A Solvable Sextic Equation

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This paper presents a solvable sextic equation under the condition that several coefficients of such polynomials are restricted to become dependent on the preceding or following coefficients. We can solve a sextic equation by restricting one or two in total seven coefficients available, and by solving a bisextic equation and a quintic equation. And we can also find the arbitrary coupling coefficients that generate a new solvable sextic equation as well.

A. A solvable sextic equation

One of the solvable sextic equations is the de Moivre's sextic that comes from de Moivre's theorem, which is given as follows.
If $x = \sqrt[6]{\alpha} - \frac{s}{6\sqrt[6]{\alpha}}$, we have

$$x^6 + sx^4 + \frac{1}{4}s^2x^2 + t \tag{1}$$

$$= \alpha + t - \frac{1}{108}s^3 + \frac{s^6}{46656\alpha} \tag{2}$$

$$= 0.$$

From (2), we get

$$\alpha = -\frac{1}{2}t + \frac{1}{216}s^3 \pm \sqrt{\frac{1}{4}t^2 - \frac{1}{216}s^3t}. \tag{3}$$

With this, we get the solution of the de Moivre sextic,

$$x = \sqrt[6]{-\frac{1}{2}t + \frac{1}{216}s^3 \pm \sqrt{\frac{1}{4}t^2 - \frac{1}{216}s^3t}} - \sqrt[6]{-\frac{1}{2}t + \frac{1}{216}s^3 \mp \sqrt{\frac{1}{4}t^2 - \frac{1}{216}s^3t}}, \tag{4}$$

where one of $\pm$ in the left side square root changes to the opposite sign on the right side alternately. The solution of the de Moivre's sextic equation does not guarantee that all radicals of the sextic equation can be found.

The other one is

$$x^6 + d_4x^4 + d_2x^2 + d_0 = 0. \tag{5}$$

This sextic provides a solution

$$x = \pm\sqrt{-\frac{1}{3}d_4 + \sqrt[3]{-\frac{1}{2}d_0 + \frac{1}{6}d_2d_4 - \frac{1}{27}{d_4}^3 - \sqrt{D}} + \sqrt[3]{-\frac{1}{2}d_0 + \frac{1}{6}d_2d_4 - \frac{1}{27}{d_4}^3 + \sqrt{D}}}, \tag{6}$$

where D represents the discriminant,

$$D = \left(-\frac{1}{2}d_0 + \frac{1}{6}d_2d_4 - \frac{1}{27}{d_4}^3\right)^2 + \left(\frac{1}{3}d_2 - \frac{1}{9}{d_4}^2\right)^3. \tag{7}$$

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There is no solvable sextic equation in general, with the exceptions of the above including $x^6+c_3x^3+c_0=0$. There are three ways to factor a sextic equation to derive a solution by splitting the sextic equation into lower polynomials. One of them is a sextic equation factored by two cubics,

$$(x^3-b_2x^2+a_1x+a_0)(x^3+b_2x^2+b_1x+b_0)=0. \tag{8}$$

The other solution is a sextic equation factored into a quartic and a quadratic, i.e.,

$$(x^4-b_1x^3+a_2x^2+a_1x+a_0)(x^2+b_1x+b_0)=0. \tag{9}$$

We can get solutions of the sextic equation factorized to a cubic equation as well as a quadratic factor of the above factorization.

B. A solvable sextic equation factored by two cubics

A solvable sextic equation factored by two cubic equations may be written as follows;

$$(x^3-b_2x^2+a_1x+a_0)(x^3+b_2x^2+b_1x+b_0)=x^6+c_4x^4+c_3x^3+c_2x^2+c_1x+c_0. \tag{10}$$

After eliminating the coefficients a_i, then b_0 becomes

$$b_0=(b_1c_4+3b_1{b_2}^2+b_2c_3-{b_1}^2-{b_2}^4-{b_2}^2c_4-c_2)/(2b_2), \quad b_2\neq 0, \tag{11}$$

and, we get the following two simultaneous equations with respect to b_1,

$$\begin{aligned}&\frac{{b_1}^3}{b_2}+\left(-\frac{3}{2}b_2-\frac{3c_4}{2b_2}\right){b_1}^2+\left(2b_2c_4+\frac{3}{2}{b_2}^3+\frac{c_2}{b_2}+\frac{{c_4}^2}{2b_2}\right)b_1\\&+\left(-\frac{1}{2}b_2c_2-\frac{1}{2}b_2{c_4}^2+\frac{1}{2}c_3c_4-\frac{1}{2}{b_2}^5-\frac{c_2c_4}{2b_2}+\frac{1}{2}{b_2}^2c_3-{b_2}^3c_4\right)-c_1=0,\end{aligned} \tag{12}$$

$$\begin{aligned}c_0=&-\frac{{b_1}^4}{4{b_2}^2}+\left(\frac{1}{2}+\frac{c_4}{2{b_2}^2}\right){b_1}^3+\left(-\frac{1}{2}c_4+\frac{3}{4}{b_2}^2-\frac{c_2}{2{b_2}^2}-\frac{{c_4}^2}{4{b_2}^2}\right){b_1}^2\\&+\left(\frac{1}{2}c_2+b_2c_3-{b_2}^4+\frac{c_2c_4}{2{b_2}^2}-{b_2}^2c_4\right)b_1\\&+\left(-\frac{1}{2}b_2c_3c_4+\frac{1}{4}{b_2}^6+\frac{1}{4}{c_3}^2-\frac{{c_2}^2}{4{b_2}^2}+\frac{1}{4}{b_2}^2{c_4}^2-\frac{1}{2}{b_2}^3c_3+\frac{1}{2}{b_2}^4c_4\right)\end{aligned} \tag{13}$$

From the equation (12), we get a solution of b_1 after manipulation as follows

$$\begin{aligned}b_1=&\ \frac{1}{2}c_4+\frac{1}{2}{b_2}^2+\sqrt[3]{-\frac{1}{2}\left(-b_2c_1+\frac{1}{2}b_2c_3c_4+\frac{1}{2}{b_2}^3c_3\right)-\frac{1}{8}\sqrt{D_1}}\\&+\sqrt[3]{-\frac{1}{2}\left(-b_2c_1+\frac{1}{2}b_2c_3c_4+\frac{1}{2}{b_2}^3c_3\right)+\frac{1}{8}\sqrt{D_1}}\end{aligned} \tag{14}$$

where D_1 is given as the discriminant of the cubic equation,

$$\begin{aligned}D_1=&\ {b_2}^{12}+2c_4{b_2}^{10}+\left(4c_2+\frac{1}{3}{c_4}^2\right){b_2}^8+\left(\frac{16}{3}c_2c_4+4{c_3}^2-\frac{28}{27}{c_4}^3\right){b_2}^6\\&+\left(-16c_1c_3-\frac{8}{9}c_2{c_4}^2+\frac{16}{3}{c_2}^2-\frac{1}{9}{c_4}^4+8{c_3}^2c_4\right){b_2}^4\\&+\left(-16c_1c_3c_4-\frac{16}{9}c_2{c_4}^3+16{c_1}^2+\frac{2}{9}{c_4}^5+\frac{32}{9}{c_2}^2c_4+4{c_3}^2{c_4}^2\right){b_2}^2\\&+\left(\frac{4}{9}c_2{c_4}^4+\frac{64}{27}{c_2}^3-\frac{1}{27}{c_4}^6-\frac{16}{9}{c_2}^2{c_4}^2\right).\end{aligned} \tag{15}$$

Though this equation with respect to b_2 is a 12th degree equation, it can be solved using a sextic equation. An equation that can be solved by a sextic equation among 12th degree equations may be called a bisextic equation.

C. A solvable Sextic Equation factored by a quadratic equation

Another solvable sextic equation may be given using the factors of a quartic equation and a quadratic equation as follows,

$$\begin{aligned} & (x^4 - v_1 x^3 + a_2 x^2 + a_1 x + a_0)(x^2 + v_1 x + v_0) \\ = \; & x^6 + d_4 x^4 + d_3 x^3 + d_2 x^2 + d_1 x + d_0 = 0. \end{aligned} \tag{16}$$

Developing the above, we get

$$\begin{aligned} a_2 &= d_4 - v_0 + {v_1}^2, \\ a_1 &= d_3 + 2v_0 v_1 - v_1 d_4 - {v_1}^3, \\ a_0 &= d_2 - v_0 d_4 - 3v_0 {v_1}^2 - v_1 d_3 + {v_0}^2 + {v_1}^4 + {v_1}^2 d_4. \end{aligned} \tag{17}$$

Plugging the above coefficients, we have

$$\begin{aligned} & x^6 + d_4 x^4 + d_3 x^3 + d_2 x^2 + (-2v_0 v_1 d_4 + v_0 d_3 - 4v_0 {v_1}^3 + v_1 d_2 + {v_1}^5 + 3{v_0}^2 v_1 - {v_1}^2 d_3 + {v_1}^3 d_4)x \\ & - v_0 v_1 d_3 + v_0 d_2 + v_0 {v_1}^2 d_4 + v_0 {v_1}^4 + {v_0}^3 - {v_0}^2 d_4 - 3{v_0}^2 {v_1}^2 = 0. \end{aligned} \tag{18}$$

This sextic equation has a quadratic factor

$$x^2 + v_1 x + v_0 = 0. \tag{19}$$

From (18), we have two simultaneous equations with respect to v_1,

$$d_1 = {v_1}^5 + (-4v_0 + d_4){v_1}^3 - d_3 {v_1}^2 + (d_2 - 2v_0 d_4 + 3{v_0}^2)v_1 + v_0 d_3, \tag{20}$$

$$d_0 = v_0 {v_1}^4 + (v_0 d_4 - 3{v_0}^2){v_1}^2 - v_0 d_3 v_1 + v_0 d_2 + {v_0}^3 - {v_0}^2 d_4. \tag{21}$$

By using the equation (20), we may get a solution of a quintic by the use of the de Moivre's quintic equation as follows with two conditions that the coefficient of ${v_1}^2$ term $d_3 = 0$ and the two coefficients of ${v_1}^3$ term and v_1 term are equal to each other. That is, the square of the former is equal to 5 times of the latter as in the de Moivre's quintic, and the coefficient d_0 of (21) dependent on the preceding coefficients of the sextic equation,

$$(-4v_0 + d_4)^2 - 5(d_2 - 2v_0 d_4 + 3{v_0}^2) = {v_0}^2 + 2d_4 v_0 + {d_4}^2 - 5d_2. \tag{22}$$

From (22), we have

$$v_0 = -d_4 \pm \sqrt{5d_2}, \tag{23}$$

then the equation (20) provides

$${v_1}^5 + \left(5d_4 - 4\sqrt{5d_2}\right){v_1}^3 + \left(16d_2 + 5{d_4}^2 - 8d_4\sqrt{5d_2}\right)v_1 - d_1 = 0, \tag{24}$$

and we have a solution of v_1

$$v_1 = \sqrt[5]{\frac{1}{2}d_1 - \sqrt{D_2}} + \sqrt[5]{\frac{1}{2}d_1 + \sqrt{D_2}}, \tag{25}$$

where, D_2 is the discriminant of the resolvent quintic (20),

$$D_2 = \frac{1}{4}{d_1}^2 + \left(d_4 - \frac{4\sqrt{5d_2}}{5}\right)^5. \tag{26}$$

The sextic equation (16) has a quadratic factor

$$x^2 + \left(\sqrt[5]{\frac{1}{2}d_1 - \sqrt{D_2}} + \sqrt[5]{\frac{1}{2}d_1 + \sqrt{D_2}} \right) x - d_4 + \sqrt{5d_2} = 0, \tag{27}$$

with D_2 of (26).

From (27), we have two roots, which are the two radicals of the sextic equation (16)

$$\begin{aligned} x \;=\; & -\frac{1}{2}\left(\sqrt[5]{\frac{1}{2}d_1 - \sqrt{D_2}} + \sqrt[5]{\frac{1}{2}d_1 + \sqrt{D_2}} \right) \\ & \pm\sqrt{d_4 - \sqrt{5d_2} + \frac{1}{4}\left(\sqrt[5]{\frac{1}{2}d_1 - \sqrt{D_2}} + \sqrt[5]{\frac{1}{2}d_1 + \sqrt{D_2}} \right)^2}. \end{aligned} \tag{28}$$

This process is also one of the ways to find the conditional solution of a sextic equation.

D. Derivation of a solvable sextic equation factored by two cubic equations

For solving a bisextic equation, replace ${b_2}^2$ in the discriminant D_1 of (15) with y, and after reducing the coefficient of y^5 term by ${b_2}^2 = y - c_4/3$, we have

$$\begin{aligned} F(D_1) \;=\; & y^6 + \left(4c_2 - \frac{4}{3}{c_4}^2\right) y^4 + 4{c_3}^2 y^3 + \left(-16c_1c_3 - \frac{32}{9}c_2{c_4}^2 + \frac{16}{3}{c_2}^2 + \frac{16}{27}{c_4}^4 + 4{c_3}^2 c_4\right) y^2 \\ & + \left(-\frac{16}{3}c_1c_3c_4 + 16{c_1}^2\right) y \\ & + \left(\frac{32}{9}c_1c_3{c_4}^2 + \frac{64}{81}c_2{c_4}^4 + \frac{64}{27}{c_2}^3 - \frac{64}{729}{c_4}^6 - \frac{16}{3}{c_1}^2 c_4 - \frac{64}{27}{c_2}^2{c_4}^2 - \frac{16}{27}{c_3}^2{c_4}^3\right). \end{aligned} \tag{29}$$

If $F(D_1)$ has a quadratic factor, $(y^2 + v_1 y + v_0)$, then we get by using the equation (20),

$$\begin{aligned} & {v_1}^5 + \left(4c_2 - 4v_0 - \frac{4}{3}{c_4}^2\right) {v_1}^3 - 4{c_3}^2{v_1}^2 \\ & + \left(-16c_1c_3 - 8c_2v_0 - \frac{32}{9}c_2{c_4}^2 + \frac{16}{3}{c_2}^2 + \frac{16}{27}{c_4}^4 + 3{v_0}^2 + 4{c_3}^2c_4 + \frac{8}{3}{c_4}^2 v_0\right) v_1 \\ & + \frac{16}{3}c_1c_3c_4 - 16{c_1}^2 + 4{c_3}^2 v_0 = 0. \end{aligned} \tag{30}$$

If the above quintic equation is to be a de Moivre's quintic, we can solve the quintic. In order to do so, the coefficient c_3 of ${v_1}^2$ term is equal to zero, and the square of the coefficient of ${v_1}^3$ term is equal to five times the coefficient of v_1 term. With these conditions, we have

$$v_0 = -4c_2 + \frac{4}{3}{c_4}^2 \pm 4\sqrt{-\frac{10}{9}c_2{c_4}^2 + \frac{5}{3}{c_2}^2 + \frac{5}{27}{c_4}^4}. \tag{31}$$

And we get v_1,

$$v_1 = \sqrt[5]{8{c_1}^2 - \sqrt{D_3}} + \sqrt[5]{8{c_1}^2 + \sqrt{D_3}}, \tag{32}$$

where D_3 represents

$$D_3 = 64{c_1}^4 + \left(4c_2 - \frac{4}{3}{c_4}^2 - \frac{16}{5}\sqrt{-\frac{10}{9}c_2{c_4}^2 + \frac{5}{3}{c_2}^2 + \frac{5}{27}{c_4}^4}\right)^5. \tag{33}$$

From the quadratic factor of the equation (29), we get two roots of the sextic

$$y = -\frac{1}{2}v_1 \pm \sqrt{\frac{1}{4}{v_1}^2 - v_0}. \tag{34}$$

With the result, b_2 of (15) is given as follows;

$$b_2 = \pm\sqrt{-\frac{1}{3}c_4 + \left(-\frac{1}{2}v_1 \pm \sqrt{\frac{1}{4}{v_1}^2 - v_0}\right)}. \tag{35}$$

Now, we can get a solution of a sextic equation by solving the cubic factors $x^3 + b_2x^2 + b_1x + b_0$ of (10), by using b_2 of (35), b_1 of (14) with the discriminant D_1 of (15), and b_0 of (11). Writing down the solution of x in a row as a single equation is too lengthy and difficult to distinguish at a glance, so it would be more convenient to arrange each variable separately.
In the case of trying to get a solvable sextic equation that can be determined to be solved by applying arbitrary coefficients, b_2, b_1 and D_1, the constant coefficient c_0 of (13) may return a different value other than expected.

Example # 1
Let the sextic equation (10) be $b_2 = -1$, $D_1 = 9/64$, $c_4 = 3/2$, $c_3 = 0$, $c_2 = -3/2$, and $c_1 = 3/16$, then we have a sextic equation

$$x^6 + \frac{3}{2}x^4 - \frac{3}{2}x^2 + \frac{3}{16}x - \frac{1483}{1024} + \frac{447\sqrt[3]{3}}{1024} + \frac{141\sqrt[3]{3^2}}{1024} = 0,$$

which has two cubic factors

$$x^3 - x^2 + \left(\frac{5}{4} - \frac{1}{4}\sqrt[3]{3} - \frac{1}{4}\sqrt[3]{3^2}\right)x - \frac{43}{32} + \frac{11}{32}\sqrt[3]{3} + \frac{9}{32}\sqrt[3]{3^2} = 0,$$
$$x^3 + x^2 + \left(\frac{5}{4} + \frac{1}{4}\sqrt[3]{3} + \frac{1}{4}\sqrt[3]{3^2}\right)x + \frac{43}{32} + \frac{5}{32}\sqrt[3]{3} + \frac{7}{32}\sqrt[3]{3^2} = 0.$$

Example # 2
Let $b_2 = 1$, $c_4 = -3$, $c_3 = 0$, $c_2 = 3$, and $c_1 = 4$, then we get $D_1 = 256$ and $b_1 = -1 + \sqrt[3]{2^2}$. The sextic equation becomes

$$x^6 - 3x^4 + 3x^2 + 4x - 1 - \sqrt[3]{2^2} = 0.$$

And two cubic factors are

$$x^3 + x^2 + \left(-1 + \sqrt[3]{2^2}\right)x - 1 - \sqrt[3]{2} + \sqrt[3]{2^2} = 0,$$
$$x^3 - x^2 + \left(-1 - \sqrt[3]{2^2}\right)x + 1 + \sqrt[3]{2} + \sqrt[3]{2^2} = 0.$$

Example # 3
Let $b_2 = 1$, $c_4 = 0$, $c_3 = 0$, $c_2 = -3/4$, and $c_1 = 1/4$, then $D_1 = 1$, and we get

$$x^6 - \frac{3}{4}x^2 + \frac{1}{4}x - \frac{1}{4} - \frac{1}{16\sqrt[3]{2^2}} + \frac{3}{4\sqrt[3]{2}} = 0,$$

and,

$$x^3 + x^2 + \left(\frac{1}{2} + \frac{1}{\sqrt[3]{2^2}}\right)x + \frac{1}{2} + \frac{1}{\sqrt[3]{2^2}} - \frac{1}{4\sqrt[3]{2}} = 0,$$
$$x^3 - x^2 + \left(\frac{1}{2} - \frac{1}{\sqrt[3]{2^2}}\right)x - \frac{1}{2} + \frac{1}{\sqrt[3]{2^2}} + \frac{1}{4\sqrt[3]{2}} = 0.$$

Example # 4
From the equation (16) and (28), let $d_4 = 0$, $d_3 = 0$, $d_2 = 5$, $d_1 = 64$, then $D_2 = 0$, and the equation becomes

$$x^6 + 5x^2 + 64x + 230 = 0,$$

and the quadratic factor provides

$$x^2 + 4x + 5 = 0.$$

E. Factoring a sextic equation into two cubic equations

Another one is directly factoring a sextic equation into two cubic equations. For this, we may apply the following formula, which has the form of $A^2 - B^2 = (A - B)(A + B)$,

$$\begin{aligned}&(x^3 + a_1x + a_0)^2 - w(x^2 + b_1x + b_0)^2 \\ &= x^6 + d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0 \\ &= 0,\end{aligned} \tag{36}$$

where w is a coupling constant.
This result is also the same as the cases mentioned above, but when w is less than 0, it becomes an imaginary number, and the sextic equation would have all 6 imaginary roots.

Example # 5
Let coefficients of the sextic equation (36) be $b_1 = -3/2$, $d_4 = -12$, $d_3 = 0$, $d_2 = 0$, and $d_1 = 1953/32$, then we have $w = 5/2$, $b_0 = 271/80$, and $d_0 = -37441/2560$. Therefore, the sextic equation becomes

$$x^6 - 12x^4 + \frac{1953}{32}x - \frac{37441}{2560} = 0.$$

This sextic equation has two cubic factors as

$$x^3 - \frac{\sqrt{5}}{\sqrt{2}}x^2 + \left(-\frac{19}{4} + \frac{3\sqrt{5}}{2\sqrt{2}}\right)x - \frac{15}{4} - \frac{271}{16\sqrt{10}} = 0,$$
$$x^3 + \frac{\sqrt{5}}{\sqrt{2}}x^2 + \left(-\frac{19}{4} - \frac{3\sqrt{5}}{2\sqrt{2}}\right)x - \frac{15}{4} + \frac{271}{16\sqrt{10}} = 0.$$

The End

[1] Sextic_equation, URL: https://en.wikipedia.org/wiki/ Access Date: Nov. 11, 2022
[2] Cayley's sextic, URL: https://en.wikipedia.org/wiki/ Access Date: Jul. 25, 2023
[3] Kulkarni RG, Solving sextic equations, URL: http://euclid.trentu.ca/aejm/V3N1/Kulkarni.V3N1.pdf Access Date: Jul. 25, 2023
[4] Hagedorn T.R., General formulas for Solving Solvable Sextic Equations, URL: https://core.ac.uk/download/pdf-/82641766.pdf, 1999 Access Date: Jul. 24, 2023