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Analytic and Parameter-Free Formula for the Neutrino Mixing Matrix

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Abstract

A parameter-free analytic expression for the PMNS matrix is derived which fits numerically all the measured matrix components to within 1 or 2 standard deviations. Results are proven within the microscopic model and also lead to a prediction of the leptonic Jarlskog invariant $J_{PMNS} = -0.0106$. An outlook is given to the treatment of the CKM matrix.

As well known there is a mixing between the flavor and mass eigenstates of the 3 neutrino species, and this can be described by a unitary matrix, the PMNS neutrino mixing matrix[1, 2]. The experimentally relevant quantities are the absolute values of the matrix elements, which describe the amount of admixture of the flavor into mass eigenstates, and the leptonic Jarlskog invariant which describes any possible CP violation in the leptonic sector.

Since the discovery of neutrino oscillations, many models of neutrino mass and mixings have been constructed. The most straightforward approach is to incorporate Dirac neutrino masses into the Standard Model by introducing three right-handed neutrinos coupled to a Higgs field analogously to the quarks and charged leptons.

Unfortunately, within the SM the values of the mixing parameters cannot be predicted.

The Formula

The formula to be derived is

$$\begin{aligned} V_{PMNS} &= \exp\left\{ \frac{i}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \right\} \\ &= \begin{bmatrix} 0.8467 - i0.0300 & -0.1489 + i0.4861 & 0.1532 - i0.00051 \\ -0.1489 - i0.4861 & 0.5446 + i0.4568 & -0.00433 - i0.4858 \\ 0.1532 - i0.00051 & -0.00433 - i0.4858 & 0.6892 - i0.5153 \end{bmatrix} \end{aligned} \tag{1}$$

This is a complex, symmetric and unitary matrix, and the absolute values of the matrix elements can be calculated numerically and compared to measurements (the latter including one standard error)

$$\begin{bmatrix} 0.843 & 0.510 & 0.153 \\ 0.510 & 0.711 & 0.486 \\ 0.153 & 0.486 & 0.861 \end{bmatrix} \quad vs. \quad \begin{bmatrix} 0.80 - 0.85 & 0.51 - 0.58 & 0.142 - 0.155 \\ 0.23 - 0.51 & 0.46 - 0.69 & 0.63 - 0.78 \\ 0.26 - 0.53 & 0.47 - 0.70 & 0.61 - 0.76 \end{bmatrix} \tag{2}$$

By inspection one concludes that all experimental values are fitted to within 1 or 2 standard deviations.

The first row, which is best measured, is also best fitting. Concerning the other rows, the experimental results in (2) are non-symmetric, though with very large errors.

It will be described later, how (1) can be improved by additional non-symmetric contributions.

A prediction for the leptonic Jarlskog invariant[3] can be calculated from (1) as

$$J_{PMNS} = \Im(V_{e1}V_{\mu 2}\bar{V}_{e2}\bar{V}_{\mu 1}) = -0.0106 \tag{3}$$

This value is large as compared to the Jarlskog parameter of the CKM matrix[4]. J_{PMNS} has not been measured so far, although there are experimental indications that leptonic CP violation is indeed rather large[5].

Motivation and Proof

The model, on which the proof is based[6, 7], starts from a fundamental isospin doublet field $\Psi = (U, D)$ consisting of two SO(3,1) Dirac fields U and D. Ordinary matter quarks and leptons are considered as excitations of isospin vectors

$$\vec{Q}_L = \frac{1}{4}\Psi^\dagger(1-\gamma_5)\vec{\tau}\Psi \qquad \vec{Q}_R = \frac{1}{4}\Psi^\dagger(1+\gamma_5)\vec{\tau}\Psi \tag{4}$$

of the Ψ-field, namely as fluctuations δ of the ground state values

$$\vec{Q}_{Li} = \langle\vec{Q}_{Li}\rangle + \delta\vec{Q}_{Li} + O(\delta^2) \qquad \vec{Q}_{Ri} = \langle\vec{Q}_{Ri}\rangle + \delta\vec{Q}_{Ri} + O(\delta^2) \tag{5}$$

Masses can then be calculated using Hamiltonians H which involve interactions of the isospin vectors (4) and then diagonalizing the equations

$$\frac{d\vec{Q}_{L,R}}{dt} = i\,[H, \vec{Q}_{L,R}] \tag{6}$$

Assuming a suitable tetrahedral configuration for the isospin vectors, 24 eigenvalues arise from (6), which are interpreted as the quark and lepton masses.

While the masses correspond to the eigenvalues, CKM and PMNS mixings can be deduced from the eigenvectors. The relation between the eigenvectors, the mass eigenstates and the weak interaction eigenstates are clarified in the following discussion. Thereby, the result (1) for the PMNS matrix will be obtained.

The first step is to explicitly represent the quark and lepton mass states in terms of the eigenstate vectors $\delta\vec{Q}$. More in detail, the following definitions are used:

$$|\vec{S}\rangle = \delta\vec{Q}_L \qquad |\vec{T}\rangle = \delta\vec{Q}_R \tag{7}$$

Dirac's notation with bra and ket states is applied here to make the mixing relations more transparent. In fact, (7) are orthonormal vector states and can be used to write down the equations for the neutrino mass eigenstates, as obtained from the diagonalization procedure[7]

$$\begin{aligned}
|\nu_{e,m}\rangle &= \frac{1}{\sqrt{6}}[(|S_x\rangle + |T_x\rangle) + (|S_y\rangle + |T_y\rangle) + (|S_z\rangle + |T_z\rangle)] \\
|\nu_{\mu,m}\rangle &= \frac{1}{\sqrt{6}}[(|S_x\rangle + |T_x\rangle) + \omega(|S_y\rangle + |T_y\rangle) + \bar{\omega}(|S_z\rangle + |T_z\rangle)] \\
|\nu_{\tau,m}\rangle &= \frac{1}{\sqrt{6}}[(|S_x\rangle + |T_x\rangle) + \bar{\omega}(|S_y\rangle + |T_y\rangle) + \omega(|S_z\rangle + |T_z\rangle)] \qquad (8)
\end{aligned}$$

The corresponding result for the charged leptons is

$$\begin{aligned}
|e_m\rangle &= \frac{1}{\sqrt{6}}[(|T_x\rangle - |S_x\rangle) + (|T_y\rangle - |S_y\rangle) + (|T_z\rangle - |S_z\rangle)] \\
|\mu_m\rangle &= \frac{1}{\sqrt{6}}[(|T_x\rangle - |S_x\rangle) + \omega(|T_y\rangle - |S_y\rangle) + \bar{\omega}(|T_z\rangle - |S_z\rangle)] \\
|\tau_m\rangle &= \frac{1}{\sqrt{6}}[(|T_x\rangle - |S_x\rangle) + \bar{\omega}(|T_y\rangle - |S_y\rangle) + \omega(|T_z\rangle - |S_z\rangle)] \qquad (9)
\end{aligned}$$

The appearance of the complex numbers

$$\omega = -\frac{1 - i\sqrt{3}}{2} \qquad\qquad \bar{\omega} = -\frac{1 + i\sqrt{3}}{2} \qquad (10)$$

corresponding to rotations by 120 and 240 degrees are an effect of the underlying tetrahedral symmetry. They turn the expressions (8) and (9) into symmetry adapted functions.

The lepton mass states actually can be brought to the much more compact form

$$\begin{bmatrix} |\nu_{em}\rangle \\ |\nu_{\mu m}\rangle \\ |\nu_{\tau m}\rangle \end{bmatrix} = Z \begin{bmatrix} |V_x\rangle \\ |V_y\rangle \\ |V_z\rangle \end{bmatrix} \qquad\qquad \begin{bmatrix} |e_m\rangle \\ |\mu_m\rangle \\ |\tau_m\rangle \end{bmatrix} = Z \begin{bmatrix} |A_x\rangle \\ |A_y\rangle \\ |A_z\rangle \end{bmatrix} \qquad (11)$$

by using the quantities

$$|\vec{V}\rangle = \frac{1}{\sqrt{2}}(|\vec{S}\rangle + |\vec{T}\rangle) \qquad\qquad |\vec{A}\rangle = \frac{1}{\sqrt{2}}(|\vec{T}\rangle - |\vec{S}\rangle) \qquad (12)$$

and the Z_3 Fourier transform matrices

$$Z = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{bmatrix} \qquad\qquad Z^\dagger = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \omega \\ 1 & \omega & \bar{\omega} \end{bmatrix} \qquad (13)$$

It is interesting to note that the eigenfunctions (8), (9) and (11) are stable against variations of all the isospin couplings one may use in the Hamiltonian H in (6). As a consequence, the neutrino mixing matrix does not depend on any fermion mass values. This implies a stable and unambiguous prediction for the PMNS matrix and is in contrast to the CKM matrix in the quark sector, where a mass dependence shows up.

As well known, the defining equation for the PMNS matrix is

$$\begin{bmatrix} \langle\nu_{ew}| & \langle\nu_{\mu w}| & \langle\nu_{\tau w}| \end{bmatrix} W_\mu^+ \begin{bmatrix} |e_w\rangle \\ |\mu_w\rangle \\ |\tau_w\rangle \end{bmatrix} = \begin{bmatrix} \langle\nu_{em}| & \langle\nu_{\mu m}| & \langle\nu_{\tau m}| \end{bmatrix} W_\mu^+ V_{PMNS} \begin{bmatrix} |e_m\rangle \\ |\mu_m\rangle \\ |\tau_m\rangle \end{bmatrix} \quad (14)$$

where the index w denotes weak interaction eigenstates, and it is understood that we talk about left handed fields only. The mixing matrix is formally given by

$$V_{PMNS} = V_N V_L^\dagger = \begin{bmatrix} V_{1e} & V_{1\mu} & V_{1\tau} \\ V_{2e} & V_{2\mu} & V_{2\tau} \\ V_{3e} & V_{3\mu} & V_{3\tau} \end{bmatrix} \quad (15)$$

where

$$V_N = \begin{bmatrix} \langle\nu_{em}| \\ \langle\nu_{\mu m}| \\ \langle\nu_{\tau m}| \end{bmatrix} \begin{bmatrix} |\nu_{ew}\rangle & |\nu_{\mu w}\rangle & |\nu_{\tau w}\rangle \end{bmatrix} \qquad V_L^\dagger = \begin{bmatrix} \langle e_w| \\ \langle\mu_w| \\ \langle\tau_w| \end{bmatrix} \begin{bmatrix} |e_m\rangle & |\mu_m\rangle & |\tau_m\rangle \end{bmatrix} \quad (16)$$

Replacing the mass eigenstates by the isospin excitations according to (11) one obtains

$$V_{PMNS} = Z \left\{ \begin{bmatrix} \langle V_x| \\ \langle V_y| \\ \langle V_z| \end{bmatrix} \begin{bmatrix} |\nu_{ew}\rangle & |\nu_{\mu w}\rangle & |\nu_{\tau w}\rangle \end{bmatrix} \begin{bmatrix} \langle e_w| \\ \langle\mu_w| \\ \langle\tau_w| \end{bmatrix} \begin{bmatrix} |A_x\rangle & |A_y\rangle & |A_z\rangle \end{bmatrix} \right\} Z^\dagger \quad (17)$$

By inspection one sees that (17) exactly compensates all the matrix transformations in (14) and (11) so as to maintain lepton universality and keep the weak current diagonal in the weak eigenstates.

The brace in (17) comprises a matrix of expectation values of the form

$$Y := \begin{bmatrix} \langle V_x| \\ \langle V_y| \\ \langle V_z| \end{bmatrix} \mathcal{O} \begin{bmatrix} |A_x\rangle & |A_y\rangle & |A_z\rangle \end{bmatrix} \quad (18)$$

where the inner product

$$\mathcal{O} := \begin{bmatrix} |\nu_{ew}\rangle & |\nu_{\mu w}\rangle & |\nu_{\tau w}\rangle \end{bmatrix} \begin{bmatrix} \langle e_w| \\ \langle \mu_w| \\ \langle \tau_w| \end{bmatrix} \tag{19}$$

is a dyadic 1-dimensional operator which acts between the complex 3-dimensional spaces of charged lepton ($\sim \vec{S}-\vec{T}$) and antineutrino ($\sim \vec{S}+\vec{T}$) states. One may say that it contains all information about what the charged W-boson does to the lepton fields: it changes isospin, mixes families and so on. Weak SU(2) and tetrahedral symmetry force $\mathcal{O}$ to have the form

$$\begin{aligned} \mathcal{O} \;=\; & |S_x\rangle\langle T_x| + |S_y\rangle\langle T_y| + |S_z\rangle\langle T_z| - |T_x\rangle\langle S_x| - |T_y\rangle\langle S_y| - |T_z\rangle\langle S_z| \\ & + \frac{i}{\sqrt{3}}[|S_y\rangle\langle S_z| + |S_z\rangle\langle S_y| - |T_y\rangle\langle T_z| - |T_z\rangle\langle T_y|] \\ & + \frac{i}{\sqrt{3}}[\omega\,|S_x\rangle\langle S_y| + \bar{\omega}\,|S_y\rangle\langle S_x| - \omega\,|T_x\rangle\langle T_y| - \bar{\omega}\,|T_y\rangle\langle T_x|] \\ & + \frac{i}{\sqrt{3}}[\bar{\omega}\,|S_x\rangle\langle S_z| + \omega\,|S_z\rangle\langle S_x| - \bar{\omega}\,|T_x\rangle\langle T_z| - \omega\,|T_z\rangle\langle T_x|] \end{aligned} \tag{20}$$

In order to derive (20) one has to note that SU(2) invariance allows the appearance of scalar products and scalar triple products only. The coefficients of these products are then dictated by the tetrahedral symmetry of the isospin vectors. For example, to derive the triple product coefficients one should remember that the W^+-boson is defined in the 3 internal dimensions in an analogous manner as a plus circularly polarized wave in 3 spatial dimensions, namely by means of an (internal) polarization vector $\vec{e}_+ = (\vec{e}_1 + i\vec{e}_2)/\sqrt{2}$ which is perpendicular to the axis of quantization, in this case given by $\sim (1,1,1)$.

$$\vec{e}_1 = \frac{1}{\sqrt{2}}(0,1,-1) \qquad\qquad \vec{e}_2 = \frac{1}{\sqrt{6}}(-2,1,1) \tag{21}$$

Allowed contributions to $\mathcal{O}$ are then of the form

$$\begin{aligned} \varepsilon_{ijk}\frac{1}{\sqrt{2}}(\vec{e}_1 + i\vec{e}_2)_i\,|Q'_j\rangle\langle Q_k| = \frac{i}{\sqrt{3}}\,[\,|Q'_y\rangle\langle Q_z| - |Q'_z\rangle\langle Q_y| \\ -\omega(|Q'_x\rangle\langle Q_z| - |Q'_z\rangle\langle Q_x|) + \bar{\omega}(|Q'_x\rangle\langle Q_y| - |Q'_y\rangle\langle Q_x|\,)\,] \end{aligned} \tag{22}$$

for the ket and bra states belonging to any 2 internal angular momenta Q and Q'. These contributions are anti-hermitean, and care must be taken in the definition of the complex triple product when using complex conjugation in the determination of $\mathcal{O}$.

Note that $\mathcal{O}$ as given in (20) is universal in the sense that it depends only on properties of the Ψ field, and therefore will appear in identical form within the quark sector and the calculation of the CKM matrix. This fact reflects the quark lepton universality of the W-boson interactions.

Inserting (20) into (18) one obtains

$$Y = \begin{bmatrix} \langle V_x| \\ \langle V_y| \\ \langle V_z| \end{bmatrix} \mathcal{O} \begin{bmatrix} |A_x\rangle & |A_y\rangle & |A_z\rangle \end{bmatrix} = I + X \tag{23}$$

i.e. a sum of a hermitean part (the unit matrix I) and an anti-hermitean matrix

$$X = -\frac{i}{\sqrt{3}} \begin{bmatrix} 0 & \bar{\omega} & \omega \\ \omega & 0 & 1 \\ \bar{\omega} & 1 & 0 \end{bmatrix} \tag{24}$$

The invariant structure which gives the unit matrix in (23) is the scalar product, while the invariant structure belonging to the anti-hermitean contribution X is the scalar triple product. The unit matrix corresponds to no mixing at all, so the origin of a non-trivial PMNS matrix is to be found solely in the triple product terms (22).

Since the result (23) is not unitary, an exponentiation suggests itself which gives a unitary PMNS matrix of the form

$$\begin{aligned} V_{PMNS} &= Ze^{X}Z^{\dagger} = e^{ZXZ^{\dagger}} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{bmatrix} \exp\left\{ \frac{-i}{\sqrt{3}} \begin{bmatrix} 0 & \bar{\omega} & \omega \\ \omega & 0 & 1 \\ \bar{\omega} & 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \omega \\ 1 & \omega & \bar{\omega} \end{bmatrix} \\ &= \begin{bmatrix} 0.8467 - i0.0300 & -0.1489 + i0.4861 & 0.1532 - i0.00051 \\ -0.1489 - i0.4861 & 0.5446 + i0.4568 & -0.00433 - i0.4858 \\ 0.1532 - i0.00051 & -0.00433 - i0.4858 & 0.6892 - i0.5153 \end{bmatrix} \end{aligned} \tag{25}$$

identical to what was claimed in (1).

One should mention that expansion of the exponential in (1) is convergent, and that truncation of the series at second order already gives a good approximation

$$V_{PMNS} \approx 1 + ZXZ^\dagger + \frac{1}{2}(ZXZ^\dagger)^2 = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} + \frac{i}{\sqrt{3}} & \frac{1}{6} \\ -\frac{1}{6} + \frac{i}{\sqrt{3}} & \frac{1}{2} + \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{1}{6} & -\frac{i}{\sqrt{3}} & \frac{2}{3} - \frac{i}{\sqrt{3}} \end{bmatrix} \tag{26}$$

with absolute values

$$|V_{PMNS}| \approx \begin{bmatrix} 0.83 & 0.51 & 0.16 \\ 0.51 & 0.69 & 0.48 \\ 0.17 & 0.48 & 0.84 \end{bmatrix} \tag{27}$$

Actually, this is the order to which (1) is proven exactly, assuming unitarity together with the arguments in connection with (22). Comparing (27) to (2), the approximation seems good for the absolute values. There is no trustworthy outcome, however, for the Jarlskog invariant, as can be seen by working out (3) for the matrix (26).

Outlook to Quark Mixing

Mixing in the quark sector has been known since the time of Cabibbo[8]. Although the mixing percentages are smaller, it is much better measured than in the lepton sector. On the other hand, concerning theory, the predictions for the CKM mixing elements in the present model are somewhat more difficult to obtain, though parts of the arguments for leptons can be taken over to the quark sector. The idea is again that the mixing matrix counterbalances the deviation of the mass eigenstates from the weak eigenstates in such a way that the charged current effectively acts diagonal on the isospin operators (7). The main complication is the appearance of mass dependent factors in the quark eigenstates, see below.

The CKM matrix is standardly defined analogously to the PMNS matrix Eqs. (15) and (16)

$$\begin{aligned} V_{CKM} &= V_U V_D^\dagger = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \\ &= \begin{bmatrix} \langle u_m|u_w\rangle & \langle u_m|c_w\rangle & \langle u_m|t_w\rangle \\ \langle c_m|u_w\rangle & \langle c_m|c_w\rangle & \langle c_m|t_w\rangle \\ \langle t_m|u_w\rangle & \langle t_m|c_w\rangle & \langle t_m|t_w\rangle \end{bmatrix} \begin{bmatrix} \langle d_w|d_m\rangle & \langle d_w|s_m\rangle & \langle d_w|b_m\rangle \\ \langle s_w|d_m\rangle & \langle s_w|s_m\rangle & \langle s_w|b_m\rangle \\ \langle b_w|d_m\rangle & \langle b_w|s_m\rangle & \langle b_w|b_m\rangle \end{bmatrix} \end{aligned} \tag{28}$$

where m denotes mass eigenstates (the physical states) and w weak interaction eigenstates.

Solving the eigenvalue problem (6) leads to mass eigenstates for the up-type quarks

$$\begin{aligned} u_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_1^2}}[(|S_x\rangle + \epsilon_1 |T_x\rangle) + (|S_y\rangle + \epsilon_1 |T_y\rangle) + (|S_z\rangle + \epsilon_1 |T_z\rangle)] \\ c_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_2^2}}[(|S_x\rangle + \epsilon_2 |T_x\rangle) + \omega(|S_y\rangle + \epsilon_2 |T_y\rangle) + \bar{\omega}(|S_z\rangle + \epsilon_2 |T_z\rangle)] \\ t_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_3^2}}[(|S_x\rangle + \epsilon_3 |T_x\rangle) + \bar{\omega}(|S_y\rangle + \epsilon_3 |T_y\rangle) + \omega(|S_z\rangle + \epsilon_3 |T_z\rangle)] \end{aligned} \tag{29}$$

and for the down quarks

$$\begin{aligned} d_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_1^2}}[(|T_x\rangle - \epsilon_1 |S_x\rangle) + (|T_y\rangle - \epsilon_1 |S_y\rangle) + (|T_z\rangle - \epsilon_1 |S_z\rangle)] \\ s_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_2^2}}[(|T_x\rangle - \epsilon_2 |S_x\rangle) + \omega(|T_y\rangle - \epsilon_2 |S_y\rangle) + \bar{\omega}(|T_z\rangle - \epsilon_2 |S_z\rangle)] \\ b_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_3^2}}[(|T_x\rangle - \epsilon_3 |S_x\rangle) + \bar{\omega}(|T_y\rangle - \epsilon_3 |S_y\rangle) + \omega(|T_z\rangle - \epsilon_3 |S_z\rangle)] \end{aligned} \tag{30}$$

Three coefficients $\epsilon_{1,2,3}$ appear in these equations, which depend on the quark and even on the lepton masses. They can be calculated within the model. So far I have not been able to derive analytical formulas for them, but will present numerical results below.

Note the lepton eigenfunctions (8) and (9) are recovered by chosing $\epsilon_3 = \epsilon_2 = \epsilon_1 = 1$. It should be stressed, however, that this is only formally true, because the quark states (29) and (30) are defined in a different space than the lepton states. The point is that for simplicity reference has been made so far to only one of the four isospins I, II, III and IV on the tetrahedral structure. While the contributions from I-IV to the lepton states are identical and of the form I+II+III+IV, the generic form of the quark states turns out to be 3×I-II-III-IV, 3×II-I-III-IV and 3×III-II-IV for the 3 colors, respectively.

Using the plain quark and lepton mass values given by the particle data group[4] the calculations reported here yield

$$\epsilon_1 = 0.0140 \qquad \epsilon_2 = 0.0130 \qquad \epsilon_3 = 0.00171 \tag{31}$$

With running masses[9] at m_Z one obtains larger values

$$\epsilon_1 = 0.115 \qquad \epsilon_2 = 0.071 \qquad \epsilon_3 = 0.0039 \tag{32}$$

exhibiting a hierarchy $\epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll 1$. As shown below, this leads to the desired hierarchy in the mixing of the quark families, i.e. it implies that the mixing is small and decreases with the generation number.

Knowing the eigenstates (29) and (30) one may write down the CKM matrix in an analogous fashion as the PMNS matrix (17) for leptons

$$V_{CKM} = \left\{ RZ \begin{bmatrix} \langle S_x| \\ \langle S_y| \\ \langle S_z| \end{bmatrix} + REZ \begin{bmatrix} \langle T_x| \\ \langle T_y| \\ \langle T_z| \end{bmatrix} \right\} \begin{bmatrix} |u_w\rangle & |c_w\rangle & |t_w\rangle \end{bmatrix} \begin{bmatrix} \langle d_w| \\ \langle s_w| \\ \langle b_w| \end{bmatrix} \times$$
$$\times \left\{ \begin{bmatrix} |T_x\rangle & |T_y\rangle & |T_z\rangle \end{bmatrix} Z^\dagger R - \begin{bmatrix} |S_x\rangle & |S_y\rangle & |S_z\rangle \end{bmatrix} Z^\dagger ER \right\} \tag{33}$$

where the matrices

$$E := \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \qquad R := \begin{bmatrix} \frac{1}{\sqrt{1+\epsilon_1^2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+\epsilon_2^2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1+\epsilon_3^2}} \end{bmatrix} \tag{34}$$

have been introduced.

Just as in the case of leptons (19) there is a 1-dimensional dyadic transformation

$$\mathcal{O} = \begin{bmatrix} |u_w\rangle & |c_w\rangle & |t_w\rangle \end{bmatrix} \begin{bmatrix} \langle d_w| \\ \langle s_w| \\ \langle b_w| \end{bmatrix} \tag{35}$$

which operates between the 3-dimensional spaces of up- and down-type quark states. Due to quark-lepton universality, when expressed in terms of operators $\vec{S}$ and $\vec{T}$, the operator $\mathcal{O}$ for quarks must be identical to what was used for leptons in (20).

Therefire, using (20) as input one may calculate V_{CKM} given in (33) to be

$$V_{CKM} = I + RZXZ^\dagger ER + REZXZ^\dagger R \to \exp\{RZXZ^\dagger ER + REZXZ^\dagger R\} \tag{36}$$

where I is the 3×3 unit matrix arising from the scalar product terms in (20). The other terms in (36) are the anti-hermitean contributions from the triple product in (22) and (20). They replace the expression $ZXZ^\dagger$ in (25) for leptons.

To derive (36) one should note that the $O(E^0)$ and $O(E^2)$ terms in (33) exactly combine to give the unit matrix in (36).

One may evaluate the absolute values $|V_{CKM}|$ using (36) and the specific numbers given in (31) or (32). Unfortunately, ϵ_1 is too small to reproduce the phenomenological value of the Cabibbo angle. Its value should be about 0.33 rather than 0.11.

Taking ϵ_1 as a free parameter and choosing a value of 0.33 one indeed obtains reasonable numbers for the absolute values of the CKM matrix elements which compare quite well to the experimental results

$$\begin{bmatrix} 0.974 & 0.222 & 0.00473 \\ 0.222 & 0.974 & 0.0422 \\ 0.00473 & 0.0422 & 0.9991 \end{bmatrix} vs. \begin{bmatrix} 0.9735 - 0.9738 & 0.224 - 0.226 & 0.0036 - 0.0040 \\ 0.217 - 0.225 & 0.977 - 0.998 & 0.040 - 0.043 \\ 0.0077 - 0.0083 & 0.0375 - 0.0400 & 0.997 - 1.04 \end{bmatrix} \quad (37)$$

However, such a large ϵ_1 is obtained from my programs only, if I would bring down both the up- and the down-quark mass values to about ≈ 1 GeV.

It thus seems that for the light quarks the linear approximation (5) used throughout this work is not valid. Next-to-leading effects from the heavy families severely modify the contributions from the light quark masses. I am still working on the problem to find a satisfactory solution.

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