2)e^{-ik(x-y)} \\ &= \eta^{\mu\nu}\delta^{(4)}(x-y). \end{aligned}$$

b) From the above steps, we find that the Fourier transform of the Feynman propagator is

$$\begin{aligned} D_F^{\mu\nu}(k) &= \frac{-\eta^{\mu\nu}}{k^2+i\epsilon} = \frac{1}{k^2+i\epsilon}\sum_\lambda (-\eta_{\lambda\lambda})\epsilon^\mu(\boldsymbol{k},\lambda)\epsilon^\nu(\boldsymbol{k},\lambda) \\ &= \underbrace{\frac{1}{k^2+i\epsilon}\sum_{\lambda=1}^{2}\epsilon^\mu(\boldsymbol{k},\lambda)\epsilon^\nu(\boldsymbol{k},\lambda)}_{=D^{\mu\nu}_{F(\text{trans})}(k)} + \frac{1}{k^2+i\epsilon}\epsilon^\mu(\boldsymbol{k},3)\epsilon^\nu(\boldsymbol{k},3) - \frac{1}{k^2+i\epsilon}\epsilon^\mu(\boldsymbol{k},0)\epsilon^\nu(\boldsymbol{k},0) \end{aligned}$$

On the other hand, reminding that $n = (1\,0\,0\,0)$,

$$\begin{aligned} \epsilon^\mu(\boldsymbol{k},3)\epsilon^\nu(\boldsymbol{k},3) - \epsilon^\mu(\boldsymbol{k},0)\epsilon^\nu(\boldsymbol{k},0) &= \frac{k^\mu-(kn)n^\mu}{\sqrt{(kn)^2-k^2}}\frac{k^\nu-(kn)n^\nu}{\sqrt{(kn)^2-k^2}} - n^\mu n^\nu \\ &= \frac{k^\mu k^\nu - (kn)k^\mu n^\nu - (kn)k^\nu n^\mu + k^2 n^\mu n^\nu}{(kn)^2-k^2} \\ &= \frac{k^\mu k^\nu - (kn)(k^\mu n^\nu + k^\nu n^\mu)}{(kn)^2-k^2} + \frac{k^2 n^\mu n^\nu}{(kn)^2-k^2}, \end{aligned}$$

so that

$$\begin{aligned} D_F^{\mu\nu}(k) &= D^{\mu\nu}_{F(\text{trans})}(k) + \frac{1}{k^2+i\epsilon}\frac{k^2 n^\mu n^\nu}{(kn)^2-k^2} + \frac{1}{k^2+i\epsilon}\frac{k^\mu k^\nu - (kn)(k^\mu n^\nu + k^\nu n^\mu)}{(kn)^2-k^2} \\ &= D^{\mu\nu}_{F(\text{trans})}(k) + \underbrace{\frac{n^\mu n^\nu}{(kn)^2-k^2}}_{=D^{\mu\nu}_{F(\text{Coul})}(k)} + \underbrace{\frac{1}{k^2+i\epsilon}\frac{k^\mu k^\nu - (kn)(k^\mu n^\nu + k^\nu n^\mu)}{(kn)^2-k^2}}_{=D^{\mu\nu}_{F(\text{res})}(k)} \\ &= D^{\mu\nu}_{F(\text{trans})}(k) + D^{\mu\nu}_{F(\text{Coul})}(k) + D^{\mu\nu}_{F(\text{res})}(k). \end{aligned}$$

The transverse part takes into account the transverse photons, which reflect the physical dof. On the other hand, in our particular frame of reference the Coulomb term in position space is

$$\begin{aligned} D^{\mu\nu}_{F(\text{Coul})}(x-y) &= \int \frac{d^4k}{(2\pi)^4} D^{\mu\nu}_{F(\text{Coul})}(k)e^{ik(x-y)} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{\delta^{\mu 0}\delta^{\nu 0}}{|\boldsymbol{k}|^2}e^{ik(x-y)} \\ &= \delta^{\mu 0}\delta^{\nu 0}\,\delta(x^0-y^0)\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{|\boldsymbol{k}|^2} \end{aligned}$$

The integral is the anti-Fourier transform of the Coulomb potential, so that

$$D^{\mu\nu}_{F(\text{Coul})}(x-y) = \delta^{\mu 0}\delta^{\nu 0}\,\delta(x^0-y^0)\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}.$$

In the interaction theory, the contribution to the transition amplitude due to a photonic propagator is given by terms of the form

$$j_\mu \, iD_F^{\mu\nu} \, j'_\nu,$$

where j_μ and j'_ν are currents (*e.g.*, in Quantum Electrodynamics $j_\mu = j'_\mu = -e\bar{\psi}\gamma_\mu\psi$). Thus, $D^{\mu\nu}_{F(\mathrm{Coul})}$ propagates instantaneously the interaction with the charge distributions j_0 and j'_0 only (which is a Coulomb interaction). The residual term does not give any contribution, since the continuity equation in momentum space becomes

$$k^\mu j_\mu = k^\nu j'_\nu = 0,$$

so that $j_\mu \, iD^{\mu\nu}_{F(\mathrm{res})} \, j'_\nu = 0$.

c) Expanding the potential field $\hat{A}^\mu$ into plane waves, we find

$$\begin{aligned}[\hat{A}^\mu(x), \hat{A}^\nu(y)] = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}} \sum_{\lambda,\lambda'} \times \\ \times \Big[\epsilon^\mu(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda)e^{-ikx} + \epsilon^\mu(\boldsymbol{k},\lambda)\hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}, \\ \epsilon^\nu(\boldsymbol{k}',\lambda')\hat{a}(\boldsymbol{k}',\lambda')e^{-ik'y} + \epsilon^\nu(\boldsymbol{k}',\lambda')\hat{a}^\dagger(\boldsymbol{k}',\lambda')e^{ik'y}\Big]\end{aligned}$$

$$\begin{aligned}[\hat{A}^\mu(x), \hat{A}^\nu(y)] &= \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \frac{d^3k'}{\sqrt{(2\pi)^3 2\omega_{k'}}} \sum_{\lambda,\lambda'} \epsilon^\mu(\boldsymbol{k},\lambda)\epsilon^\nu(\boldsymbol{k}',\lambda') \times \\ &\quad \times \Big(\Big[\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k}',\lambda')\Big]e^{-ikx}e^{ik'y} + \Big[\hat{a}^\dagger(\boldsymbol{k},\lambda), \hat{a}(\boldsymbol{k}',\lambda')\Big]e^{ikx}e^{-ik'y}\Big) \\ &= -\eta^{\mu\nu} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big(e^{-ik(x-y)} - e^{ik(x-y)}\Big) \\ &= -i\eta^{\mu\nu} D(x-y),\end{aligned}$$

where we have set

$$\begin{aligned}D(x-y) &= -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big(e^{-ik(x-y)} - e^{ik(x-y)}\Big) \\ &= D^{(+)}(x-y) + D^{(-)}(x-y) = 2\Re\Big(D^{(+)}(x-y)\Big).\end{aligned}$$

Let us write $D^{(+)}(z)$ in spherical coordinates, using the fact that $\omega_k = |k|$ and setting $r = |z|$,

$$\begin{aligned}D^{(+)}(z) &= -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-i(\omega_k t - \boldsymbol{k}\cdot\boldsymbol{z})} \\ &= -i \int_0^{+\infty} \frac{d\omega_k}{(2\pi)^2} \frac{\omega_k^2}{2\omega_k} e^{-i\omega_k t} \int_{-1}^{+1} d(\cos\vartheta)\, e^{i\omega_k r\cos\vartheta} \\ &= -\frac{1}{8\pi^2}\frac{1}{r} \int_0^{+\infty} d\omega_k \left(e^{i\omega_k(r-t)} - e^{-i\omega_k(r+t)}\right) \\ &= \lim_{\epsilon\to 0^+} -\frac{1}{8\pi^2}\frac{1}{r} \int_0^{+\infty} d\omega_k \left(e^{i\omega_k(r-t+i\epsilon)} - e^{-i\omega_k(r+t-i\epsilon)}\right) \\ &= \lim_{\epsilon\to 0^+} -i\frac{1}{8\pi^2}\frac{1}{r}\left(\frac{1}{r-t+i\epsilon} + \frac{1}{r+t-i\epsilon}\right).\end{aligned}$$

From this we can find

$$\begin{aligned}D(z) = 2\Re\Big(D^{(+)}(z)\Big) &= \lim_{\epsilon\to 0^+} -\frac{1}{4\pi^2}\frac{1}{r}\Re\left(\frac{i}{r-t+i\epsilon} + \frac{i}{r+t-i\epsilon}\right) \\ &= \lim_{\epsilon\to 0^+} -\frac{1}{4\pi^2}\frac{1}{r}\left(\frac{\epsilon}{(r-t)^2+\epsilon^2} - \frac{\epsilon}{(r+t)^2+\epsilon^2}\right).\end{aligned}$$

Using the expression for the delta
$$\delta(a)=\lim_{\epsilon\to 0^+}\frac{1}{\pi}\frac{\epsilon}{a^2+\epsilon^2},$$
we can write
$$D(z)=-\frac{1}{4\pi}\frac{1}{r}\big(\delta(r-t)-\delta(r+t)\big).$$
Using the relation
$$\delta(r^2-t^2)=\frac{1}{2r}\big(\delta(r-t)+\delta(r+t)\big)=\frac{1}{2r}\epsilon(t)\big(\delta(r-t)-\delta(r+t)\big).$$
We obtain
$$D(z)=-\frac{1}{2\pi}\epsilon(t)\,\delta(r^2-t^2)=-\frac{1}{2\pi}\epsilon(t)\,\delta(z^2).$$
We can finally write
$$D(x-y)=-\frac{1}{2\pi}\epsilon(t)\,\delta\big((x-y)^2\big)\,.$$
As a result, the electromagnetic potential field components commute everywhere except that on the light cone. This is not a physical result yet, since the potential is not a physical quantity. Nevertheless, from this relation we will prove that the components of the electromagnetic field commute outside the light cone, that is the measurements of these observables do not interfere between each other, except in the case of light-like separation. The massless hypothesis guarantees that the photons can travel only at the speed of light.

d) Now, the electric and magnetic fields are
$$\hat{E}^i=\partial^i\hat{A}^0-\partial^0\hat{A}^i$$
$$\hat{B}^i=\epsilon^{ijk}\partial_j\hat{A}_k.$$

Thus, we can compute the commutation relations between the components.
$$\begin{aligned}\Big[\hat{E}^i(x),\hat{E}^j(y)\Big]&=\Big[\partial_x^0\hat{A}^i(x),\partial_y^0\hat{A}^j(y)\Big]-\Big[\partial_x^i\hat{A}^0(x),\partial_y^0\hat{A}^j(y)\Big]-\Big[\partial_x^0\hat{A}^i(x),\partial_y^j\hat{A}^0(y)\Big]+\Big[\partial_x^i\hat{A}^0(x),\partial_y^j\hat{A}^0(y)\Big]\\&=\partial_x^0\partial_y^0\Big[\hat{A}^i(x),\hat{A}^j(y)\Big]-\partial_x^i\partial_y^0\Big[\hat{A}^0(x),\hat{A}^j(y)\Big]-\partial_x^0\partial_y^j\Big[\hat{A}^i(x),\hat{A}^0(y)\Big]+\partial_x^i\partial_y^j\Big[\hat{A}^0(x),\hat{A}^0(y)\Big]\\&=-i\Big(\eta^{ij}\partial_x^0\partial_y^0+\partial_x^i\partial_y^j\Big)D(x-y).\end{aligned}$$
On the other hand, the function D satisfies $\partial_x^\mu D(x-y)=-\partial_y^\mu D(x-y)$, so that setting simply $\partial_x^\mu D(x-y)=\partial^\mu D(x-y)$
$$\begin{aligned}\Big[\hat{E}^i(x),\hat{E}^j(y)\Big]&=i\Big(\eta^{ij}\partial^0\partial^0+\partial^i\partial^j\Big)D(x-y)\\&=-i\Big(\delta^{ij}\partial^0\partial^0-\partial^i\partial^j\Big)D(x-y).\end{aligned}$$

Analogously,
$$\begin{aligned}\Big[\hat{B}_i(x),\hat{B}_j(y)\Big]&=\Big[\epsilon_{ikl}\,\partial_x^k\hat{A}^l(x),\epsilon_{jmn}\,\partial_y^m\hat{A}^n(y)\Big]=\epsilon_{ikl}\,\epsilon_{jmn}\,\partial_x^k\partial_y^m\Big[\hat{A}^l(x),\hat{A}^n(y)\Big]\\&=-i\,\epsilon_{ikl}\,\epsilon_{jmn}\,(-\delta^{ln})\,\partial_x^k\partial_y^m D(x-y)=i\sum_n\epsilon_{ikn}\,\epsilon_{jmn}\,\partial_x^k\partial_y^m D(x-y)\\&=-i(\delta_{ij}\delta_{km}-\delta_{im}\delta_{jk})\,\partial_x^k\partial_x^m D(x-y)\\&=-i\Big(\delta_{ij}\nabla_x^2-\partial_i^x\partial_j^x\Big)D(x-y),\end{aligned}$$

where in the third step we used again the fact that $\partial_x^\mu D(x-y)=-\partial_y^\mu D(x-y)$. But the function D satisfies the wave equation: $\nabla_x^2 D(x-y)=\partial_x^0\partial_x^0 D(x-y)$, so that
$$\begin{aligned}\Big[\hat{B}_i(x),\hat{B}_j(y)\Big]&=-i\Big(\delta_{ij}\partial_x^0\partial_x^0-\partial_i^x\partial_j^x\Big)D(x-y)\\&=-i\Big(\delta_{ij}\partial^0\partial^0-\partial_i\partial_j\Big)D(x-y).\end{aligned}$$

Last,

$$\begin{aligned}\left[\hat{E}_i(x), \hat{B}_j(y)\right] &= -\left[\partial_0^x \hat{A}_i(x), \epsilon_{jmn}\, \partial_y^m \hat{A}^n(y)\right] + \left[\partial_i^x \hat{A}_0(x), \epsilon_{jmn}\, \partial_y^m \hat{A}^n(y)\right] \\ &= -\epsilon_{jmn} \partial_0^x \partial_y^m \left[\hat{A}_i(x), \hat{A}^n(y)\right] + \epsilon_{jmn} \partial_i^x \partial_y^m \left[\hat{A}_0(x), \hat{A}^n(y)\right] \\ &= i\, \epsilon_{jmn} \delta_i^n\, \partial^0 \partial^m D(x-y) \\ &= i\, \epsilon_{jmi}\, \partial^0 \partial^m D(x-y).\end{aligned}$$

The commutators are all second derivatives of the function $D(x-y)$, which vanishes everywhere but on the light cone. Thus, the commutators vanishes too in such regions, so that the measurements of these observables do not interfere between each other, except in the case of light-like separation. This fact proves that microcausality condition holds true.

e) The quantization in the Coulomb gauge

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(x) = 0$$

is simpler than the quantization in the Lorenz gauge, because the redundant dof are suppressed from the very beginning. Nevertheless, the theory will not be manifestly Lorentz invariant.

With the fixed gauge, A^0 is no longer a dynamical variable, in the sense that it is simply determined by the charge distribution j^0. If there is no source, we have $A^0 = 0$. On the other hand, the gauge condition in momentum space becomes

$$\boldsymbol{k} \cdot \boldsymbol{A}(\boldsymbol{k}, t) = 0,$$

so that $\boldsymbol{A}$ is perpendicular to the momentum. Splitting the vector potential as

$$\boldsymbol{A} = \boldsymbol{A}_\perp + \boldsymbol{A}_\parallel,$$

we can rewrite the orthogonality condition as $\boldsymbol{A}_\parallel = 0$. Note that $\boldsymbol{A}_\parallel$ reflects only one dof (namely, the direction of $\boldsymbol{k}$), while $\boldsymbol{A}_\perp$ represents the physical dof, those we will work with. The conjugate momenta will be

$$\boldsymbol{\pi}_\perp = \boldsymbol{E}_\perp = -(\partial^0 \boldsymbol{A}_\perp - \boldsymbol{\nabla} A^0_\perp) = -\partial^0 \boldsymbol{A}_\perp.$$

Thus, $\boldsymbol{\pi}_\perp = \boldsymbol{E}_\perp$ has two dof and the issue encountered in the Lorenz gauge quantization is resolved. However, the quantization cannot be performed simply as

$$\boldsymbol{A}_\perp, \boldsymbol{\pi}_\perp \longrightarrow \hat{\boldsymbol{A}}_\perp, \hat{\boldsymbol{\pi}}_\perp$$

by implementing the commutation relations

$$\begin{aligned}&[\hat{A}^i_\perp(t, \boldsymbol{x}), \hat{\pi}^j_\perp(t, \boldsymbol{y})] = i\, \delta^{ij}\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \\ &[\hat{A}^i_\perp(t, \boldsymbol{x}), \hat{A}^j_\perp(t, \boldsymbol{y})] = [\hat{\pi}^i_\perp(t, \boldsymbol{x}), \hat{\pi}^j_\perp(t, \boldsymbol{y})] = 0.\end{aligned}$$

In fact, the first condition does not fulfill the Coulomb gauge:

$$\begin{aligned}0 = [\boldsymbol{\nabla}_x \cdot \hat{\boldsymbol{A}}_\perp(t, \boldsymbol{x}), \hat{\pi}^j_\perp(t, \boldsymbol{y})] &= \partial_i^x [\hat{A}^i_\perp(t, \boldsymbol{x}), \hat{\pi}^j_\perp(t, \boldsymbol{y})] \\ &= i\, \delta^{ij}\, \partial_i^x \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \neq 0.\end{aligned}$$

This is a consequence of the fact that $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{\pi}}$ are not totally independent, but they must satisfy the transversality condition. In order to solve this problem, it necessary to implement the projection of the commutation relation

$$[\hat{A}^i(t, \boldsymbol{x}), \hat{\pi}^j(t, \boldsymbol{y})] = i\, \delta^{ij}\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

into the plane orthogonal to the momentum. The projector operator acts as

$$P_\perp \boldsymbol{v} = \boldsymbol{v}_\perp = \boldsymbol{v} - \frac{\boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{v})}{|\boldsymbol{k}|^2},$$

so that

$$v_{\perp i} = \underbrace{\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right)}_{=(P_\perp)_{ij}} v^j.$$

Thus, the standard normalization

$$\delta_{ij}\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) = \int \frac{d^3k}{(2\pi)^3} \delta_{ij}\, e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

becomes

$$\delta^{(3)}_{\perp ij}(\boldsymbol{x}-\boldsymbol{y}) = \int \frac{d^3k}{(2\pi)^3} \left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}.$$

The commutation relations

$$[\hat{A}^i_\perp(t,\boldsymbol{x}), \hat{\pi}^j_\perp(t,\boldsymbol{y})] = i\, \delta^{(3)}_{\perp ij}(\boldsymbol{x}-\boldsymbol{y})$$
$$[\hat{A}^i_\perp(t,\boldsymbol{x}), \hat{A}^j_\perp(t,\boldsymbol{y})] = [\hat{\pi}^i_\perp(t,\boldsymbol{x}), \hat{\pi}^j_\perp(t,\boldsymbol{y})] = 0$$

are in fact the correct ones.

Let us drop out the symbol $\perp$. For the expansion of the field $\hat{\boldsymbol{A}}$ in plane waves, we need only the transverse polarization vectors:

$$\hat{\boldsymbol{A}}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\Big(\hat{a}(\boldsymbol{k},\lambda)e^{-ikx} + \hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}\Big),$$

so that

$$\hat{\boldsymbol{E}}(x) = -\partial^0 \hat{\boldsymbol{A}}(x) = i\int \frac{d^3k}{\sqrt{(2\pi)^3}} \sqrt{\frac{\omega_k}{2}} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\Big(\hat{a}(\boldsymbol{k},\lambda)e^{-ikx} - \hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}\Big)$$
$$\hat{\boldsymbol{B}}(x) = \int d^3k\, \boldsymbol{k}\times\hat{\boldsymbol{E}}(\boldsymbol{k},t) = i\int \frac{d^3k}{\sqrt{(2\pi)^3}} \sqrt{\frac{\omega_k}{2}} \sum_{\lambda=1}^{2} \boldsymbol{k}\times\boldsymbol{\epsilon}(\boldsymbol{k},\lambda)\Big(\hat{a}(\boldsymbol{k},\lambda)e^{-ikx} - \hat{a}^\dagger(\boldsymbol{k},\lambda)e^{ikx}\Big).$$

The operators $\hat{a}(\boldsymbol{k},\lambda)$ and their adjoints satisfy the commutation relations

$$[\hat{a}(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k}',\lambda')] = \delta_{\lambda\lambda'}\, \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$$
$$[\hat{a}(\boldsymbol{k},\lambda), \hat{a}(\boldsymbol{k}',\lambda')] = [\hat{a}^\dagger(\boldsymbol{k},\lambda), \hat{a}^\dagger(\boldsymbol{k}',\lambda')] = 0.$$

In the calculation is crucial the presence of the transverse delta function, which agrees with the relation

$$\sum_{\lambda=1}^{2} \epsilon_i(\boldsymbol{k},\lambda)\epsilon_j(\boldsymbol{k},\lambda) = \delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}.$$

The normal ordered Hamiltonian and the normal ordered momentum will be the correct ones

$$\hat{H} = \int d^3k \sum_{\lambda=1}^{2} \omega_k\, \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda)$$
$$\hat{\boldsymbol{P}} = \int d^3k \sum_{\lambda=1}^{2} \boldsymbol{k}\, \hat{a}^\dagger(\boldsymbol{k},\lambda)\hat{a}(\boldsymbol{k},\lambda).$$

Last, the Feynman propagator in the Coulomb gauge turns out to be the transverse part of discussed previously:

$$\langle 0|T(\hat{A}^i_\perp(x)\hat{A}^j_\perp(y))|0\rangle = iD^{ij}_{(\mathrm{trans})}(x-y).$$

Interacting fields

15 Interacting quantum fields

a) Define the Schrödinger, Heisenberg and interaction pictures, and write down the precise relations among the three (for the state vectors and for the operators). What is the advantage of the interaction picture over the other two, when dealing with interacting quantum fields?

b) Define the time evolution operator $\hat{U}(t, t_0)$ in the interaction picture, which satisfies the equation:

$$i\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}^I_{\text{int}}(t)\hat{U}(t, t_0), \qquad \hat{U}(t_0, t_0) = 1.$$

Write the equation in integral form. Perform a perturbation expansion of the equation and show how to arrive at the final result:

$$\hat{U}(t, t_0) = \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n \; T(\hat{H}^I_{\text{int}}(t_1) \dots \hat{H}^I_{\text{int}}(t_n))$$

c) State Wick's theorem. Prove it with a simple nontrivial example. Why is it important?

d) Write down the Lagrangian density of QED. Write down the Fourier expansion of the photonic and fermionic fields in the interaction picture. Write down the first order contribution $\hat{S}^{(1)}$ to the scattering matrix for QED in terms of the fields. Show that it can be split in eight different parts, each corresponding to a different physical situations. Explain what these physical situations are.

e) Why do the above process give no contribution to the scattering matrix? Show it explicitly for one case.

---

a) In the Schrödinger picture, states evolve according to the Schrödinger equation, while operators (whose classical analogue are time independent) do not change in time:

$$|\alpha, t\rangle^{\text{S}} = e^{-i\hat{H}t} |\alpha, 0\rangle^{\text{S}}$$
$$\hat{O}^{\text{S}}(t) = \hat{O}^{\text{S}} = \hat{O}(0).$$

The first equation arises from the Schrödinger equation

$$i\frac{\partial}{\partial t}|\alpha, t\rangle^{\text{S}} = \hat{H} |\alpha, t\rangle^{\text{S}}.$$

On the opposite, in the Heisenberg picture the states do not change in time, while the operators evolve:

$$|\alpha, t\rangle^{\text{H}} = |\alpha\rangle^{\text{H}} = |\alpha, 0\rangle$$
$$\hat{O}^{\text{H}}(t) = e^{i\hat{H}t}\hat{O}^{\text{H}}(0)e^{-i\hat{H}t}.$$

The second equation arises from the Heisenberg equation

$$i\frac{\partial}{\partial t}\hat{O}^{\text{H}}(t) = [\hat{O}^{\text{H}}(t), \hat{H}].$$

The interaction picture is in between the previous two. It is used when the Hamiltonian can be divided in two parts

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}.$$

Usually, $\hat{H}_0$ is the free part of the Hamiltonian, while $\hat{H}_{\text{int}}$ is the interacting one. The evolution is defined as

$$|\alpha, t\rangle^{\text{I}} = e^{i\hat{H}_0 t} |\alpha, t\rangle^{\text{S}}$$
$$\hat{O}^{\text{I}}(t) = e^{i\hat{H}_0 t}\hat{O}^{\text{S}}e^{-i\hat{H}_0 t}.$$

Here the free Hamiltonian does not evolve, *i.e.* it is in the Schrödinger picture: $\hat{H}_0 = \hat{H}_0^{\mathrm{S}}$. The equation of motion for the states will be

$$\begin{aligned}
i\frac{\partial}{\partial t}|\alpha, t\rangle^{\mathrm{I}} &= i(i\hat{H}_0)e^{i\hat{H}_0 t}\,|\alpha, t\rangle^{\mathrm{S}} + i\,e^{i\hat{H}_0 t}\frac{\partial}{\partial t}|\alpha, t\rangle^{\mathrm{S}} \\
&= -\hat{H}_0\,|\alpha, t\rangle^{\mathrm{I}} + e^{i\hat{H}_0 t}\hat{H}\,|\alpha, t\rangle^{\mathrm{S}} \\
&= -\hat{H}_0\,|\alpha, t\rangle^{\mathrm{I}} + e^{i\hat{H}_0 t}\hat{H}e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}\,|\alpha, t\rangle^{\mathrm{S}} \\
&= -\hat{H}_0\,|\alpha, t\rangle^{\mathrm{I}} + \hat{H}^{\mathrm{I}}(t)\,|\alpha, t\rangle^{\mathrm{I}}.
\end{aligned}$$

From the fact that

$$\hat{H}_0^{\mathrm{I}}(t) = e^{i\hat{H}_0^{\mathrm{S}} t}\hat{H}_0^{\mathrm{S}}e^{-i\hat{H}_0^{\mathrm{S}} t} = \hat{H}_0^{\mathrm{S}} = \hat{H}_0,$$

we find

$$i\frac{\partial}{\partial t}|\alpha, t\rangle^{\mathrm{I}} = \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t)\,|\alpha, t\rangle^{\mathrm{I}},$$

which means that the states evolve with the interacting Hamiltonian in the interacting picture. On the other hand, for the observables we obtain

$$\begin{aligned}
i\frac{\partial}{\partial t}\hat{O}^{\mathrm{I}}(t) &= i\,(i\hat{H}_0)e^{i\hat{H}_0 t}\hat{O}^{\mathrm{S}}e^{-i\hat{H}_0 t} + i\,e^{i\hat{H}_0 t}\hat{O}^{\mathrm{S}}e^{-i\hat{H}_0 t}(-i\hat{H}_0) \\
&= -\hat{H}_0\hat{O}^{\mathrm{I}} + \hat{O}^{\mathrm{I}}\hat{H}_0 = [\hat{O}^{\mathrm{I}}, \hat{H}_0]
\end{aligned}$$

i.e. the observables evolve with the free Hamiltonian $\hat{H}_0$.

The three pictures are related as follows:

$$\begin{aligned}
|\alpha, 0\rangle^{\mathrm{S}} = |\alpha, 0\rangle^{\mathrm{H}} = |\alpha, 0\rangle^{\mathrm{I}} = |\alpha, 0\rangle \\
\hat{O}^{\mathrm{S}}(0) = \hat{O}^{\mathrm{H}}(0) = \hat{O}^{\mathrm{I}}(0) = \hat{O}(0)
\end{aligned}$$

for the initial states and operators;

$$\begin{aligned}
|\alpha, t\rangle^{\mathrm{S}} &= e^{-i\hat{H}t}\,|\alpha\rangle^{\mathrm{H}} \\
\hat{O}^{\mathrm{S}} &= e^{-i\hat{H}t}\hat{O}^{\mathrm{H}}(t)e^{i\hat{H}t}
\end{aligned}$$

for the Schrödinger-Heisenberg correspondence;

$$\begin{aligned}
|\alpha, t\rangle^{\mathrm{I}} &= e^{i\hat{H}_0 t}\,|\alpha, t\rangle^{\mathrm{S}} \\
\hat{O}^{\mathrm{I}}(t) &= e^{i\hat{H}_0 t}\hat{O}^{\mathrm{S}}e^{-i\hat{H}_0 t}
\end{aligned}$$

for the interaction-Schrödinger correspondence, and

$$\begin{aligned}
|\alpha, t\rangle^{\mathrm{I}} &= e^{i\hat{H}_0 t}e^{-i\hat{H}t}\,|\alpha\rangle^{\mathrm{H}} \\
\hat{O}^{\mathrm{I}}(t) &= e^{i\hat{H}_0 t}e^{-i\hat{H}t}\hat{O}^{\mathrm{H}}(t)e^{i\hat{H}t}e^{-i\hat{H}_0 t}
\end{aligned}$$

for the interaction-Heisenberg correspondence.

The three pictures are all equivalent, in the sense that the expectation values are the same:

$${}^{\mathrm{S}}\langle\alpha, t|\hat{O}^{\mathrm{S}}|\alpha, t\rangle^{\mathrm{S}} = {}^{\mathrm{H}}\langle\alpha|\hat{O}^{\mathrm{H}}(t)|\alpha\rangle^{\mathrm{H}} = {}^{\mathrm{I}}\langle\alpha, t|\hat{O}^{\mathrm{I}}(t)|\alpha, t\rangle^{\mathrm{I}}.$$

In fact,

$$\begin{aligned}
{}^{\mathrm{S}}\langle\alpha, t|\hat{O}^{\mathrm{S}}|\alpha, t\rangle^{\mathrm{S}} &= {}^{\mathrm{H}}\langle\alpha|e^{-i\hat{H}t}e^{-i\hat{H}t}\hat{O}^{\mathrm{H}}(t)e^{i\hat{H}t}e^{i\hat{H}t}|\alpha\rangle^{\mathrm{H}} \\
&= {}^{\mathrm{H}}\langle\alpha|\hat{O}^{\mathrm{H}}(t)|\alpha\rangle^{\mathrm{H}}
\end{aligned}$$

and analogously

$$
{}^{\mathrm{I}}\langle\alpha,t|\hat{O}^{\mathrm{I}}(t)|\alpha,t\rangle^{\mathrm{I}} = {}^{\mathrm{S}}\langle\alpha,t|e^{i\hat{H}_0 t}e^{-i\hat{H}_0 t}\hat{O}^{\mathrm{S}}e^{i\hat{H}_0 t}e^{-i\hat{H}_0 t}|\alpha,t\rangle^{\mathrm{S}}
$$
$$
= {}^{\mathrm{S}}\langle\alpha,t|\hat{O}^{\mathrm{S}}|\alpha,t\rangle^{\mathrm{S}}.
$$

The interaction picture has the advantage that the operators evolve freely, so that the theory of the free of Klein-Gordon, Dirac and Maxwell fields can still be used even if the fields are interacting, and the states evolve with the interaction Hamiltonian only, so that we need only to worry about the evolution of the states.

The formal solution for the equation for the states in the interaction picture is

$$
\begin{aligned}
|\alpha,t\rangle^{\mathrm{I}} &= e^{i\hat{H}_0 t}\,|\alpha,t\rangle^{\mathrm{S}} = e^{i\hat{H}_0 t}e^{-i\hat{H}(t-t_0)}\,|\alpha,t_0\rangle^{\mathrm{S}} \\
&= e^{i\hat{H}_0 t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0 t_0}\,|\alpha,t_0\rangle^{\mathrm{I}}.
\end{aligned}
$$

Setting

$$
\hat{U}(t,t_0) = e^{i\hat{H}_0 t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0 t_0},
$$

we find that

$$
|\alpha,t\rangle^{\mathrm{I}} = \hat{U}(t,t_0)\,|\alpha,t_0\rangle^{\mathrm{I}}.
$$

Moreover, $\hat{U}(t_0,t_0) = \mathbb{1}$ and

$$
\begin{aligned}
i\frac{\partial}{\partial t}|\alpha,t\rangle^{\mathrm{I}} &= i\frac{\partial}{\partial t}\hat{U}(t,t_0)\,|\alpha,t_0\rangle^{\mathrm{I}} \\
&= \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t)\,|\alpha,t\rangle^{\mathrm{I}} = \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t)\hat{U}(t,t_0)\,|\alpha,t_0\rangle^{\mathrm{I}},
\end{aligned}
$$

so that

$$
i\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t)\hat{U}(t,t_0).
$$

Solving the equation for $\hat{U}$ is equivalent to solving the equation for the states. In addition, in this case we do not have to worry about the initial state (the initia condition is always $\hat{U}(t_0,t_0) = \mathbb{1}$). In integral form, the equation becomes

$$
\hat{U}(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} d\tau\,\hat{H}^{\mathrm{I}}_{\mathrm{int}}(\tau)\hat{U}(\tau,t_0).
$$

Thus, we find

$$
\begin{aligned}
\hat{U}(t,t_0) &= \mathbb{1} - i\int_{t_0}^{t} dt_1\,\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\,\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2)\hat{U}(t_2,t_0) \\
&= \cdots = \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n\,\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\cdots\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n).
\end{aligned}
$$

The integrals can be rewritten changing the integration extremes. In particular, we can prove by induction that

$$
\begin{aligned}
\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n\,\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\cdots\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n) &= \frac{1}{n!}\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_2\cdots\int_{t_0}^{t} dt_n\,T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\cdots\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big) \\
&= \frac{1}{n!}\int_{[t_0,t]^n} dt_1\,dt_2\cdots dt_n\,T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\cdots\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big),
\end{aligned}
$$

where T is the time-ordered product

$$
T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\cdots\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big) = \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_{i_1})\cdots\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_{i_n}), \qquad t_{i_1}\geq\cdots\geq t_{i_n}.
$$

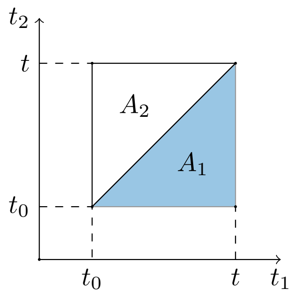


Figure 1: Decomposition of A in the case $n = 2$.

For equal-times this definition is consistent, since the operators commute. Now, for $n = 1$ the proof is trivial, because $T(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)) = \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)$. We now prove it in the simple case $n = 2$. We want to compute

$$I_2 = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2).$$

The integration extends over a triangular area in the $t_1 - t_2$ plane. As sketched in Fig 1, when the boundaries are suitably chosen one may as well integrate first over the variable t_1 and then over t_2.

$$I_2 = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2) = \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1).$$

In the second step the integration variables have been renamed, $t_1 \leftrightarrow t_2$. Adding up the two alternative but equivalent forms of integration, we arrive at

$$\begin{aligned} 2\int_{t_0}^{t} dt_1 & \int_{t_0}^{t_1} dt_2\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2) \\ &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2) + \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \\ &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1)\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_2)\Big). \end{aligned}$$

The same procedure can be extended to all the higher-order multiple integrals. Let us suppose the thesis to hold for $n - 1$. We want to compute

$$I_n = \int_{[t_0,t]^n} dt_1\, dt_2 \cdots dt_n\, T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big).$$

We can see that the cube $[t_0, t]^n$ is decomposable in

$$[t_0, t]^n = \bigcup_{i=1}^{n} A_i,$$

where $A_i = \{\, (t_1, \ldots, t_n) \in [t_0, t]^n \mid t_i \geq t_j\ \forall j = 1, \ldots, n \,\}$. Further, $A_i \cap A_j$ has zero measure for $i \neq j$. Thus,

$$\begin{aligned} I_n &= \sum_{i=1}^{n} \int_{A_i} dt_1\, dt_2 \cdots dt_n\, T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big) \\ &= \sum_{i=1}^{n} \int_{A_i} dt_1\, dt_2 \cdots dt_n\, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i) T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \widetilde{\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i)} \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big), \end{aligned}$$

where the factors with the tilde do not appear. The integral over A_i can be written explicitly as

$$\int_{A_i} dt_1 \cdots dt_n \, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i) T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \widetilde{\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i)} \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big) =$$
$$= \int_{t_0}^{t} dt_i \int_{[t_0,t_i]^{n-1}} dt_1 \cdots \widetilde{dt_i} \cdots dt_n \, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i) T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \widetilde{\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i)} \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big)$$
$$= (n-1)! \int_{t_0}^{t} dt_i \int_{t_0}^{t_i} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \widetilde{\int_{t_0}^{t_{i-1}} dt_i} \cdots \int_{t_0}^{t_{n-1}} dt_n \, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i) \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \widetilde{\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i)} \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)$$

where in the second step the theorem for the case $n-1$ has been used. With a family of changes of variables

$$\begin{aligned} t_i &\longmapsto t_1 \\ t_1 &\longmapsto t_2 \\ &\vdots \\ t_{i-1} &\longmapsto t_i, \end{aligned}$$

we obtain

$$\int_{A_i} dt_1 \cdots dt_n \, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i) T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \widetilde{\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_i)} \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big) =$$
$$= (n-1)! \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n).$$

Eventually, the thesis is proved:

$$\begin{aligned} I_n &= \sum_{i=1}^{n} (n-1)! \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n) \\ &= n! \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n). \end{aligned}$$

Hence, we have the so-called Dyson series for the time-evolution operator

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} dt_1 \, dt_2 \cdots dt_n \, T\Big(\hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_1) \cdots \hat{H}^{\mathrm{I}}_{\mathrm{int}}(t_n)\Big),$$

where the zeroth term has been set equal to $\mathbb{1}$.

c) Wick's theorem allows to express the time-ordered product of operators as normal ordered product plus a sum over all possible contraction of the product:

$$T\Big(\hat{O}_1 \cdots \hat{O}_n\Big) = :\hat{O}_1 \cdots \hat{O}_n: + \sum_{i<j} :\hat{O}_1 \cdots \underbrace{\hat{O}_i \cdots \hat{O}_j} \cdots \hat{O}_n: +$$
$$+ \sum_{\substack{i<j,\, h<k, \\ i<h,\, h \neq j \neq k}} :\hat{O}_1 \cdots \underbrace{\hat{O}_i \cdots \hat{O}_h \cdots \hat{O}_j} \cdots \hat{O}_k \cdots \hat{O}_n: + \text{higher contractions.}$$

A contraction is only defined for fields of the same type (both bosonic or fermionic) as follows:

$$\underbrace{\hat{A}\hat{B}} = \langle 0| T\big(\hat{A}\hat{B}\big) |0\rangle \,.$$

For a single contraction in a normal-ordered product, we have

$$:\hat{O}_1 \cdots \underbrace{\hat{O}_i \cdots \hat{O}_j} \cdots \hat{O}_n: = (-1)^{\lambda} :\hat{O}_1 \cdots \widetilde{\hat{O}_i} \cdots \widetilde{\hat{O}_j} \cdots \hat{O}_n: \underbrace{\hat{O}_i \hat{O}_j},$$

where λ is the number of fermionic changes done to move $\hat{O}_i$ on the left besides $\hat{O}_j$. For double contractions in normal ordered products,

$$:\hat{O}_1 \cdots \underbrace{\hat{O}_i \cdots \hat{O}_h \cdots \hat{O}_j}_{} \cdots \hat{O}_k \cdots \hat{O}_n: = (-1)^{\lambda+\mu} :\hat{O}_1 \cdots \widetilde{\hat{O}_i} \cdots \widetilde{\hat{O}_h} \cdots \widetilde{\hat{O}_j} \cdots \widetilde{\hat{O}_k} \cdots \hat{O}_n: \ \underbrace{\hat{O}_i \hat{O}_j} \underbrace{\hat{O}_h \hat{O}_k}$$

where λ and μ are the number of fermionic changes done to move $\hat{O}_i$ on the left besides $\hat{O}_j$ and $\hat{O}_h$ also on the left besides $\hat{O}_k$ respectively. The formula can be generalized to higher contractions. We will prove it for the time ordered product of two fields $\hat{A}$ and $\hat{B}$.

We assume that the field are both bosonic or both fermionic (otherwise no contraction can be taken). We know that

$$T\big(\hat{A}(x)\hat{B}(y)\big) = \Theta(x^0 - y^0)\hat{A}(x)\hat{B}(y) + \epsilon\,\Theta(y^0 - x^0)\hat{B}(y)\hat{A}(x),$$

with $\epsilon = \pm 1$ for bosonic and fermionic fields respectively. Let us assume that $x^0 > y^0$. Dropping out the space-time dependence, we find

$$\begin{aligned} T\big(\hat{A}(x)\hat{B}(y)\big) &= \hat{A}\hat{B} = \hat{A}^{(+)}\hat{B}^{(+)} + \hat{A}^{(+)}\hat{B}^{(-)} + \hat{A}^{(-)}\hat{B}^{(+)} + \hat{A}^{(-)}\hat{B}^{(-)} \\ &= \hat{A}^{(+)}\hat{B}^{(+)} + \epsilon\hat{B}^{(-)}\hat{A}^{(+)} + \hat{A}^{(-)}\hat{B}^{(+)} + \hat{A}^{(-)}\hat{B}^{(-)} + [\hat{A}^{(+)}, \hat{B}^{(-)}]_{\mp} \\ &= :\hat{A}\hat{B}: + [\hat{A}^{(+)}, \hat{B}^{(-)}]_{\mp}, \end{aligned}$$

where $[\cdot,\cdot]_-$ denotes the commutator, and $[\cdot,\cdot]_+$ the anti-commutator. On the other hand, the field $[\hat{A}^{(+)}, \hat{B}^{(-)}]_{\mp}$ is a scalar function (multiple of the identity operator) and in particular

$$\begin{aligned} [\hat{A}^{(+)}, \hat{B}^{(-)}]_{\mp} &= \langle 0|[\hat{A}^{(+)}, \hat{B}^{(-)}]_{\mp}|0\rangle = \langle 0|\hat{A}^{(+)}\hat{B}^{(-)}|0\rangle \\ &= \langle 0|\hat{A}\hat{B}|0\rangle = \langle 0|T\big(\hat{A}\hat{B}\big)|0\rangle = \underbrace{\hat{A}\hat{B}}. \end{aligned}$$

For $y^0 > x^0$ the calculations are analogue, so we have the final result

$$T\big(\hat{A}(x)\hat{B}(y)\big) = :\hat{A}\hat{B}: + \underbrace{\hat{A}\hat{B}}.$$

d) The Lagrangian of the Quantum Electrodynamic is

$$\mathcal{L}_{\text{QED}} = \underbrace{\bar{\psi}\left(\frac{i}{2}\overleftrightarrow{\not{\partial}} - m\right)\psi}_{\mathcal{L}_{\text{Dirac}}} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2}_{\mathcal{L}_{\text{e.m.}}} \underbrace{-e\bar{\psi}\gamma_\mu\psi A^\mu}_{\mathcal{L}_{\text{int}}}.$$

The Fourier expansion of the fields is the same as that for the free fields:

$$\begin{aligned} \hat{\psi}(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}\sum_s \Big(\hat{b}(\boldsymbol{p},s)u(\boldsymbol{p},s)e^{-ipx} + \hat{d}^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s)e^{ipx}\Big) \\ \hat{\bar{\psi}}(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}\sum_s \Big(\hat{d}(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s)e^{-ipx} + \hat{b}^\dagger(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)e^{ipx}\Big) \\ \hat{A}^\mu(x) &= \int \frac{d^3k}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}\sum_\lambda \Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx} + \hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big). \end{aligned}$$

The interaction Hamiltonian in the quantized theory is

$$\hat{\mathcal{H}}_{\text{int}} = e :\hat{\bar{\psi}}\gamma_\mu\hat{\psi}\hat{A}^\mu:,$$

so that the scattering matrix $\hat{S} = \hat{U}(+\infty, -\infty)$ is

$$\hat{S} = \sum_{n=0}^{\infty}\frac{(-ie)^n}{n!}\int d^4x_1 \cdots d^4x_n\, T\Big(:\hat{\bar{\psi}}(x_1)\gamma_\mu\hat{\psi}(x_1)\hat{A}^\mu(x_1): \cdots :\hat{\bar{\psi}}(x_n)\gamma_{\mu_n}\hat{\psi}(x_n)\hat{A}^{\mu_n}(x_n):\Big).$$

The first order term in the Dyson series is

$$\hat{S}^{(1)} = -ie \int d^4x \, :\hat{\bar{\psi}}(x)\gamma_\mu \hat{\psi}(x)\hat{A}^\mu(x): \, .$$

Due to the fact that

$$\hat{\psi} \propto \hat{b} + \hat{d}^\dagger, \qquad \hat{\bar{\psi}} \propto \hat{d} + \hat{b}^\dagger, \qquad \hat{A}^\mu \propto \hat{a} + \hat{a}^\dagger,$$

we obtain eight different terms in $\hat{S}^{(1)}$. Namely, we have (be careful to normal ordering):

$$\hat{S}^{(1)} = -ie \int d^4x \frac{d^3p'}{\sqrt{(2\pi)^3}} \frac{d^3p''}{\sqrt{(2\pi)^3}} \frac{d^3k'}{\sqrt{(2\pi)^3}} \sqrt{\frac{m}{\omega_{p'}}} \sqrt{\frac{m}{\omega_{p''}}} \frac{1}{\sqrt{2\omega_{k'}}} \sum_{s',s'',\lambda'} \times$$

$$\Big(\hat{b}(\boldsymbol{p}',s')\hat{d}(\boldsymbol{p}'',s'')\hat{a}(\boldsymbol{k}',\lambda')\, u(\boldsymbol{p}',s')\gamma_\mu \bar{v}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(p'+p''+k')x} \qquad \text{(a)}$$

$$+ \hat{b}(\boldsymbol{p}',s')\hat{d}(\boldsymbol{p}'',s'')\hat{a}^\dagger(\boldsymbol{k}',\lambda')\, u(\boldsymbol{p}',s')\gamma_\mu \bar{v}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(p'+p''-k')x} \qquad \text{(b)}$$

$$+ \hat{b}^\dagger(\boldsymbol{p}'',s'')\hat{b}(\boldsymbol{p}',s')\hat{a}(\boldsymbol{k}',\lambda')\, u(\boldsymbol{p}',s')\gamma_\mu \bar{u}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(p'-p''+k')x} \qquad \text{(c)}$$

$$+ \hat{b}^\dagger(\boldsymbol{p}'',s'')\hat{b}(\boldsymbol{p}',s')\hat{a}^\dagger(\boldsymbol{k}',\lambda')\, u(\boldsymbol{p}',s')\gamma_\mu \bar{u}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(p'-p''-k')x} \qquad \text{(d)}$$

$$- \hat{d}^\dagger(\boldsymbol{p}',s')\hat{d}(\boldsymbol{p}'',s'')\hat{a}(\boldsymbol{k}',\lambda')\, v(\boldsymbol{p}',s')\gamma_\mu \bar{v}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(-p'+p''+k')x} \qquad \text{(e)}$$

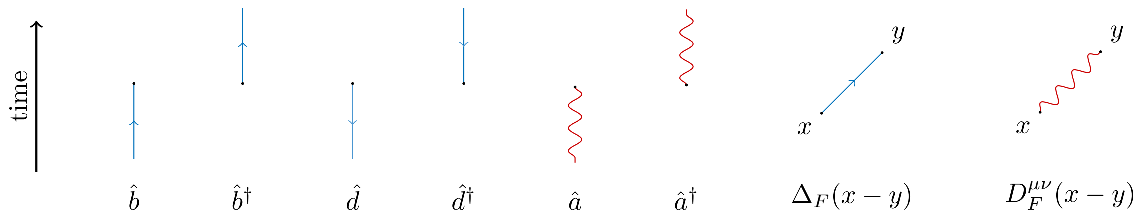
$$- \hat{d}^\dagger(\boldsymbol{p}',s')\hat{d}(\boldsymbol{p}'',s'')\hat{a}^\dagger(\boldsymbol{k}',\lambda')\, v(\boldsymbol{p}',s')\gamma_\mu \bar{v}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(-p'+p''-k')x} \qquad \text{(f)}$$

$$+ \hat{d}^\dagger(\boldsymbol{p}',s')\hat{b}^\dagger(\boldsymbol{p}'',s'')\hat{a}(\boldsymbol{k}',\lambda')\, v(\boldsymbol{p}',s')\gamma_\mu \bar{u}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(-p'-p''+k')x} \qquad \text{(g)}$$

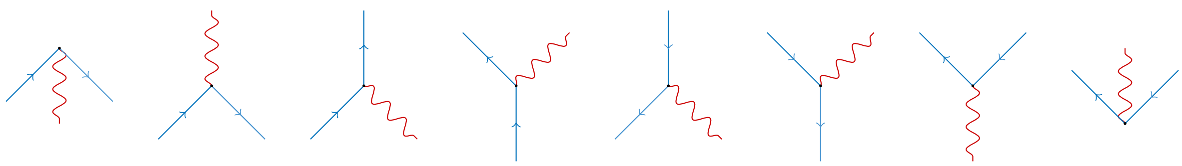
$$+ \hat{d}^\dagger(\boldsymbol{p}',s')\hat{b}^\dagger(\boldsymbol{p}'',s'')\hat{a}^\dagger(\boldsymbol{k}',\lambda')\, v(\boldsymbol{p}',s')\gamma_\mu \bar{u}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{i(p'+p''+k')x} \Big). \qquad \text{(h)}$$

Thus, we will have non-zero matrix elements only if the initial and final states contain the correct number of creation and annihilation operators, so that the number of $\hat{a}$ matches the number of $\hat{a}^\dagger$ in $\hat{S}^{(1)}$, and so on.

Let us give the following graphical representation of the terms in the expansion:



Then the eight terms of the expansion can be represented as follows:



(a) (b) (c) (d) (e) (f) (g) (h)

which correspond to:

a) Annihilation of a fermion, an anti-fermion and a photon;

b) Annihilation of a fermion and an anti-fermion, by creating a photon;

c) Photon absorption by a fermion;

d) Photon emission by a fermion;

e) Photon absorption by an anti-fermion;

f) Photon emission by an anti-fermion;

g) Annihilation of a photon, creating a fermion-antifermion pair;

h) Spontaneous creation of an fermion, an anti-fermion and a photon.

e) At the end of the day, all these processes are kinematically forbidden. For example, for (a) we must have the vacuum as final state, while a fermion f_{p_1,s_1}, an antifermion $\bar{f}_{p_2,s_2}$ and a photon $\gamma_{k,\lambda}$ as initial state. All other choices give a null contribution. Thus, the only relevant matrix element in this case is

$$
\begin{aligned}
\langle 0|\hat{S}^{(1)}|f_{p_1,s_1}\,\bar{f}_{p_2,s_2}\,\gamma_{k,\lambda}\rangle &= -ie\int d^4x\,\frac{d^3p'}{\sqrt{(2\pi)^3}}\frac{d^3p''}{\sqrt{(2\pi)^3}}\frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_{p'}}}\sqrt{\frac{m}{\omega_{p''}}}\frac{1}{\sqrt{2\omega_{k'}}}\sum_{s',s'',\lambda'}\times\\
&\qquad \langle 0|\,\hat{b}(\boldsymbol{p}',s')\hat{d}(\boldsymbol{p}'',s'')\hat{a}(\boldsymbol{k}',\lambda')\;\hat{b}^\dagger(\boldsymbol{p}_1,s_1)\hat{d}^\dagger(\boldsymbol{p}_2,s_2)\hat{a}^\dagger(\boldsymbol{k},\lambda)\,|0\rangle\times\\
&\qquad u(\boldsymbol{p}',s')\gamma_\mu\bar{v}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(p'+p''+k')x}\\
&= -ie\int d^4x\,\frac{d^3p'}{\sqrt{(2\pi)^3}}\frac{d^3p''}{\sqrt{(2\pi)^3}}\frac{d^3k'}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_{p'}}}\sqrt{\frac{m}{\omega_{p''}}}\frac{1}{\sqrt{2\omega_{k'}}}\sum_{s',s'',\lambda'}\times\\
&\qquad \delta^{(3)}(\boldsymbol{p}_1-\boldsymbol{p}')\delta^{(3)}(\boldsymbol{p}_2-\boldsymbol{p}'')\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{s_1s'}\delta_{s_2s''}\delta_{\lambda\lambda'}\times\\
&\qquad u(\boldsymbol{p}',s')\gamma_\mu\bar{v}(\boldsymbol{p}'',s'')\epsilon^\mu(\boldsymbol{k}',\lambda')e^{-i(p'+p''+k')x}\\
&= \frac{-ie}{\sqrt{(2\pi)^9}}\int d^4x\sqrt{\frac{m}{\omega_{p_1}}}\sqrt{\frac{m}{\omega_{p_2}}}\frac{1}{\sqrt{2\omega_k}}u(\boldsymbol{p}_1,s_1)\gamma_\mu\bar{v}(\boldsymbol{p}_2,s_2)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-i(p_1+p_2+k)x}\\
&= \frac{-ie}{\sqrt{(2\pi)^9}}\sqrt{\frac{m}{\omega_{p_1}}}\sqrt{\frac{m}{\omega_{p_2}}}\frac{1}{\sqrt{2\omega_k}}u(\boldsymbol{p}_1,s_1)\gamma_\mu\bar{v}(\boldsymbol{p}_2,s_2)\epsilon^\mu(\boldsymbol{k},\lambda)\;(2\pi)^4\,\delta^{(4)}(p_1+p_2+k)
\end{aligned}
$$

In general, for each of the eight processes previously described, we will end up with Dirac deltas of the form

$$\delta^4(\pm p_1 \pm p_2 \pm k).$$

However the system consisting of the 4-momentum conservation and dispersion relations

$$\begin{cases}\pm p_1 \pm p_2 \pm k = 0\\ k^2 = 0\\ p_1^2 = p_2^2 = m^2\end{cases}$$

has no solution for all the eight combinations of signs. For example, for (a) we have

$$\begin{cases}p_1 + p_2 + k = 0\\ k^2 = 0\\ p_1^2 = p_2^2 = m^2\end{cases} \implies p_1^2 + p_2^2 + 2p_1p_2 = k^2 \implies m^2 + p_1p_2 = 0.$$

$$
\begin{gathered}
m^2 + \sqrt{m^2+|\boldsymbol{p}_1|^2}\sqrt{m^2+|\boldsymbol{p}_2|^2} - \boldsymbol{p}_1\cdot\boldsymbol{p}_2 = 0\\
\sqrt{m^2+|\boldsymbol{p}_1|^2}\sqrt{m^2+|\boldsymbol{p}_2|^2} = \boldsymbol{p}_1\cdot\boldsymbol{p}_2 - m^2.
\end{gathered}
$$

Taking the square of both sides

$$m^4 + m^2(|\boldsymbol{p}_1|^2 + |\boldsymbol{p}_2|^2) + |\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2 = m^4 - 2m^2\boldsymbol{p}_1\cdot\boldsymbol{p}_2 + |\boldsymbol{p}_1\cdot\boldsymbol{p}_2|^2$$

$$\underbrace{m^2|\boldsymbol{p}_1+\boldsymbol{p}_2|^2}_{\geq 0} = \underbrace{|\boldsymbol{p}_1\cdot\boldsymbol{p}_2|^2 - |\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2}_{\leq 0 \text{ for Cauchy-Schwarz}}.$$

The only possibility is that $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, but in this case the momentum conservation tells us that $\boldsymbol{k} = \boldsymbol{0}$ (a photon with null momentum), which is not possible.

16 Quantum Electrodynamics

a) Write down the Lagrangian density of QED. Write down the Fourier expansion of the photonic and fermionic fields in the interaction picture. Write down the second order contribution $\hat{S}^{(2)}$ to the scattering matrix for QED in terms of the fields. Explain how Wick's theorem is modified by the presence of normal ordering.

b) Write down the eight relevant terms of the Wick expansion of $\hat{S}^{(2)}$. With the help of Feynman diagrams, explain what they represent physically.

c) Write down Feynman rules of QED in coordinate space.

d) Write down Feynman rules of QED in momentum space.

e) Using Feynman rules, compute the scattering matrix for the scattering in which two electrons with momenta and spin p_1, s_1 and p_2, s_2 in the initial state are scattered into the final state with p_1', s_1' and p_2', s_2'.

---

a) The Lagrangian of the Quantum Electrodynamic is

$$\mathcal{L}_{\text{QED}} = \underbrace{\bar{\psi}\left(\frac{i}{2}\overleftrightarrow{\not{\partial}} - m\right)\psi}_{\mathcal{L}_{\text{Dirac}}} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2}_{\mathcal{L}_{\text{e.m.}}} \underbrace{-e\bar{\psi}\gamma_\mu\psi A^\mu}_{\mathcal{L}_{\text{int}}}.$$

The Fourier expansion of the fields in the quantize theory is the same as that for the free fields:

$$\begin{aligned}
\hat{\psi}(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}\sum_s \Big(\hat{b}(\boldsymbol{p},s)u(\boldsymbol{p},s)e^{-ipx} + \hat{d}^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s)e^{ipx}\Big)\\
\hat{\bar{\psi}}(x) &= \int \frac{d^3p}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}\sum_s \Big(\hat{d}(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s)e^{-ipx} + \hat{b}^\dagger(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)e^{ipx}\Big)\\
\hat{A}^\mu(x) &= \int \frac{d^3k}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}\sum_\lambda \Big(\hat{a}(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{-ikx} + \hat{a}^\dagger(\boldsymbol{k},\lambda)\epsilon^\mu(\boldsymbol{k},\lambda)e^{ikx}\Big).
\end{aligned}$$

The interaction Hamiltonian in the quantized theory is

$$\hat{\mathcal{H}}_{\text{int}} = e :\hat{\bar{\psi}}\gamma_\mu\hat{\psi}\hat{A}^\mu:,$$

so that the scattering matrix $\hat{S} = \hat{U}(+\infty, -\infty)$ is

$$\hat{S} = \sum_{n=0}^{\infty}\frac{(-ie)^n}{n!}\int d^4x_1\cdots d^4x_n\, T\Big(:\hat{\bar{\psi}}(x_1)\gamma_\mu\hat{\psi}(x_1)\hat{A}^\mu(x_1):\,\cdots :\hat{\bar{\psi}}(x_n)\gamma_{\mu_n}\hat{\psi}(x_n)\hat{A}^{\mu_n}(x_n):\Big).$$

The second order term in the Dyson series is

$$\hat{S}^{(2)} = \frac{(-ie)^2}{2}\int d^4x\, d^4y\, T\Big(:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)::\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y):\Big).$$

We can use Wick's theorem in the simplified version for time ordered product of factors within a normal order: contractions of fields within the same normal order do not enter. In fact, they would produce the divergent factors $\Delta_F(0)$ and $D_F^{\mu\nu}(0)$, which are removed by the normal order.

b) Thus, using Wick's theorem

$$\hat{S}^{(2)} = \frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y): + \qquad \text{(a) no contraction}$$

$$\frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y): + \qquad \text{(b)}$$

$$\frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y): + \qquad \text{(c)}$$

$$\frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y): + \qquad \text{(d)}$$

(b), (c), (d): single contractions

$$\frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y): + \qquad \text{(e)}$$

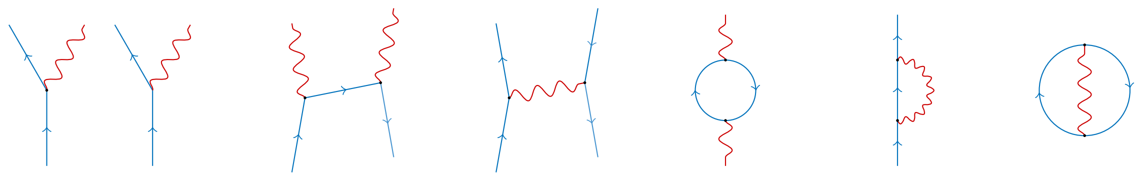
$$\frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y): + \qquad \text{(f)}$$

$$\frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y): + \qquad \text{(g)}$$

(e), (f), (g): double contractions

$$\frac{(-ie)^2}{2}\int d^4x\, d^4y \;:\hat{\bar{\psi}}(x)\gamma_\mu\hat{\psi}(x)\hat{A}^\mu(x)\hat{\bar{\psi}}(y)\gamma_\nu\hat{\psi}(y)\hat{A}^\nu(y):\,. \qquad \text{(h) triple contraction}$$

We will have non-zero matrix elements only if the initial and final states contain the correct number of creation and annihilation operators, so that the number of $\hat{a}$ matches the number of $\hat{a}^\dagger$ in $\hat{S}^{(2)}$, and so on. Thus, we can graphically represent the above terms as follows.



(a) (b),(c) (d) (e) (f),(g) (h)

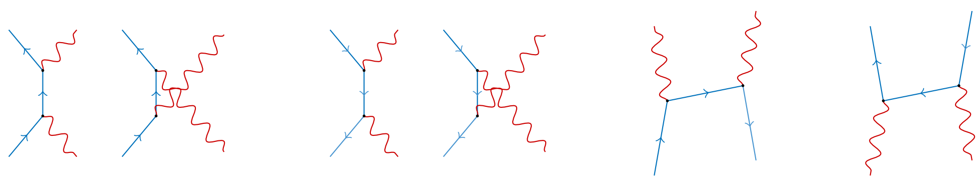
These diagrams are representatives of a class of diagrams, with the same number and type (bosonic or fermionic) of internal and external lines. For example, (a) corresponds to $64 = 8 \times 8$ diagrams, coming from the eight terms in the expansion of

$$:(\hat{b} + \hat{d}^\dagger)(\hat{d} + \hat{b}^\dagger)(\hat{a} + \hat{a}^\dagger): \,.$$

The diagrams represent the following physical situations:

a) All terms are kinematically forbidden, because of the 4-momentum conservation together with the dispersion relations. Thus, they do not represent physical situations.

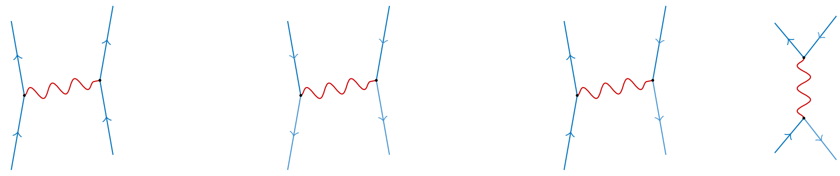
b) The terms which are not kinematically suppressed are four: fermion-photon scattering (bi), antifermion-photon scattering (bii), fermion-antifermion pair annihilation into two photons (biii), a fermion-antifermion pair creation by two photons (biv).



(bi) (bii) (biii) (biv)

c) The same as (b), as we can see with the change of variables $x \leftrightarrow y$ and indices $\mu \leftrightarrow \nu$.

d) The terms which are not kinematically suppressed are three: fermion-fermion scattering (di), antifermion-antifermion scattering (dii) and fermion-antifermion scattering (diii). Again in each case there is a digram and an exchange one, with the fermions or antifermions exchanged. Thus, a minus sign appear.



(di) (dii) (diii)

e) The diagrams represent an interaction of a photon with the Dirac sea, resulting in fermion-antifermion creation and annihilation.

f) The diagrams represent an interaction of a fermion or antifermion with the photonic vacuum, resulting in a creation and annihilation of a virtual photon.

g) The same as (f).

h) Here we have a vacuum fluctuation. However, the contribution to the S-matrix due to all bubble diagrams is a phase factor, which does not change the physics. Thus, it can be neglected.

c) d) To evaluate the S-matrix elements at a given order, we can use the Feynman rules for QED. They allow to write directly the algebraic expression for the matrix elements in both coordinate and momentum space, by resorting to a graphical representation of the different elements entering the Dyson series, without having to start from the Dyson expansion, thus saving a lot of work.

Feynman rules in coordinate space

1) To evaluate the matrix elements of $\hat{S}^{(n)}$, all topologically distinct diagrams with n vertices are drawn. To every vertex, we assign a coordinate x_i. The algebraic expression is obtained as follows.

2) Assign to each vertex the term $-ie\gamma_{\mu_i}$.

3) To each internal fermionic line between the vertices x_i and x_j, assign $i\Delta_F(x_i - x_j)$.

4) To each internal photonic line between the vertices x_i and x_j, assign $iD_F^{\mu_i\mu_j}(x_i - x_j)$.

5) To each external fermionic line of momentum p and spin s, assign:
   - $N_p u(\boldsymbol{p}, s)e^{-ipx_i}$ for an incoming fermion going to x_i;
   - $N_p \bar{u}(\boldsymbol{p}, s)e^{ipx_i}$ for an outgoing fermion coming from x_i;
   - $N_p \bar{v}(\boldsymbol{p}, s)e^{-ipx_i}$ for an incoming antifermion going to x_i;
   - $N_p v(\boldsymbol{p}, s)e^{ipx_i}$ for an outgoing antifermion coming from x_i.

   The normalization factors is set to be

$$N_p = \frac{1}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}.$$

6) To each external photonic line of momentum k and polarization λ, assign:
   - $N_k \epsilon^{\mu_i}(\boldsymbol{k}, \lambda)e^{-ikx_i}$ for an incoming photon going to x_i;
   - $N_k \epsilon^{\mu_i}(\boldsymbol{k}, \lambda)e^{ikx_i}$ for an outgoing photon coming from x_i.

   The normalization factors is set to be

$$N_k = \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}.$$

7) Integrate over $d^4x_1 \cdots d^4x_n$.

8) To each closed fermionic loop, assign a factor -1.

Feynman rules in momentum space

i) To evaluate the matrix elements of $\hat{S}^{(n)}$, all topologically distinct diagrams with n vertices are drawn. To each internal fermionic line assign a momentum p_l and to each internal photonic line, assign a direction and momentum k_l. The algebraic expression is obtained as follows.

ii) Assign to each vertex the term $-ie\gamma_{\mu_i}$.

iii) To each internal fermionic line, assign

$$i\Delta_F(p_l) = i\frac{\not{p}_l + m}{p_l^2 - m^2 + i\epsilon}.$$

iv) To each internal photonic line between the vertices x_i and x_j, assign

$$iD_F^{\mu_i\mu_j}(k_l) = \frac{-i\eta^{\mu_i\mu_j}}{k_l^2 + i\epsilon}.$$

v) To each external fermionic line of momentum p and spin s, assign:

- $N_p u(\boldsymbol{p}, s)$ for an incoming fermion going to x_i;
- $N_p \bar{u}(\boldsymbol{p}, s)$ for an outgoing fermion coming from x_i;
- $N_p \bar{v}(\boldsymbol{p}, s)$ for an incoming antifermion going to x_i;
- $N_p v(\boldsymbol{p}, s)$ for an outgoing antifermion coming from x_i.

The normalization factors is set to be

$$N_p = \frac{1}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{\omega_p}}.$$

vi) To each external photonic line of momentum k and polarization λ, assign:

- $N_k \epsilon^{\mu_i}(\boldsymbol{k}, \lambda)$ for an incoming photon going to x_i;
- $N_k \epsilon^{\mu_i}(\boldsymbol{k}, \lambda)$ for an outgoing photon coming from x_i.

The normalization factors is set to be

$$N_k = \frac{1}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\omega_k}}.$$

vii) To each vertex with incoming/outgoing fermion/antifermion of momentum p_i, incoming/outgoing fermion/antifermion of momentum p_j and incoming/outgoing photon of momentum k_l, assign

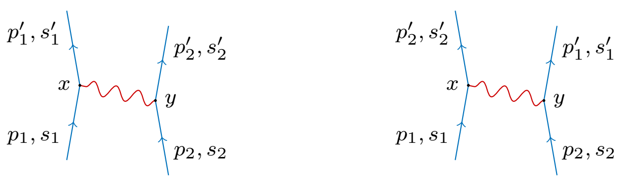
$$(2\pi)^4\delta^{(4)}(\pm p_i \pm p_j \pm k_l).$$

viii) Integrate over $\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4k_1}{(2\pi)^4}\cdots$.

ix) To each closed fermionic loop, assign a factor -1.

Note that the $n!$ factor in the Dyson series does not appear. This is a consequence of the permutation of the vertices.

e) Let us consider the Møller scattering of two electrons. It is a second order process, so that we must draw a diagram with two vertices. The only two possible Feynman diagrams are



The two diagrams are related by an exchange of identical particles. Thus, they are related by a minus sign. Using Feynman rules, we obtain

$$S_{fi} = \overbrace{\int d^4x\, d^4y}^{(7)}\ \overbrace{N_{p_1'}N_{p_1}\bar{u}(\boldsymbol{p}_1', s_1')(-ie\gamma_\mu)u(\boldsymbol{p}_1, s_1)\, e^{-i(p_1-p_1')x}}^{(2)+(5)}\ \overbrace{iD_F^{\mu\nu}(x-y)}^{(4)} \times$$
$$\times \underbrace{N_{p_2'}N_{p_2}\bar{u}(\boldsymbol{p}_2', s_2')(-ie\gamma_\nu)u(\boldsymbol{p}_2, s_2)\, e^{-i(p_2-p_2')y}}_{(2)+(5)} - \text{exchange}\Big(p_1', s_1' \longleftrightarrow p_2', s_2'\Big)$$
$$= -e^2 \int d^4x\, d^4y\, N_{p_1'}N_{p_1}N_{p_2'}N_{p_2}\, \bar{u}(\boldsymbol{p}_1', s_1')\gamma_\mu u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_2', s_2')\gamma_\nu u(\boldsymbol{p}_2, s_2) \times$$
$$\times\, e^{-i(p_1-p_1')x} e^{-i(p_2-p_2')y}\, iD_F^{\mu\nu}(x-y) \ - \ \text{exchange}\Big(p_1', s_1' \longleftrightarrow p_2', s_2'\Big).$$

Fourier transforming the Feynman propagator and integrating over the coordinates, gives

$$S_{fi} = -e^2 \int d^4x\, d^4y\, N_{p_1'}N_{p_1}N_{p_2'}N_{p_2}\, \bar{u}(\boldsymbol{p}_1', s_1')\gamma_\mu u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_2', s_2')\gamma_\nu u(\boldsymbol{p}_2, s_2) \times$$
$$\times\, e^{-i(p_1-p_1')x} e^{-i(p_2-p_2')y} \int \frac{d^4k}{(2\pi)^4} iD_F^{\mu\nu}(k)\, e^{-ik(x-y)} \ - \ \text{exchange}\Big(p_1', s_1' \longleftrightarrow p_2', s_2'\Big)$$
$$= -e^2 \int d^4y \frac{d^4k}{(2\pi)^4}\, N_{p_1'}N_{p_1}N_{p_2'}N_{p_2}\, \bar{u}(\boldsymbol{p}_1', s_1')\gamma_\mu u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_2', s_2')\gamma_\nu u(\boldsymbol{p}_2, s_2) \times$$
$$\times\, (2\pi)^4 \delta^{(4)}(p_1 - p_1' + k)\, e^{-i(p_2-p_2')y} iD_F^{\mu\nu}(k)\, e^{iky} \ - \ \text{exchange}\Big(p_1', s_1' \longleftrightarrow p_2', s_2'\Big)$$
$$= -e^2 \int \frac{d^4k}{(2\pi)^4}\, N_{p_1'}N_{p_1}N_{p_2'}N_{p_2}\, \bar{u}(\boldsymbol{p}_1', s_1')\gamma_\mu u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_2', s_2')\gamma_\nu u(\boldsymbol{p}_2, s_2) \times$$
$$\times\, (2\pi)^4 \delta^{(4)}(p_1 - p_1' + k)\, (2\pi)^4 \delta^{(4)}(p_2 - p_2' - k) iD_F^{\mu\nu}(k) \ - \ \text{exchange}\Big(p_1', s_1' \longleftrightarrow p_2', s_2'\Big).$$

We would have obtained the same expression from the diagrams with the prescription of Feynman rules in momentum space:

$$S_{fi} = \overbrace{\int \frac{d^4k}{(2\pi)^4}}^{\text{(viii)}}\ \overbrace{N_{p_1'}N_{p_1}\bar{u}(\boldsymbol{p}_1', s_1')(-ie\gamma_\mu)u(\boldsymbol{p}_1, s_1)}^{\text{(ii)+(v)}}\ \overbrace{(2\pi)^4\delta^{(4)}(p_1 - p_1' + k)}^{\text{(vii)}}\ \overbrace{iD_F^{\mu\nu}(k)}^{\text{(iv)}} \times$$
$$\times \underbrace{N_{p_2'}N_{p_2}\bar{u}(\boldsymbol{p}_2', s_2')(-ie\gamma_\nu)u(\boldsymbol{p}_2, s_2)}_{\text{(ii)+(v)}}\ \underbrace{(2\pi)^4\delta^{(4)}(p_2 - p_2' - k)}_{\text{(vii)}} - \text{exchange}\Big(p_1', s_1' \longleftrightarrow p_2', s_2'\Big).$$

The final expression is

$$S_{fi} = -e^2\, N_{p_1'}N_{p_1}N_{p_2'}N_{p_2}\, \bar{u}(\boldsymbol{p}_1', s_1')\gamma_\mu u(\boldsymbol{p}_1, s_1)\bar{u}(\boldsymbol{p}_2', s_2')\gamma_\nu u(\boldsymbol{p}_2, s_2) \times$$
$$\times\, (2\pi)^4 \delta^{(4)}(p_2 - p_2' - p_1' + p_1)\, iD_F^{\mu\nu}(p_1' - p_1) \ - \ \text{exchange}\Big(p_1', s_1' \longleftrightarrow p_2', s_2'\Big).$$

17 Electron-muon scattering

a) Introduce the scattering cross section σ from physical arguments. Define the $\mathcal{M}$-matrix $\mathcal{M}_{fi}$ in terms of the scattering matrix S_{fi}. Show how the cross section is defined in terms of $\mathcal{M}_{fi}$. Specialize the formula to the case of a two-body scattering process.

b) Consider the $e^+\ e^- \to \mu^+\ \mu^-$ scattering event. Draw the Feynman diagram to order e^2 and, using Feynman rules, write down the expression for the $\mathcal{M}$-matrix (scattering amplitude).

c) Write $|\mathcal{M}|^2$ and perform the sum over the spins of the electron and the muon. Show details. You will arrive at an expression containing the trace of the γ matrices.

d)Perform the trace under the assumption that the mass of the electron can be neglected. Show details.

e)Write the resulting expression for the cross section in the center-of-mass frame.

---

a) Let us consider a scattering process where a beam is scattered by a target. The number of scattering events per unit of time W will be proportional to the number density of the target n, the flux of the incident beam Φ, the overlap area A and the thickness of the target z. We define the cross section as the proportionality constant:

$$\sigma = \frac{W}{n\,\Phi A z}.$$

The scattering matrix can be expressed as

$$\hat{S} = \mathbb{1} + i\hat{T}.$$

In such a way, we separate the non-interacting term from the S-matrix. The T-matrix can be further split into a kinematic part and a dynamical part:

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) \left(\prod_i \frac{\chi_i}{(2\pi)^3 2\omega_{p_i}}\right)^{\frac{1}{2}} \left(\prod_f \frac{\chi_f}{(2\pi)^3 2\omega_{p_f}}\right)^{\frac{1}{2}} \mathcal{M}_{fi}.$$

Here χ_i or χ_f is 1 for bosons and $2m$ for fermions. Thus, the dynamical part is contained in the matrix elements $\mathcal{M}_{fi}$. If the interaction is non trivial, $\delta_{fi} = 0$ and the transition amplitude becomes

$$|S_{fi}|^2 = (2\pi)^8 \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) \prod_i \frac{\chi_i}{(2\pi)^3 2\omega_{p_i}} \prod_f \frac{\chi_f}{(2\pi)^3 2\omega_{p_f}} |\mathcal{M}_{fi}|^2.$$

The Dirac delta product can be easily treated by using the box normalization:

$$(2\pi)^4 \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) = \lim_{t,V\to\infty} \int_{t,V} d^4x\, e^{\left(\sum_i p_i - \sum_f p'_f\right)x} \longrightarrow (2\pi)^4 \delta^{(4)}(0) = tV.$$

The singularity $\delta^{(4)}(0)$ signals that working with definite energy and momentum implies infinite time and space separation of the initial and final states by the uncertainty principle. In particular, for large time t and volume V

$$\begin{aligned}(2\pi)^8 \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) &= (2\pi)^8 \delta^{(4)}(0) \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) \\ &= tV\,(2\pi)^4 \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big).\end{aligned}$$

Also the normalization factors must be chanced when using the box normalization:

$$\prod \frac{\chi}{(2\pi)^3 2\omega_p} \longrightarrow \prod \frac{\chi}{V 2\omega_p}.$$

Dividing the expression for $|S_{fi}|^2$ by the elapsed time t, we get the transition amplitude

$$W = V\,(2\pi)^4 \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) \prod_i \frac{\chi_i}{V 2\omega_{p_i}} \prod_f \frac{\chi_f}{V 2\omega_{p_f}} |\mathcal{M}_{fi}|^2.$$

The differential cross section will be

$$d\sigma = \frac{W\, d\Pi}{n\Phi A z},$$

where $d\Pi$ must take into account the infinitesimal variation of the final states. The correct expression turns out to be

$$d\Pi = \prod_f \frac{V\, d^3 p_f}{(2\pi)^3},$$

so that the integration over all possible final momenta is 1. The expression for the cross section becomes

$$d\sigma = \frac{1}{n\Phi Az} V\,(2\pi)^4 \delta^{(4)}\Big(\sum_i p_i - \sum_f p'_f\Big) \prod_i \frac{\chi_i}{V 2\omega_{p_i}} \prod_f \frac{\chi_f\, d^3 p_f}{(2\pi)^3 2\omega_{p_f}} |\mathcal{M}_{fi}|^2.$$

In the case of two-body scattering $1\ 2 \to 1'\ 2'\ 3' \cdots$, we can consider one incoming particle as the beam and the other incoming particle as the target. Thus, by introducing the relative velocity $v_{\rm rel}$, we can write

$$n = \frac{1}{V}, \qquad \Phi = \frac{v_{\rm rel}}{V}.$$

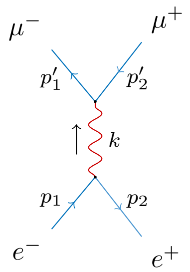
Note that the box normalization limit $t, V \to \infty$ agrees with our assumptions about the scattering process, which could happen at any time and everywhere in the space. Hence $Az = V$ and the cross section becomes

$$\sigma = \frac{V\,W}{v_{\rm rel}}.$$

Thus,

$$d\sigma = \frac{\chi_1 \chi_2}{4\omega_{p_1}\omega_{p_2} v_{\rm rel}} (2\pi)^4 \delta^{(4)}\Big(p_1 + p_2 - \sum_f p'_f\Big) \prod_f \frac{\chi_f\, d^3 p_f}{(2\pi)^3 2\omega_{p_f}} |\mathcal{M}_{fi}|^2.$$

b) Let us consider the $e^+\, e^- \to \mu^+\, \mu^-$ scattering. We have just the Feynman diagram



Using Feynman rules in momentum space, the scattering amplitude will be

$$\begin{aligned} i\mathcal{M}_{s_1 s_2 s'_1 s'_2} &= \bar v(\boldsymbol{p}_2, s_2)(-ie\gamma_\mu) u(\boldsymbol{p}_1, s_1) \bar u(\boldsymbol{p}'_2, s'_1)(-ie\gamma_\nu) v(\boldsymbol{p}'_1, s'_1)\, iD_F^{\mu\nu}(k) \\ &= \frac{ie^2}{k^2}\, \bar v(\boldsymbol{p}_2, s_2)\gamma^\mu u(\boldsymbol{p}_1, s_1) \bar u(\boldsymbol{p}'_2, s'_2)\gamma_\mu v(\boldsymbol{p}'_1, s'_1), \end{aligned}$$

where we have used the expression

$$D_F^{\mu\nu}(k) = \frac{-\eta^{\mu\nu}}{k^2}.$$

c) The above expression depends on the spin of the particles. The average of the square modulus of the scattering amplitude over all possible spin states is

$$|\mathcal{M}|^2 = \frac{1}{4} \sum_{s_1, s_2, s'_1, s'_2} |\mathcal{M}_{s_1 s_2 s'_1 s'_2}|^2.$$

On the other hand,

$$\begin{aligned} \big(\bar v(\boldsymbol{p}_2, s_2)\gamma^\mu u(\boldsymbol{p}_1, s_1)\big)^* &= u^\dagger(\boldsymbol{p}_1, s_1)\big(\gamma^\mu\big)^\dagger \big(v^\dagger(\boldsymbol{p}_2, s_2)\gamma^0\big)^\dagger = u^\dagger(\boldsymbol{p}_1, s_1)\big(\gamma^\mu\big)^\dagger \big(\gamma^0\big)^\dagger v(\boldsymbol{p}_2, s_2) \\ &= u^\dagger(\boldsymbol{p}_1, s_1)\big(\gamma^0\gamma^\mu\big)^\dagger v(\boldsymbol{p}_2, s_2) = u^\dagger(\boldsymbol{p}_1, s_1)\gamma^0\gamma^\mu v(\boldsymbol{p}_2, s_2) \\ &= \bar u(\boldsymbol{p}_1, s_1)\gamma^\mu v(\boldsymbol{p}_2, s_2) \end{aligned}$$

and similarly

$$\big(\bar u(\boldsymbol{p}'_2, s'_2)\gamma_\mu v(\boldsymbol{p}'_1, s'_1)\big)^* = \bar v(\boldsymbol{p}'_1, s'_1)\gamma_\mu u(\boldsymbol{p}'_2, s'_2).$$

Thus, we arrive at the expression

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{4k^4} \sum_{s_1,s_2,s_1',s_2'} \bar{v}(\boldsymbol{p}_2,s_2)\gamma^\mu u(\boldsymbol{p}_1,s_1)\bar{u}(\boldsymbol{p}_2',s_2')\gamma_\mu v(\boldsymbol{p}_1',s_1')\bar{u}(\boldsymbol{p}_1,s_1)\gamma^\nu v(\boldsymbol{p}_2,s_2)\bar{v}(\boldsymbol{p}_1',s_1')\gamma_\nu u(\boldsymbol{p}_2',s_2') \\
&= \frac{e^4}{4k^4}\left(\sum_{s_1,s_2} \bar{v}(\boldsymbol{p}_2,s_2)\gamma^\mu u(\boldsymbol{p}_1,s_1)\bar{u}(\boldsymbol{p}_1,s_1)\gamma^\nu v(\boldsymbol{p}_2,s_2)\right)\times \\
&\qquad\times\left(\sum_{s_1',s_2'} \bar{u}(\boldsymbol{p}_2',s_2')\gamma_\mu v(\boldsymbol{p}_1',s_1')\bar{v}(\boldsymbol{p}_1',s_1')\gamma_\nu u(\boldsymbol{p}_2',s_2')\right).
\end{aligned}
$$

We can use now the completeness relations

$$
\sum_s u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s) = \frac{\not{p}+m}{2m}, \qquad \sum_s v(\boldsymbol{p},s)\bar{v}(\boldsymbol{p},s) = \frac{\not{p}-m}{2m}.
$$

By writing the factors in components and using the above expression, we find

$$
\begin{aligned}
&\sum_{s_1,s_2} \big(\bar{v}(\boldsymbol{p}_2,s_2)\big)_\alpha \big(\gamma^\mu\big)_{\alpha\beta}\big(u(\boldsymbol{p}_1,s_1)\big)_\beta \big(\bar{u}(\boldsymbol{p}_1,s_1)\big)_\rho \big(\gamma^\nu\big)_{\rho\sigma}\big(v(\boldsymbol{p}_2,s_2)\big)_\sigma = \\
&\qquad = \frac{1}{4m_e^2}\big(\not{p}_2 - m_e\big)_{\sigma\alpha}\big(\gamma^\mu\big)_{\alpha\beta}\big(\not{p}_1+m_e\big)_{\beta\rho}\big(\gamma^\nu\big)_{\rho\sigma} \\
&\qquad = \frac{1}{4m_e^2}\,\mathrm{tr}\Big((\not{p}_2 - m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\Big).
\end{aligned}
$$

Here the sum over repeated matrix indices is omitted. Furthermore, we have used the expression for the trace of a product:

$$
\mathrm{tr}\,\big(ABCD\big) = A_{ij}B_{jk}C_{kl}D_{li}.
$$

Thus,

$$
|\mathcal{M}|^2 = \frac{e^4}{4k^4}\frac{1}{4m_e^2\,4m_\mu^2}\,\mathrm{tr}\Big((\not{p}_2 - m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\Big)\,\mathrm{tr}\Big((\not{p}_2' - m_\mu)\gamma_\mu(\not{p}_1'+m_\mu)\gamma_\nu\Big).
$$

d) The traces can be expressed in terms of the traces of γ matrices:

$$
\begin{aligned}
\mathrm{tr}\,\gamma^\mu &= 0 \\
\mathrm{tr}\,\gamma^\mu\gamma^\nu &= 4\eta^{\mu\nu} \\
\mathrm{tr}\,\gamma^\mu\gamma^\lambda\gamma^\nu &= 0 \\
\mathrm{tr}\,\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu &= 4(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\sigma}\eta^{\mu\nu} + \eta^{\rho\nu}\eta^{\mu\sigma}).
\end{aligned}
$$

The first relation holds for any representation of the matrices, and it can be proved by introducing the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Such matrix has the following properties:

$$
\big(\gamma^5\big)^2 = \mathbb{1}, \qquad \{\gamma^5,\gamma^\mu\} = 0.
$$

Thus, the trace of a γ matrix can be computed as

$$
\begin{aligned}
\mathrm{tr}\,\gamma^\mu &= \mathrm{tr}\,\gamma^5\gamma^5\gamma^\mu \\
&= -\,\mathrm{tr}\,\gamma^5\gamma^\mu\gamma^5 && \text{for the anticommutation relation} \\
&= -\,\mathrm{tr}\,\gamma^5\gamma^5\gamma^\mu && \text{for cyclicity} \\
&= -\,\mathrm{tr}\,\gamma^\mu.
\end{aligned}
$$

It is clear that the proof applies to all odd products of γ matrices. The second one follows from the cyclic property of the matrices and the anticommutation relation:

$$
\begin{aligned}
\mathrm{tr}\,\gamma^\mu\gamma^\nu &= \mathrm{tr}\,(2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2\eta^{\mu\nu}\,\mathrm{tr}\,\mathbb{1} - \mathrm{tr}\,\gamma^\nu\gamma^\mu \\
&= 8\eta^{\mu\nu} - \mathrm{tr}\,\gamma^\mu\gamma^\nu,
\end{aligned}
$$

so that $\operatorname{tr}\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$. Again, the proof can be generalized to all even products of γ matrices, with the alternate sum of cyclic permutations of the indices. With the help of this trace identity, we find

$$\begin{aligned}
\operatorname{tr}\Big((\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\Big) &= \operatorname{tr}\Big(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\Big) - m_e\operatorname{tr}\Big(\gamma^\mu\not{p}_1\gamma^\nu\Big) + m_e\operatorname{tr}\Big(\not{p}_2\gamma^\mu\gamma^\nu\Big) - m_e^2\operatorname{tr}\Big(\gamma^\mu\gamma^\nu\Big)\\
&= p_{2\rho}p_{1\sigma}\operatorname{tr}\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - m_e p_{1\lambda}\operatorname{tr}\gamma^\mu\gamma^\lambda\gamma^\nu + m_e p_{2\lambda}\operatorname{tr}\gamma^\lambda\gamma^\mu\gamma^\nu - m_e^2\operatorname{tr}\gamma^\mu\gamma^\nu\\
&= 4\Big(p_2^\mu p_1^\nu - p_1\cdot p_2\eta^{\mu\nu} + p_2^\nu p_1^\mu - m_e^2\eta^{\mu\nu}\Big)\\
&= 4\Big(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - \eta^{\mu\nu}(m_e^2 + p_1\cdot p_2)\Big)
\end{aligned}$$

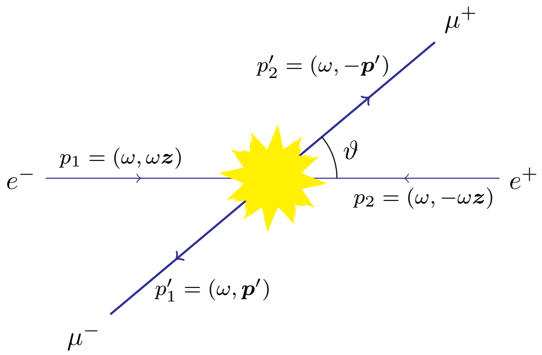
Performing similar calculations for the second trace, we arrive at the expression

$$|\mathcal{M}|^2 = \frac{4e^4}{k^4}\frac{1}{4m_e^2\,4m_\mu^2}\Big(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - \eta^{\mu\nu}(m_e^2 + p_1\cdot p_2)\Big)\Big(p'_{2\mu}p'_{1\nu} + p'_{2\nu}p'_{1\mu} - \eta_{\mu\nu}(m_\mu^2 + p'_1\cdot p'_2)\Big)$$

Now, since $\frac{m_e}{m_\mu} \simeq \frac{1}{200}$, we can neglect the terms containing the electron mass. Note that the mass term at the denominator will be simplified in the scattering cross section, since the denominator is exactly $\chi_1\chi_2\chi'_1\chi'_2$. Setting $m = m_\mu$ and $\lambda = \chi_1\chi_2\chi'_1\chi'_2$ we obtain

$$\begin{aligned}
\lambda|\mathcal{M}|^2 &= \frac{4e^4}{k^4}\Big(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - \eta^{\mu\nu}p_1\cdot p_2\Big)\Big(p'_{2\mu}p'_{1\nu} + p'_{2\nu}p'_{1\mu} - \eta_{\mu\nu}(m^2 + p'_1\cdot p'_2)\Big)\\
&= \frac{4e^4}{k^4}\Big(2(p_1\cdot p'_1)(p_2\cdot p'_2) + 2(p_1\cdot p'_2)(p_2\cdot p'_1) - 2(p_1\cdot p_2)(m^2 + p'_1\cdot p'_2) +\\
&\qquad - 2(p'_1\cdot p'_2)(p_1\cdot p_2) + 4(p_1\cdot p_2)(m^2 + p'_1\cdot p'_2)\Big)\\
&= \frac{8e^4}{k^4}\Big((p_1\cdot p'_1)(p_2\cdot p'_2) + (p_1\cdot p'_2)(p_2\cdot p'_1) + m^2(p_1\cdot p_2)\Big).
\end{aligned}$$

e) In the CoM frame, implementing the conservation of momentum that comes from the S-matrix and under the assumption $m_e = 0$, we have



Thus, taking into account that $k^4 = (p_1 + p_2)^4 = 16\omega^4 = s^2$, where s is the invariant relativistic, the above expression becomes

$$\begin{aligned}
\lambda|\mathcal{M}|^2 &= \frac{8e^4}{16\omega^4}\Big((\omega^2 + \omega|\boldsymbol{p}'|\cos\vartheta)^2 + (\omega^2 - \omega|\boldsymbol{p}'|\cos\vartheta)^2 + m^2(2\omega^2)\Big)\\
&= \frac{e^4}{2\omega^4}\Big(2\omega^4 + 2\omega^2|\boldsymbol{p}'|^2\cos^2\vartheta + 2m^2\omega^2\Big)\\
&= e^4\left(1 + \frac{|\boldsymbol{p}'|^2}{\omega^2}\cos^2\vartheta + \frac{m^2}{\omega^2}\right)\\
&= e^4\left(1 + \frac{m^2}{\omega^2} + \left(1 - \frac{m^2}{\omega^2}\right)\cos^2\vartheta\right).
\end{aligned}$$

Let us rewrite now the infinitesimal cross section for a process $1\ 2 \to 1'\ 2'$.

$$\begin{aligned} d\sigma &= \frac{1}{(2\pi)^6 v_{\text{rel}}} \frac{\chi_1\chi_2}{4\omega_1\omega_2} \frac{\chi_1'\chi_2'}{4\omega_1'\omega_2'} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2')\, d^3p_1'\, d^3p_2'\, |\mathcal{M}|^2 \\ &= \frac{1}{16\pi^2\, v_{\text{rel}}} \frac{\chi_1\chi_2}{4\omega_1\omega_2} \frac{\chi_1'\chi_2'}{\omega_1'\omega_2'} \delta^{(4)}(p_1 + p_2 - p_1' - p_2')\, d^3p_1'\, d^3p_2'\, |\mathcal{M}|^2. \end{aligned}$$

We can use the delta function to fix the momentum $\boldsymbol{p}_2'$ and the modulus $|\boldsymbol{p}_1'|$, obtaining an expression in the infinitesimal solid angle $d\Omega$ identified by the direction of $\boldsymbol{p}_1'$. The following equalities can be considered true only under integration.

$$\begin{aligned} \delta^{(4)}(p_1 + p_2 - p_1' - p_2')\, d^3p_1'\, d^3p_2' &= \delta^{(3)}(\boldsymbol{p}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1' - \boldsymbol{p}_2')\ \delta(\omega_1 + \omega_2 - \omega_1' - \omega_2')\, d^3p_1'\, d^3p_2' \\ &= \delta(\omega_1 + \omega_2 - \omega_1' - \omega_2')\, d^3p_1' \\ &= \delta(\omega_1 + \omega_2 - \omega_1' - \omega_2')\, |\boldsymbol{p}_1'|^2\, d|\boldsymbol{p}_1'|\, d\Omega \\ &= \frac{1}{\left|\frac{d(\omega_1' + \omega_2')}{d|\boldsymbol{p}_1'|}\right|} |\boldsymbol{p}_1'|^2\, d\Omega. \end{aligned}$$

In the last two steps we have performed a transformation in spherical coordinates $d^3p_1' \to |\boldsymbol{p}_1'|^2\, d|\boldsymbol{p}_1'|\, d\Omega$ and the change of variable formula for the Dirac delta

$$\delta(f(x)) = \sum_{x_0\ |\ f(x_0)\ =\ 0} \frac{\delta(x - x_0)}{|f'(x_0)|}.$$

Furthermore, all the values must be computed for $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2'$, as a consequence of the delta in 3-momentum. Hence, we have the expression

$$d\sigma = \frac{1}{16\pi^2\, v_{\text{rel}}} \frac{\chi_1\chi_2}{4\omega_1\omega_2} \frac{\chi_1'\chi_2'}{\omega_1'\omega_2'} \frac{1}{\left|\frac{d(\omega_1' + \omega_2')}{d|\boldsymbol{p}_1'|}\right|} |\boldsymbol{p}_1'|^2\, |\mathcal{M}|^2\, d\Omega.$$

In the CoM frame for $e^-e^+ \to \mu^-\mu^+$ with $m_e = 0$, we have $v_{\text{rel}} = 2$, $\omega_1 = \omega_2 = \omega_1' = \omega_2' = \omega$ and

$$\frac{d(\omega_1' + \omega_2')}{d|\boldsymbol{p}_1'|} = \frac{d\left(2\sqrt{m^2 + |\boldsymbol{p}'|^2}\right)}{d|\boldsymbol{p}'|} = \frac{2|\boldsymbol{p}'|}{\omega}.$$

The expression becomes

$$\begin{aligned} d\sigma &= \frac{1}{16\pi^2} \frac{1}{2} \frac{1}{4\omega^2} \frac{1}{\omega^2} \frac{\omega}{2|\boldsymbol{p}'|} |\boldsymbol{p}'|^2\, \lambda |\mathcal{M}|^2\, d\Omega \\ &= \frac{1}{16\pi^2} \frac{1}{16\omega^2} \frac{|\boldsymbol{p}'|}{\omega} e^4 \left(1 + \frac{m^2}{\omega^2} + \left(1 - \frac{m^2}{\omega^2}\right) \cos^2\vartheta\right) d\Omega \\ &= \frac{e^4}{16\pi^2} \frac{1}{16\omega^2} \frac{\sqrt{\omega^2 - m^2}}{\omega} \left(1 + \frac{m^2}{\omega^2} + \left(1 - \frac{m^2}{\omega^2}\right) \cos^2\vartheta\right) d\Omega \\ &= \alpha^2 \frac{1}{16\omega^2} \sqrt{1 - \frac{m^2}{\omega^2}} \left(1 + \frac{m^2}{\omega^2} + \left(1 - \frac{m^2}{\omega^2}\right) \cos^2\vartheta\right) d\Omega, \end{aligned}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant. The cross section turns out to be

$$\begin{aligned}
\sigma &= \alpha^2 \frac{1}{16\omega^2}\sqrt{1-\frac{m^2}{\omega^2}} \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\vartheta) \left(1+\frac{m^2}{\omega^2}+\left(1-\frac{m^2}{\omega^2}\right)\cos^2\vartheta\right) \\
&= \alpha^2 \frac{1}{16\omega^2}\sqrt{1-\frac{m^2}{\omega^2}}\, 2\pi \int_{-1}^{1} du \left(1+\frac{m^2}{\omega^2}+\left(1-\frac{m^2}{\omega^2}\right)u^2\right) \\
&= \alpha^2 \frac{1}{16\omega^2}\sqrt{1-\frac{m^2}{\omega^2}}\, 4\pi \left(1+\frac{m^2}{\omega^2}+\frac{1}{3}\left(1-\frac{m^2}{\omega^2}\right)\right) \\
&= \alpha^2 \frac{1}{16\omega^2}\sqrt{1-\frac{m^2}{\omega^2}}\, 4\pi \left(\frac{4}{3}+\frac{2}{3}\frac{m^2}{\omega^2}\right) \\
&= \frac{\pi\alpha^2}{3}\frac{1}{\omega^2}\sqrt{1-\frac{m^2}{\omega^2}}\left(1+\frac{1}{2}\frac{m^2}{\omega^2}\right).
\end{aligned}$$

Remembering that $16\omega^4 = s^2$, where s is the invariant relativistic, the above expression becomes

$$\sigma = \frac{\pi\alpha^2}{3}\frac{1}{\omega^2}\sqrt{1-\frac{m^2}{\omega^2}}\left(1+\frac{1}{2}\frac{m^2}{\omega^2}\right) = \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right).$$

For particle with energies such that $s >> m^2$ we can neglect all the mass terms and finally find the famous cross section formula

$$\sigma = \frac{4\pi\alpha^2}{3s}.$$