# A proof of Fermat's conjecture

$$(x+y)^n = x^n + y^n + xy\sum_{j=0}^{n-2}\left(x^j+y^j\right)(x+y)^{n-2-j}$$

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**Abstract**

The binomial formula, set by Isaac Newton, is of utmost importance and has been extensively used in many different fields. This study aims at coming up with alternative expressions to the Newton's formula, as well as some othe results that we can obtain from them. A proof of the Fermat's conjecture seems to be possible.

# Chapter 1

# Another way to write Newton's binomial expansion.

## 1.1 Purpose of this chapter.

Newton's binomial expansion can be expressed differently. This new formulation allows in turn to perform other calculations which will highlight certain properties that the original formula may not be able to provide.

## 1.2 Another formula.

Let $n \in \mathbb{N}^*$, and $x \in \mathbb{R}^*$ and $y \in \mathbb{R}^*$. In all that follows, we assume $n \geq 3$. We can write

$$\frac{(x+y)^n - x^n}{(x+y) - x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j}\, x^j = \frac{(x+y)^n - x^n}{y}$$

and likewise

$$\frac{(x+y)^n - y^n}{(x+y) - y} = \sum_{j=0}^{n-1} (x+y)^{n-1-j}\, y^j = \frac{(x+y)^n - y^n}{x}$$

Let us add these two quantities

$$\frac{(x+y)^n - x^n}{y} + \frac{(x+y)^n - y^n}{x} = \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

and we end up with the formula

$$(x+y)^{n+1} - \left(x^{n+1} + y^{n+1}\right) = xy \sum_{j=0}^{n-1} (x+y)^{n-1-j} \left(x^j + y^j\right)$$

which, for convenience's sake, we write

$$(x+y)^n - (x^n + y^n) = xy \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.1}$$

The Newton's binomial expansion formula, that we recall here

$$(x+y)^n = \sum_{j=0}^{n} C_n^j x^{n-j} y^j \tag{1.2}$$

wherein

$$C_n^j = \frac{n!}{(n-j)!j!} \tag{1.3}$$

allows to establish the equality

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=1}^{n-1} C_n^j x^{n-j-1} y^{j-1}$$

or lastly

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j$$

## 1.3 Study of the new formula.

Let us pose

$$A_n(x,y) = \sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) \tag{1.4}$$

Let us remark first that

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{n-2-j} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right)$$

with $p \in \mathbb{N}^*$ and $p < n$, or likewise

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) = \sum_{j=0}^{p-2} (x+y)^{(n-p)+(p-2-j)} \left(x^j + y^j\right) + \sum_{j=p-1}^{n-2} (x+y)^{n-2-(j-(p-1)+p-1)} \left(x^{j-(p-1)+p-1} + y^{j-(p-1)+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right)$$

$$+ \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-2-(j+p-1)} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and

$$\sum_{j=0}^{n-2} (x+y)^{n-2-j} \left(x^j + y^j\right) =$$

$$(x+y)^{n-p} \sum_{j=0}^{p-2} (x+y)^{p-2-j} \left(x^j + y^j\right)$$

$$+ \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

and finally

$$A_n(x,y) = (x+y)^{n-p} A_p(x,y)$$

$$+ \sum_{j=0}^{n-2-(p-1)} (x+y)^{n-p-1-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

Let us now consider the case wherein $n = p+1$, then

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \sum_{j=0}^{0} (x+y)^{-j} \left(x^{j+p-1} + y^{j+p-1}\right)$$

or likewise

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + \left(x^{p-1} + y^{p-1}\right)$$

but

$$x^{p-1} + y^{p-1} = (x+y)^{p-1} - xyA_{p-1}(x,y)$$

and so

$$A_{p+1}(x,y) = (x+y) A_p(x,y) + (x+y)^{p-1} - xyA_{p-1}(x,y) \tag{1.5}$$

Let us concentrate now more specifically on $A_n(x,y)$ and let us develop this quantity from the formula 1.4 in page 2. Then

$$
\begin{aligned}
A_n(x,y) &= 3(x+y)^{n-2} + \sum_{j=0}^{n-4} (x+y)^{n-4-j}\left(x^{j+2}+y^{j+2}\right)\\
&= 3(x+y)^{n-2} + \left(x^2+y^2\right)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)\\
&= 3(x+y)^{n-2} + (x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)\\
&= 4(x+y)^{n-2} - 2xy(x+y)^{n-4} + \sum_{j=0}^{n-5} (x+y)^{n-5-j}\left(x^{j+3}+y^{j+3}\right)
\end{aligned}
$$

As we continue our calculations in the same manner, we get

$$
A_n(x,y) = 5(x+y)^{n-2} - 5xy(x+y)^{n-4} + \sum_{j=0}^{n-6} (x+y)^{n-6-j}\left(x^{j+4}+y^{j+4}\right)
$$

$$
A_n(x,y) = 6(x+y)^{n-2} - 9xy(x+y)^{n-4} + 2x^2y^2(x+y)^{n-6} + \sum_{j=0}^{n-7} (x+y)^{n-7-j}\left(x^{j+5}+y^{j+5}\right)
$$

$$
A_n(x,y) = 7(x+y)^{n-2} - 14xy(x+y)^{n-4} + 7x^2y^2(x+y)^{n-6} + \sum_{j=0}^{n-8} (x+y)^{n-8-j}\left(x^{j+6}+y^{j+6}\right)
$$

$$
A_n(x,y) = 8(x+y)^{n-2} - 20xy(x+y)^{n-4} + 16x^2y^2(x+y)^{n-6} - 2x^3y^3(x+y)^{n-8} + \sum_{j=0}^{n-9} (x+y)^{n-9-j}\left(x^{j+7}+y^{j+7}\right)
$$

$$
A_n(x,y) = 9(x+y)^{n-2} - 27xy(x+y)^{n-4} + 30x^2y^2(x+y)^{n-6} - 9x^3y^3(x+y)^{n-8} + \sum_{j=0}^{n-10} (x+y)^{n-10-j}\left(x^{j+8}+y^{j+8}\right)
$$

$$
\begin{aligned}
A_n(x,y) = {} & 10(x+y)^{n-2} - 35xy(x+y)^{n-4} + 50x^2y^2(x+y)^{n-6} - 25x^3y^3(x+y)^{n-8}\\
& + 2x^4y^4(x+y)^{n-10}\\
& + \sum_{j=0}^{n-11} (x+y)^{n-11-j}\left(x^{j+9}+y^{j+9}\right)
\end{aligned}
$$

$$A_n(x,y) = 11(x+y)^{n-2} - 44xy(x+y)^{n-4} + 77x^2y^2(x+y)^{n-6} - 55x^3y^3(x+y)^{n-8} + 11x^4y^4(x+y)^{n-10} + \sum_{j=0}^{n-12}(x+y)^{n-12-j}\left(x^{j+10}+y^{j+10}\right)$$

It is of course possible to extend our calculations as far as we desire. As $n$ is taking on the values 3, 4, 5, 6,..., we can deduct the respective new developments of $A_3(x,y)$, $A_4(x,y)$, $A_5(x,y)$, $A_6(x,y)$, etc...

Let us assume now that the following formulas hold for all natural integers less than or equal to $2k$ and $2k+1$, wherein $k \in \mathbb{N}^*$

$$A_{2k}(x,y) = \sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} \tag{1.6}$$

$$A_{2k+1}(x,y) = (x+y)\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} \tag{1.7}$$

The coefficients $D_{2k}^{j}$ and $D_{2k+1}^{j}$ are to be made explicit if possible (and will be indeed further down in this study).

Let us go back to the equation 1.5 page 3 and rewrite in the form

$$A_{2k+2}(x,y) = (x+y)A_{2k+1}(x,y) + (x+y)^{2k} - xyA_{2k}(x,y)$$

Let us develop now this relation

$$A_{2k+2}(x,y) = (x+y)^2\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} + (x+y)^{2k} - xy\sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)}$$

$$\Longleftrightarrow$$

$$A_{2k+2}(x,y) = (x+y)^2\sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^j(xy)^j(x+y)^{2(k-1-j)} + (x+y)^{2k} + \sum_{j=0}^{k-1} D_{2k}^{j}(-1)^{j+1}(xy)^{j+1}(x+y)^{2(k-1-j)}$$

Let us carry on with our calculations. We obtain in an equivalent manner

$$A_{2k+2}(x,y) = \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} + (x+y)^{2k} + \sum_{j=0}^{k-1} D_{2k}^{j} (-1)^{j+1} (xy)^{j+1} (x+y)^{2(k-1-j)}$$

$$\Longleftrightarrow$$

$$A_{2k+2}(x,y) = \sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} + (x+y)^{2k} + \sum_{j=1}^{k} D_{2k}^{j-1} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$

$$\Longleftrightarrow$$

$$A_{2k+2}(x,y) = \sum_{j=1}^{k-1} \left(D_{2k+1}^{j} + D_{2k}^{j-1}\right) (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} + D_{2k+1}^{0} (x+y)^{2k} + (x+y)^{2k} + D_{2k}^{k-1} (xy)^{k}$$

and we can write

$$A_{2k+2}(x.y) = \sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0} = D_{2k+1}^{0} + 1$$
$$D_{2k+2}^{k} = D_{2k}^{k-1}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k-1)\left(D_{2k+2}^{j} = D_{2k+1}^{j} + D_{2k}^{j-1}\right)$$

Similarly, we have

$$A_{2k+3}(x,y) = (x+y)\,A_{2k+2}(x,y) + (x+y)^{2k} - xyA_{2k+1}(x,y)$$

Let us make it more explicit

$$A_{2k+3}(x,y) = (x+y)\sum_{j=0}^{k} D_{2k+2}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-j)} + (x+y)^{2k+1} - xy\,(x+y)\sum_{j=0}^{k-1} D_{2k+1}^{j} (-1)^{j} (xy)^{j} (x+y)^{2(k-1-j)}$$

hence

$$A_{2k+3}(x,y) = \sum_{j=0}^{k} D_{2k+2}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)+1}$$
$$+ (x+y)^{2k+1}$$
$$+ \sum_{j=0}^{k-1} D_{2k+1}^{j}(-1)^{j+1}(xy)^{j+1}(x+y)^{2(k-1-j)+1}$$

which is equivalent to

$$A_{2k+3}(x,y) = (x+y)^{2k+1}$$
$$+ \sum_{j=0}^{k} D_{2k+2}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)+1}$$
$$+ \sum_{j=1}^{k-1} D_{2k+1}^{j-1}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)+1}$$

and also

$$A_{2k+3}(x,y) = \left(D_{2k+2}^{0}+1\right)(x+y)^{2k+1}$$
$$+ \sum_{j=1}^{k} \left(D_{2k+2}^{j} + D_{2k+1}^{j-1}\right)(-1)^{j}(xy)^{j}(x+y)^{2(k-j)+1}$$

and we can finally write

$$A_{2k+3}(x,y) = (x+y)\sum_{j=0}^{k} D_{2k+2}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-j)}$$

with

$$D_{2k+2}^{0} = D_{2k+1}^{0} + 1$$

and

$$(\forall j \in \mathbb{N})\,(1 \le j \le k)\left(D_{2k+3}^{j} = D_{2k+2}^{j} + D_{2k+1}^{j-1}\right)$$

This concludes our mathematical induction and we can write at last as a conclusion

$$(\forall k \in \mathbb{N}^{*})\left(A_{2k} = \sum_{j=0}^{k-1} D_{2k}^{j}(-1)^{j}(xy)^{j}(x+y)^{2(k-1-j)}\right) \tag{1.8}$$

with

$$D_{2k}^{0} = D_{2k-1}^{0} + 1 \Longleftrightarrow D_{2k}^{0} = 2k \tag{1.9}$$

and

$$D_{2k}^{k-1} = D_{2k-2}^{k-2} = \cdots = D_{4}^{1} = 2 \tag{1.10}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k-1)\left(D_{2k}^{j} = D_{2k-1}^{j} + D_{2k-2}^{j-1}\right) \tag{1.11}$$

and as well

$$(\forall k \in \mathbb{N}^{*})\left(A_{2k+1} = (x+y)\sum_{j=0}^{k-1} D_{2k+1}^{j}\,(-1)^{j}\,(xy)^{j}\,(x+y)^{2(k-1-j)}\right) \tag{1.12}$$

with

$$D_{2k+1}^{0} = D_{2k}^{0} + 1 \Longleftrightarrow D_{2k}^{0} = 2k+1 \tag{1.13}$$

and

$$(\forall j \in \mathbb{N})\,(1 \leq j \leq k)\left(D_{2k+1}^{j} = D_{2k}^{j} + D_{2k-1}^{j-1}\right) \tag{1.14}$$

## 1.4 Values taken by the coefficients $D_h^j$ wherein $(h \in \mathbb{N})$ and $(h \geq 3)$.

We have, as we just established it

$$(\forall h \in \mathbb{N})\,(h \geq 3)\left(D_h^0 = h\right)$$

Let us now take $j = 1$. We have

$$D_h^1 = D_{h-1}^1 + D_{h-2}^0$$

We can then write

$$\left.\begin{aligned} D_h^1 &= D_{h-1}^1 + D_{h-2}^0 \\ D_{h-1}^1 &= D_{h-2}^1 + D_{h-3}^0 \\ &\dots \\ &\dots \\ &\dots \\ D_5^1 &= D_4^1 + D_3^0 \end{aligned}\right\} \Longrightarrow D_h^1 = \sum_{j=0}^{h-5} D_{h-2-j}^0 + D_4^1$$

but

$$D_{h-2-j}^0 = h - 2 - j$$

and, according to the relation 1.10 established in page 7

$$D_4^1 = 2$$

hence we get

$$D_h^1 = \sum_{j=0}^{h-5} (h-2-j) + 2 = ((h-2) + (h-3) + (h-4) + \dots + 3) + 2$$

and therefore

$$2D_h^1 = h\,(h+3)$$

and finally

$$(\forall h \in \mathbb{N}^{*})\,(h \geq 3)\left(D_h^1 = \frac{h\,(h-3)}{2}\right) \tag{1.15}$$

Clearly

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^1 \in \mathbb{N}\right)$$

Making similar calculations, we find for every natural integer $h \geq 3$

$$D_h^2 = \frac{h\,(h-4)\,(h-5)}{6} \tag{1.16}$$

$$D_h^3 = \frac{h\,(h-5)\,(h-6)\,(h-7)}{24} \tag{1.17}$$

There as well

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^2 \in \mathbb{N}\right)$$

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\,\left(D_h^3 \in \mathbb{N}\right)$$

We then remark that the relations 1.11 and 1.14 established in page 8, as well as those ((see relations 1.15, 1.16 and 1.17) established in pages 8 and 9 allow us to affirm

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\left(\forall j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}\right)\left(D_h^j \in \mathbb{N}\right)$$

Let us assume now, $h$ being chosen as even, and for all $j \in \left\{0, 1, \cdots, \frac{h-2}{2}\right\}$ the formula

$$D_h^j = \frac{h\,(h-(j+2))!}{(j+1)!\,(h-2\,(j+1))!} \tag{1.18}$$

true until rank $h$, for all even natural integer lower than or equal to $h$.

Let us assume as well that, for all $j \in \left\{0, 1, \cdots, \frac{h-4}{2}\right\}$, until rank $h-1$, the formula

$$D_{h-1}^j = \frac{(h-1)\,((h-1)-(j+2))!}{(j+1)!\,((h-1)-2\,(j+1))!} \tag{1.19}$$

is true. Then

$$D_{h-1}^{j-1} = \frac{(h-1)\,(h-1-(j+1)!)}{j!\,(h-1-2j)!} = \frac{(h-1)\,(h-(j+2)!)}{j!\,(h-1-2j)!}$$

The relation 1.11 established in page 8 allows us to write

$$D_{h+1}^{j} = \frac{h\,(h-(j+2))!}{(j+1)!\,(h-2\,(j+1))!} + \frac{(h-1)\,(h-(j+2))!}{j!\,(h-1-2j)!}$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{j!}\left(\frac{h}{(j+1)\,(h-2\,(j+1))!} + \frac{(h-1)}{(h-1-2j)!}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{j!}\left(\frac{h\,(h-1-2j)+(h-1)\,(j+1)}{(h-1-2j)!\,(j+1)}\right)$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\,(h\,(h-1)-2jh+(h-1)\,j+(h-1))$$

$$\Longleftrightarrow$$

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\left(h^2-1-(h+1)\,j\right)$$

and finally

$$D_{h+1}^{j} = \frac{(h-(j+2))!}{(j+1)!\,(h-1-2j)!}\,(h+1)\,(h-1-j)$$

We can therefore write

$$D_{h+1}^{j} = \frac{(h+1)\,(h-(j+1))!}{(j+1)!\,(h+1-2\,(j+1))!} \tag{1.20}$$

We could make similar calculations if we take $h$ as odd

We verify that

$$(\forall h \in \mathbb{N}^*)\,(h \geq 3)\left(D_h^0 = h\right)$$

and, as we denote the ensemble of even natural integers as $2\mathbb{N}$

$$(\forall h = 2k \in 2\mathbb{N}^*)\,(h \geq 4)\left(D_{2k}^{k-1} = 2\right)$$

At the end of this mathematical induction, we have therefore established

$$(\forall k \in \mathbb{N}^*)\,(\forall j \in \{0,1,2,\cdots,k-1\})$$
$$\left(D_{2k+1}^{j} = \frac{(2k+1)\,(2k-(j+1))!}{(j+1)!\,(2k+1-2\,(j+1))!}\right)$$
$$\left(D_{2(k+1)}^{j} = \frac{2\,(k+1)\,(2\,(k+1)-1-(j+1))!}{(j+1)!\,(2\,(k+1)-2\,(j+1))!}\right) \tag{1.21}$$

Let us remark that for all natural integer $h$

$$h-2\,(j+1)+(j+1) = h-(j+1)$$

We can then write

$$D_h^j = \frac{h\,(h-(j+1))!}{(h-(j+1))\,(j+1)!\,(h-2\,(j+1))!}$$

and also

$$D_h^j = \frac{h}{h-(j+1)}C_{h-(j+1)}^{j+1}$$

## 1.5 Study on the coefficients $D_h^j$Etude sur les coefficients $D_h^j$.

For the following odd natural integers $h = 2k+1$, we verify the relations

$$k = 1 \Longleftrightarrow h = 2k+1 = 3$$
$$D_3^0 = 3C_0^0$$

$$k = 2 \Longleftrightarrow h = 2k+1 = 5$$
$$D_5^0 = 5C_1^0$$
$$D_5^1 = 5C_1^1$$

$$k = 3 \Longleftrightarrow h = 2k+1 = 7$$
$$D_7^0 = 7C_2^0$$
$$D_7^1 = 7C_2^1$$
$$D_7^2 = 7C_2^2$$

$$k = 4 \Longleftrightarrow h = 2k+1 = 9$$
$$D_9^0 = 9C_3^0$$
$$D_9^1 = 9C_3^1$$
$$D_9^2 = 9C_3^2 + 3C_0^0$$
$$D_9^1 = 9C_3^3$$

$$k = 5 \Longleftrightarrow h = 2k+1 = 11$$
$$D_{11}^0 = 11C_4^0$$
$$D_{11}^1 = 11C_4^1$$
$$D_{11}^2 = 11\left(C_4^2 + C_1^0\right)$$
$$D_{11}^3 = 11\left(C_4^3 + C_1^1\right)$$
$$D_{11}^4 = 11C_4^4$$

$$k = 6 \Longleftrightarrow h = 2k+1 = 13$$
$$D_{13}^0 = 13C_5^0$$
$$D_{13}^1 = 13C_5^1$$
$$D_{13}^2 = 13\left(C_5^2 + 2C_2^0\right)$$
$$D_{13}^3 = 13\left(C_5^3 + 2C_2^1\right)$$
$$D_{13}^4 = 13\left(C_5^4 + 2C_2^2\right)$$
$$D_{13}^5 = 13C_5^5$$

$k = 7 \Longleftrightarrow h = 2k + 1 = 15$

$$D_{15}^{0} = 15C_{6}^{0}$$
$$D_{15}^{1} = 15C_{6}^{1}$$
$$D_{15}^{2} = 15\left(C_{6}^{2} + 3C_{3}^{0}\right)$$
$$D_{15}^{3} = 15\left(C_{6}^{3} + 3C_{3}^{1}\right)$$
$$D_{15}^{4} = 15\left(C_{6}^{4} + 3C_{3}^{2} + 3C_{0}^{0}\right)$$
$$D_{15}^{5} = 15\left(C_{6}^{5} + 3C_{3}^{3}\right)$$
$$D_{15}^{6} = 15C_{6}^{6}$$

$k = 8 \Longleftrightarrow h = 2k + 1 = 17$

$$D_{17}^{0} = 17C_{7}^{0}$$
$$D_{17}^{1} = 17C_{7}^{1}$$
$$D_{17}^{2} = 17\left(C_{7}^{2} + 5C_{4}^{0}\right)$$
$$D_{17}^{3} = 17\left(C_{7}^{3} + 5C_{4}^{1}\right)$$
$$D_{17}^{4} = 17\left(C_{7}^{4} + 5C_{4}^{2} + C_{0}^{0}\right)$$
$$D_{17}^{5} = 17\left(C_{7}^{5} + 5C_{4}^{3} + C_{1}^{1}\right)$$
$$D_{17}^{6} = 17\left(C_{7}^{6} + 5C_{4}^{4}\right)$$
$$D_{17}^{7} = 17C_{7}^{7}$$

$k = 9 \Longleftrightarrow h = 2k + 1 = 19$

$$D_{19}^{0} = 19C_{8}^{0}$$
$$D_{19}^{1} = 19C_{8}^{1}$$
$$D_{19}^{2} = 19\left(C_{8}^{2} + 7C_{5}^{0}\right)$$
$$D_{19}^{3} = 19\left(C_{8}^{3} + 7C_{5}^{1}\right)$$
$$D_{19}^{4} = 19\left(C_{8}^{4} + 7C_{5}^{2} + 3C_{2}^{0}\right)$$
$$D_{19}^{5} = 19\left(C_{8}^{5} + 7C_{5}^{3} + 3C_{2}^{1}\right)$$
$$D_{19}^{5} = 19\left(C_{8}^{6} + 7C_{5}^{4} + 2C_{2}^{2}\right)$$
$$D_{19}^{6} = 19\left(C_{8}^{7} + 7C_{5}^{5}\right)$$
$$D_{19}^{8} = 19C_{8}^{8}$$

$k = 10 \Longleftrightarrow h = 2k + 1 = 21$

$$\begin{aligned}
D_{21}^{0} &= 21C_{9}^{0}\\
D_{21}^{1} &= 21C_{9}^{1}\\
D_{21}^{2} &= 21\left(C_{9}^{2} + 19C_{6}^{0}\right)\\
D_{21}^{3} &= 21\left(C_{9}^{3} + 19C_{6}^{1}\right)\\
D_{21}^{4} &= 21\left(C_{9}^{4} + 19C_{6}^{2} + 14C_{3}^{0}\right)\\
D_{21}^{5} &= 21\left(C_{9}^{5} + 19C_{6}^{3} + 14C_{3}^{1}\right)\\
D_{21}^{6} &= 21\left(C_{9}^{6} + 19C_{6}^{4} + 14C_{3}^{2} + 3C_{0}^{0}\right)\\
D_{21}^{7} &= 21\left(C_{9}^{7} + 19C_{6}^{5} + 14C_{3}^{3}\right)\\
D_{21}^{8} &= 21\left(C_{9}^{7} + 19C_{6}^{6}\right)\\
D_{21}^{9} &= 21C_{9}^{9}
\end{aligned}$$

We are led to assume that for all odd natural integer $2k + 1$, greater than or equal to 3, each coefficient $D_{2k+1}^{j}$ can be expressed as follows

$$D_{2k+1}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} \tag{1.22}$$

with

$$0 \leq k - 1 - 3l \leq k - 1 \tag{1.23}$$

and we will write by convention

$$(\forall j)\,(j - 2l < 0)\left(C_{k-1-3l}^{j-2l} = 0\right) \tag{1.24}$$

In order to demonstrate the validity of this formula for all natural integer $k$, we are going to develop, to the extent possible, the coefficient $F_{2k+1}^{l}$ against $k$ and $l$

For any natural integer $k$, we verify the relations

$$D_{2k+1}^{0} = (2k + 1)\, C_{k-1}^{0}$$

$$D_{2k+1}^{1} = (2k + 1)\, C_{k-1}^{1}$$

We can always write, with $k \geq 4$

$$D_{2k+1}^{2} = (2k + 1)\, C_{k-1}^{2} + \left(D_{2k+1}^{2} - (2k + 1)\, C_{k-1}^{2}\right) C_{k-4}^{0}$$

But, in accordance with the relations 1.3 and 1.21 established in pages 2 and 10

$$D_{2k+1}^{2} - (2k + 1)\, C_{k-1}^{2} = (2k + 1)\left(\frac{(2k - 3)!}{3!\,(2k + 1 - 6)!} - \frac{(k - 1)!}{2!\,(k - 3)!}\right)$$

and similarly

$$D_{2k+1}^{2} - (2k + 1)\, C_{k-1}^{2} = (2k + 1)\left(\frac{(2k - 3)!}{3!\,(2k - 5)!} - \frac{(k - 1)!}{2!\,(k - 3)!}\right)$$

and

$$D^2_{2k+1}-(2k+1)\,C^2_{k-1}=(2k+1)\left(\frac{(2k-3)\,(2k-4)}{3!}-\frac{(k-1)\,(k-2)}{2!}\right)$$

and

$$D^2_{2k+1}-(2k+1)\,C^2_{k-1}=(2k+1)\left(\frac{(2k-3)\,(k-2)}{3}-\frac{(k-1)\,(k-2)}{2}\right)$$

and also

$$D^2_{2k+1}-(2k+1)\,C^2_{k-1}=(2k+1)\left(\frac{2\,(2k-3)\,(k-2)-3\,(k-1)\,(k-2)}{6}\right)$$

and finally

$$D^2_{2k+1}-(2k+1)\,C^2_{k-1}=\frac{(2k+1)\,(k-2)\,(k-3)}{6}$$

Let us pose

$$F^1_{2k+1}=D^2_{2k+1}-(2k+1)\,C^2_{k-1}=\frac{(2k+1)\,(k-2)\,(k-3)}{3!}$$

In a similar way, we could find

$$D^3_{2k+1}=(2k+1)\left(C^3_{k-1}+F^1_{2k+1}C^1_{k-4}\right)$$

and

$$D^4_{2k+1}=(2k+1)\left(C^4_{k-1}+F^1_{2k+1}C^2_{k-4}+F^2_{2k+1}C^0_{k-7}\right)$$

which gives us

$$F^2_{2k+1}=\left(\left(D^4_{2k+1}-(2k+1)\,C^4_{k-1}\right)-\left(D^2_{2k+1}-(2k+1)\,C^2_{k-1}\right)C^2_{k-4}\right)$$

Making similar calculations as the previous ones, and with coefficients $D^j_{2k+1}$ and $C^l_{k-j}$ being made explicit, we find

$$F^2_{2k+1}=\frac{(2k+1)\,(k-3)\,(k-4)\,(k-5)\,(k-6)}{5!}$$

We are therefore led to assume that, for all natural integer $k\geq 1$, the equality

$$F^l_{2k+1}=\frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} \qquad (1.25)$$

is true, with the natural integer $l$ such that

$$0\leq l\leq \lfloor\frac{k}{3}\rfloor$$

Let us calculate the difference $F^l_{2k+1}$ and $F^l_{2k-1}$

$$F^l_{2k+1}-F^l_{2k-1}=\frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}-\frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Then

$$F^l_{2k+1} - F^l_{2k-1} = \frac{(k-2-l)!}{(2l+1)!\,(k-2-3l)!}\left(\frac{(2k+1)\,(k-1-l)-(2k-1)\,(k-1-3l)}{(k-1-3l)}\right)$$

and

$$F^l_{2k+1} - F^l_{2k-1} = \frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left((2k+1)\,(k-1-l)-(2k-1)\,(k-1-3l)\right)$$

and also

$$F^l_{2k+1} - F^l_{2k-1} = \frac{(k-2-l)!}{(2l+1)!\,(k-1-3l)!}\left(2\,(2l+1)\,(k-1)\right)$$

and finally

$$F^l_{2k+1} - F^l_{2k-1} = \frac{2\,(k-1)\,(k-2-l)!}{(2l)!\,(k-1-3l)!} \tag{1.26}$$

Our hypothesis 1.22 stated in pages 13 leads us to use a mathematical induction to show the existence of the relation

$$(\forall k \in \mathbb{N}^*)\,(k \geq 1)\,(\forall j \in \mathbb{N})\,(0 \leq j \leq k-1)\left(D^j_{2k+1} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F^l_{2k+1} C^{j-2l}_{k-1-3l}\right)$$

wherein each coefficient $F^l_{2k+1}$ is expressed by the formula 1.25 established in page 14.

Let us assume that, for all natural integer $j \leq k-2$, the relation

$$D^j_{2k-1} = \sum_{l=0}^{\lfloor \frac{k-1}{3} \rfloor} F^l_{2k-1} C^{j-2l}_{k-2-3l} \tag{1.27}$$

is true until the rank $2k-1$, with

$$F^l_{2k-1} = \frac{(2k-1)\,(k-2-l)!}{(2l+1)!\,(k-2-3l)!}$$

Let us calculate now the difference

$$D^j_{2k-1} - D^{j-1}_{2k-3} = D^j_{2k-2}$$

and also

$$D^j_{2k-2} = \sum_{l=0}^{\lfloor \frac{k-1}{3} \rfloor} F^l_{2k-1} C^{j-2l}_{k-2-3l} - \sum_{l=0}^{\lfloor \frac{k-2}{3} \rfloor} F^l_{2k-3} C^{j-2l-1}_{k-3-3l} \tag{1.28}$$

Then, we are faced with two cases.

### 1.5.1 Case 1: $\lfloor \frac{k-1}{3} \rfloor = \lfloor \frac{k-2}{3} \rfloor = m$

We have

$$\lfloor \frac{k-1}{3} \rfloor = m \Longleftrightarrow k-1 = 3m+\rho > 3m$$

The only values that $\rho$ can take a priori are 0, 1 and 2

#### 1.5.1.1 $\rho = 0$

$$\begin{aligned}\rho = 0 &\Longrightarrow k-1 = 3m \\ &\Longleftrightarrow k-2 = 3m-1 < 3m\end{aligned}$$

#### 1.5.1.2 $\rho = 1$

$$\begin{aligned}\rho = 1 &\Longrightarrow k-1 = 3m+1 \\ &\Longleftrightarrow k-2 = 3m\end{aligned}$$

#### 1.5.1.3 $\rho = 2$

$$\begin{aligned}\rho = 2 &\Longrightarrow k-1 = 3m+2 \\ &\Longrightarrow k-2 = 3m+1\end{aligned}$$

Clearly, $\rho$ cannot be equal to 0. We also notice that in this Case 1

$$2k+1 \not\equiv 0 \quad (3) \tag{1.29}$$

Let us recall that

$$\left(C_{k-2-3l}^{j-2l} = C_{k-3-3l}^{j-2l-1} + C_{k-3-3l}^{j-2l}\right) \Longleftrightarrow \left(C_{k-3-3l}^{j-2l-1} = C_{k-2-3l}^{j-2l} - C_{k-3-3l}^{j-2l}\right) \tag{1.30}$$

We then have (see the relation relation 1.28 established page 15)

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(F_{2k-1}^{l} C_{k-2-3l}^{j-2l} - F_{2k-3}^{l} C_{k-3-3l}^{j-2l-1}\right) \tag{1.31}$$

which is equivalent to

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(F_{2k-1}^{l} C_{k-2-3l}^{j-2l} - F_{2k-3}^{l} \left(C_{k-2-3l}^{j-2l} - C_{k-3-2l}^{j-2l}\right)\right)$$

and

$$D_{2k-2}^{j} = \sum_{l=0}^{m} \left(\left(F_{2k-1}^{l} - F_{2k-3}^{l}\right) C_{k-2-3l}^{j-2l} + F_{2k-3}^{l} C_{k-3-2l}^{j-2l}\right)$$

with $m = \lfloor \frac{k-2}{3} \rfloor = \lfloor \frac{k-1}{3} \rfloor$. In particular, among the natural integers $2k+1$ wherein $k$ satisfies this property, we find all the prime integers strictly greater to 3.

### 1.5.2 Case 2: $\lfloor \frac{k-1}{3} \rfloor = \lfloor \frac{k-2}{3} \rfloor + 1 = m$

We have

$$\lfloor \frac{k-1}{3} \rfloor = m \Longleftrightarrow k-1 = 3m+\rho$$

As previously,

#### 1.5.2.1 $\rho = 0$

$$\begin{aligned}\rho = 0 &\Longrightarrow k - 1 = 3m \\ &\Longleftrightarrow k - 2 = 3m - 1\end{aligned}$$

#### 1.5.2.2 $\rho = 1$

$$\begin{aligned}\rho = 1 &\Longrightarrow k - 1 = 3m + 1 \\ &\Longleftrightarrow k - 2 = 3m\end{aligned}$$

#### 1.5.2.3 $\rho = 2$

$$\begin{aligned}\rho = 2 &\Longrightarrow k - 1 = 3m + 2 \\ &\Longleftrightarrow k - 2 = 3m + 1\end{aligned}$$

And in this case, $\rho$ can only be equal to 0. We also notice

$$\begin{aligned}2k + 1 \equiv 0 &\Longleftrightarrow k \equiv 1 \quad (3) \\ &\Longleftrightarrow k - 1 \equiv 0 \quad (3)\end{aligned}$$

We then have (see the relation 1.28 established page 15)

$$\begin{aligned}D_{2k-2}^{j} &= \sum_{l=0}^{m} F_{2k-1}^{l} C_{k-2-3l}^{j-2l} - \sum_{l=0}^{m-1} F_{2k-3}^{l} C_{k-3-3l}^{j-2l-1} \\ &= F_{2k-1}^{m} C_{k-2-3(m+1)}^{j-2(m+1)} + \sum_{l=0}^{m-1} F_{2k-1}^{l} C_{k-2-3l}^{j-2l} - \sum_{l=0}^{m-1} F_{2k-3}^{l} C_{k-3-3l}^{j-2l-1} \\ &= F_{2k-1}^{m} C_{k-2-3(m+1)}^{j-2(m+1)} + \sum_{l=0}^{m-1} \left( \left( F_{2k-1}^{l} - F_{2k-3}^{l} \right) C_{k-2-3l}^{j-2l} + F_{2k-3}^{l} C_{k-3-3l}^{j-2l} \right)\end{aligned}$$

with $m = \lfloor \frac{k-1}{3} \rfloor$ and $m - 1 = \lfloor \frac{k-2}{3} \rfloor$

Let us return to Case 1 and let us take our hypothesis 1.27 stated page 15

$$\sum_{l=0}^{m} F_{2k-3}^{l} C_{k-3-3l}^{j-2l} = D_{2k-3}^{j}$$

then, in accordance with the relation relation 1.31 set out page 16

$$D_{2k-2}^{j} - D_{2k-3}^{j} = D_{2k-4}^{j-1}$$

and finally, we get the equality

$$D_{2k-4}^{j-1} = \sum_{l=0}^{\lfloor \frac{k-2}{3} \rfloor} F_{2k-4}^{l} C_{k-2-3l}^{j-2l} \tag{1.32}$$

with, in accordance with the relation 1.26 established page 15

$$F_{2k-4}^{l} = F_{2k-1}^{l} - F_{2k-3}^{l}$$

We still have to establish that the equality 1.32 in page 17 is true when $k \geq 4$ describes $\mathbb{N}$. We make sure first, by a simple calculation, that this equality indeed holds when $k$ takes successively the values 4, 5 and $6\cdots$, when $j$ takes its values in its domain.

We then assume that this equality holds for any given natural integer less or equal to $2k$, for all $j \leq (k-1)$, that is

$$D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l}$$

We can now remark that the calculations made to get the formula of $D_{h=2k}^{j}$ depending on the coefficients $F_{2k}^{l}$ and the binomial coefficients $C_{k-3l}^{j+1-2l}$ are generalizable to any value of $h$ in $\mathbb{N}$. We just have to verify by mathematical induction the correctness of the formulation of the odd index coefficients $D_{h=2k+1}^{j}$ to obtain a result that is valid, irrespective of the parity of this index $h$.

Let us go back to the initial hypothesis on the odd index coefficients (see our hypothesis 1.27 stated page 15)and let us utilize what we just established. We verify

$$D_{2k+1}^{j} = D_{2k}^{j} + D_{2k-1}^{j-1}$$

with

$$D_{2k}^{j} = \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k}^{l} C_{k-3l}^{j+1-2l}$$

and

$$D_{2k-1}^{j-1} = \sum_{l=0}^{\lfloor \frac{k-1}{3} \rfloor} F_{2k-1}^{l} C_{k-2-3l}^{j-2l}$$

Further to the calculations we just made in pages 16 and 17, we have

$$\begin{aligned}
D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k-2}^{l} C_{k-1-3l}^{j-2l} + F_{2k-1}^{l} C_{k-2-3l}^{j-2l} \\
\Longleftrightarrow D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} \left(F_{2k+1}^{l} - F_{2k-1}^{l}\right) C_{k-1-3l}^{j-2l} + F_{2k-1}^{l} C_{k-2-3l}^{j-2l} \\
\Longleftrightarrow D_{2k}^{j} &= \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} - \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor = m} F_{2k-1}^{l} C_{k-1-3l}^{j-2l-1} \\
\Longleftrightarrow D_{2k}^{j} &= D_{2k+1}^{j} - D_{2k-1}^{j-1}
\end{aligned}$$

This result is in agreement with the equality 1.14 established in page 8.

As we know how to express the coefficients $F_{2k-2}^{l}$ and $F_{2k-1}^{l}$ against $l$ and $k$, we can now calculate $F_{2k+1}^{l}$. We thus find

$$D_{2k+1}^{j}=\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}F_{2k+1}^{l}C_{k-1-3l}^{j-2l}$$

with

$$F_{2k+1}^{l}=\frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

Our mathematical induction is therefore complete for every coefficient $D_h^j$, with odd or even indices $h$.

Let us now summarize all the results we have obtained over the previous pages (see equations 1.8 and 1.12 in pages 7 and 8)

$$(\forall n\in\mathbb{N})\,(n\geq 3)\left((x^n+y^n)=x^n+y^n+xy\sum_{j=1}^{n-2}A_n\,(x,y)\right)$$

with, for $n=2k$ (see equation 1.8 in page 7)

$$A_{2k}\,(x,y)=\sum_{j=0}^{k-1}D_{2k}^{j}\,(-1)^j\,(xy)^j\,(x+y)^{2(k-1-j)}$$

and

$$(\forall k\in\mathbb{N}^*)\,(k>1)\,(\forall j\in\mathbb{N})\,(j\leq k-1)\left(D_{2k}^{j}=\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}F_{2k}^{l}C_{k-3l}^{j+1-2l}\right)$$

and

$$F_{2k}^{l}=\frac{2k\,(k-1-l)!}{(2l)!\,(k-3l)!}$$

and for $n=2k+1$ (see equation 1.12 in page 8)

$$A_{2k+1}\,(x,y)=(x+y)\sum_{j=0}^{k-1}D_{2k+1}^{j}\,(-1)^j\,(xy)^j\,(x+y)^{2(k-1-j)}$$

and

$$(\forall k\in\mathbb{N}^*)\,(\forall j\in\mathbb{N})\,(j\leq k-1)\left(D_{2k+1}^{j}=\sum_{l=0}^{\lfloor\frac{k}{3}\rfloor}F_{2k+1}^{l}C_{k-1-3l}^{j-2l}\right)$$

and

$$F_{2k+1}^{l}=\frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!}$$

## 1.6 Study of $A_{2k+1}(x,y)$ wherein $k \in \mathbb{N}^*$.

We will show in this paragraph how we can further factorize the quantity $A_{2k+1}(x,y)$. Using the previous results, we can write

$$A_{2k+1}(x,y) = (x+y)\sum_{j=0}^{k-1}\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

We then have for each $k$, and for all $j$ and all $l$

$$\begin{aligned} &F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^j (xy)^j (x+y)^{2(k-1-j)} \\ &\quad = F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j-2l+2l} (xy)^{j-2l+2l} (x+y)^{2(k-1-3l+3l-(j-2l)-2l)} \\ &= \left(F_{2k+1}^{l} C_{k-1-3l}^{j-2l} (-1)^{j-2l} (xy)^{j-2l} (x+y)^{2(k-1-3l-(j-2l))}\right) (-1)^{2l} (xy)^{2l} (x+y)^{2l} \end{aligned}$$

We can therefore write $A_{2k+1}(x.y)$ in the following manner

$$\begin{aligned} &A_{2k+1}(x,y) \\ &= (x+y) \\ &\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} (-1)^{2l} (x+y)^{2l} \\ &\sum_{j=0}^{k-1} C_{k-1-3l}^{j-2l} (-1)^{j-2l} (xy)^{j-2l} (x+y)^{2(k-1-3l-(j-2l))} \end{aligned}$$

If $j$ varies from 0 to $k-1$, then $j-2l$ varies from 0 to $k-1-2l$, and as we necessarily have

$$j-2l \leq k-1-3l$$

we get

$$\begin{aligned} &A_{2k+1}(x,y) \\ &= (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} (-1)^{2l} (x+y)^{2l} \sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-3l-j)} \end{aligned}$$

but

$$\begin{aligned} &\sum_{j=0}^{k-1-3l} C_{k-1-3l}^{j} (-1)^j (xy)^j (x+y)^{2(k-1-3l-j)} \\ &\quad = \left((x+y)^2 - xy\right)^{k-1-3l} \\ &\quad = \left(x^2+xy+y^2\right)^{k-1-3l} \end{aligned}$$

and lastly

$$A_{2k+1}(x,y) = (x+y)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l} (-1)^{2l} (x+y)^{2l} \left(x^2+xy+y^2\right)^{k-1-3l}$$

If, in addition, we assume that $2k+1$ is an odd natural integer strictly greater than 3, and not a multiple of 3 then (see the equality 1.29 page 16)

$$k-1 \not\equiv 0 \quad (3)$$

and therefore $k-1-3l$ does not vanish for any value taken by $l$. As a result, $A_{2k+1}(x,y)$ is always divisible by $\left(x^2+xy+y^2\right)$ and we can write for every natural integer $n=2k+1>3$.

$$A_{2k+1}(x,y) =$$

$$(x+y)\left(x^2+xy+y^2\right)\sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}(-1)^{2l}(x+y)^{2l}\left(x^2+xy+y^2\right)^{k-2-3l} \quad (1.33)$$

## 1.7 Various ways to express the Binomial expansion.

We are getting now close to the end of this study, the purpose of which was to express the Newton binomial expansion in other manners. As enounced (see relation 1.2 in page 2) and later established (see relation 1.1 in page 1), we have

$$\begin{aligned}(x+y)^n &= \sum_{j=0}^{n} C_n^j x^{n-j} y^j \\ &= x^n + y^n + \sum_{j=1}^{n-1} C_n^j x^{n-j} y^j \\ &= x^n + y^n + xy\sum_{j=0}^{n-2} C_n^{j+1} x^{n-2-j} y^j \\ &= x^n + y^n + xy\sum_{j=0}^{n-2} (x+y)^{n-2-j}\left(x^j+y^j\right)\end{aligned}$$

Moreover, depending on whether the natural integer $n$ is even or odd, the binomial expansion can be equally expressed as follows

$n=2k$ **pair**

$$(x+y)^{2k} = x^{2k} + y^{2k} + xy\sum_{j=0}^{k-1} D_{2k}^{j}(-1)^j (xy)^j (x+y)^{2(k-1-j)}$$

with

$$D_{2k}^{j} = \frac{2k\,(2k-1-(j+1))!}{(j+1)!\,(2k-2\,(j+1))!}$$

as established in page 7 (see equation 1.8).

$n = 2k + 1$ **impair**

$$(x+y)^{2k+1} = x^{2k+1} + y^{2k+1} + xy\,(x+y) \sum_{j=0}^{k-1} D_{2k+1}^{j}\,(-1)^j\,(xy)^j\,(x+y)^{2(k-1-j)}$$

with

$$D_{2k+1}^{j} = \frac{(2k+1)\,(2k-(j+1))!}{(j+1)!\,(2k+1-2\,(j+1))!}$$

as established in page 8 (see equation 1.12).

$n = 2k + 1 > 3$ **and** $n \not\equiv 0\ (3)$

$$(x+y)^n = x^n + y^n + \\ xy\,(x+y)\left(x^2+xy+y^2\right) \sum_{l=0}^{\lfloor \frac{k}{3} \rfloor} F_{2k+1}^{l}\,(-1)^{2l}\,(x+y)^{2l}\left(x^2+xy+y^2\right)^{k-2-3l} \tag{1.34}$$

with

$$F_{2k+1}^{l} = \frac{(2k+1)\,(k-1-l)!}{(2l+1)!\,(k-1-3l)!} \tag{1.35}$$
$$\equiv 0 \quad (n = 2k+1)$$

as established in page 21 (see equation 1.33).

Let us notice that the set of the prime integer greater than 3 is a subset of these natural integers $n$.

Outlining these results concludes this study. Let us now turn to the study of the Fermat's conjecture, which was proved by Andrew Wiles (1993/1995).

# Chapter 2

# Study of Fermat's Conjecture.

## 2.1 Subject of the chapter.

This conjecture was proved by Andrew Wiles between 1993 and 1995. However, as it has been the case for other problems in the history of Mathematics, setting up to explore other avenues that could lead to other demonstrations is not without interest. This what we are going to try and show.

## 2.2 Reminder of the conjecture.

Let the equation

$$x^n + y^n = z^n \tag{2.1}$$

with $n$ prime integer, $n > 2 \in \mathbb{N}^*$.

Pierre de Fermat (1607-1665) stated that no three non zero natural integers $x \in \mathbb{N}^*$, $y \in \mathbb{N}^*$ and $z \in \mathbb{N}^*$ could satisfy the relation 2.1. We shall assume

$$0 < x < y < z$$

This leads us to write

$$\begin{aligned} x + y \equiv z \quad (n) &\Longleftrightarrow x + y = kn + z \, (k \in \mathbb{N}^*) \\ &\Longrightarrow x > kn \end{aligned} \tag{2.2}$$

Leaving aside the case wherein $n = 3$, we are going to be interested in all the other cases wherein $n > 3$. It is possible, without loss of generality, to consider only the cases wherein $n$ is prime.

## 2.3 First point.

Let, if they exist, $x \in \mathbb{N}^*$, $y \in \mathbb{N}^*$ and $z \in \mathbb{N}^*$ which satisfy the relation 2.1 (see on this page 23). Then, it is always possible to assume that $x$, $y$ and $z$ are pairwise coprime integers. We have

$$\begin{aligned} z^n &= x^n + y^n \\ &= x^n - (-1)^n y^n \\ &= (x - (-1) y) \sum_{j=0}^{n-1} x^{n-1-j} (-1)^j y^j \end{aligned}$$

In the case wherein $z \not\equiv 0 \ (n)$, $(x - (-1) y)$ and $\sum_{j=0}^{n-1} x^{n-1-j} (-1)^j y^j$ are two coprime quantities and we have

$$(\exists h \in \mathbb{N}^*) (x + y = h^n)$$

and

$$z^n \equiv 0 \Longrightarrow z \equiv 0 \quad (h)$$

Finally, the relation 1.34 and the formula 1.35 stated in page 22 allow us to write

$$((x + y) - z) \sum_{j=0}^{n-1} (x + y)^{n-1-j} z^j = n\lambda xy (x + y) \left(x^2 + xy + y^2\right) \tag{2.3}$$

with $\lambda \in \mathbb{N}^*$.

It is obvious that the natural integers $x$, $y$, $(x + y)$ and $\left(x^2 + xy + y^2\right)$ are pairwise coprime.

## 2.4 Second point.

Let $n$ and $p$ be two odd prime integers, distinct or not from each other. Let us place ourselves in $\mathbb{Z}/p\mathbb{Z}$, the set of integers modulo $p$, equipped with the addition and the multiplication. This set, equipped with these two laws, is a commutative field and each and every of its element $u$ has an inverse $u^{-1}$. Let us also consider $x$, $y$ and $z$, solutions, if they exist, of the equation 2.1 page 23 as stated above (see section 2.2 page 23).

We first remark that $x^2 + xy + y^2$ is always an odd natural integer. We can now choose $p$ such that

$$x^2 + xy + y^2 \equiv 0 \quad (p) \tag{2.4}$$

We then have

$$x^2 + xy + y^2 = (x + y)^2 - xy$$

and so in $\mathbb{Z}/p\mathbb{Z}$

$$(x + y)^2 \equiv xy \quad (p)$$

Moreover, it is clear that if $p \neq n$ and if

$$(x + y) - z \equiv 0 \Longleftrightarrow (x + y) \equiv z \quad (p) \tag{2.5}$$

then

$$\sum_{j=0}^{n-1} (x + y)^{n-1-j} z^j \not\equiv 0 \quad (p) \tag{2.6}$$

and reciprocally.

It is also clear that, given the assumptions made on $x$, $y$ and $z$, we have

$$\begin{aligned} &x \not\equiv 0 \quad (p) \\ &y \not\equiv 0 \quad (p) \\ &x + y \not\equiv 0 \quad (p) \\ \Longrightarrow &z \not\equiv 0 \quad (p) \end{aligned} \tag{2.7}$$

The three first inequalities are easy to establish. In the case of $z$

$$\begin{aligned} z \equiv 0 &\Longrightarrow z^n \equiv 0 \quad (p) \\ &\Longleftrightarrow x^n + y^n \equiv 0 \quad (p) \\ &\Longleftrightarrow (x+y)^n \equiv 0 \quad (p) \\ &\Longleftrightarrow x + y \equiv 0 \quad (p) \end{aligned}$$

But we just showed that

$$x + y \not\equiv 0 \quad (p)$$

and so

$$z \not\equiv 0 \quad (p) \tag{2.8}$$

## 2.5 Third point.

Let us consider first any odd prime integer $p$ and $\mathbb{Z}/p\mathbb{Z}$ the set of integer modulo $p$, equipped with the addition and the multiplication. This set is a commutative field and each and every of its element $u$ has an inverse $u^{-1}$.

Now, we have in $\mathbb{Z}/p\mathbb{Z}$

$$\begin{aligned} (x+y)^n &\equiv x^n + y^n \quad (p) \\ &\equiv z^n \quad (p) \end{aligned} \tag{2.9}$$

Let us write

$$\begin{aligned} x^2 + xy + y^2 \equiv 0 &\Longrightarrow x + y \equiv -y^2 x^{-1} \quad (p) \\ &\Longrightarrow x + y \equiv -x^2 y^{-1} \quad (p) \end{aligned}$$

then

$$\begin{aligned} (x+y)^n - z^n \equiv 0 &\Longrightarrow (-1)^n \left(y^2 x^{-1}\right)^n - z^n \equiv 0 \quad (p) \\ &\Longrightarrow (-1)^n \left(x^2 y^{-1}\right)^n - z^n \equiv 0 \quad (p) \end{aligned}$$

and

$$(-1)^n y^{2n} \equiv x^n z^n \Longleftrightarrow (-1)^n x^{2n} \equiv y^n z^n \quad (p)$$

which allows us to write

$$z^{2n} \equiv (-1)^n \left(x^{2n} + y^{2n}\right) \quad (p)$$
$$z^{2n} \equiv x^n y^n \quad (p)$$
$$x^{2n} + x^n y^n + y^{2n} \equiv 0 \quad (p)$$

## 2.6 A proof of the conjecture.

We have (see relation 2.9 page 25)

$$\frac{(x+y)^n}{z^n} \equiv 1 \quad (p)$$

Let us now remark

$$\begin{aligned}
\frac{x+y}{z}\sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^j &\equiv \sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^{j+1} \quad (p)\\
&\equiv \sum_{j+1-1=0}^{n-1}\left(\frac{x+y}{z}\right)^{j+1} \quad (p)\\
&\equiv \sum_{j=1}^{n}\left(\frac{x+y}{z}\right)^{j} \quad (p)\\
&\equiv 1+\sum_{j=1}^{n-1}\left(\frac{x+y}{z}\right)^{j} \quad (p)\\
&\equiv \sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^{j} \quad (p)
\end{aligned}$$

and so, necessarily

$$\left(\frac{x+y}{z}\sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^j \equiv \sum_{j=0}^{n-1}\left(\frac{x+y}{z}\right)^j \quad (p)\right)$$
$$\Longleftrightarrow$$
$$\left(\frac{x+y}{z} \equiv 1 \quad (p)\right)$$

Let us now consider the following two cases

### 2.6.1 $x^2 + xy + y^2 \not\equiv 0 \ (n)$

Let $p$ be a prime integer distinct from $n$. Let us get back to the formula 1.34 en page 22 and let us write

$$(x+y)^n - (x^n + y^n) \equiv 0 \quad (p)$$

Clearly

$$(x+y) \equiv z \quad (p) \Longrightarrow \sum_{j=0}^{n-1}(x+y)^j z^{n-1-j} \equiv nz^{n-1} \not\equiv 0 \quad (p)$$

and we must have

$$x^2 + xy + y^2 \leq (x+y) - z$$

but, whatever $x$, $y$ and $z$ positive numbers and greater than 1

$$(x+y) - z > 0 \Longrightarrow (x+y) - z < x^2 + xy + y^2$$

and we end up with an impossibility.

### 2.6.2 $x^2 + xy + y^2 \equiv 0 \ (n)$

So let us assume

$$x^2 + xy + y^2 \equiv 0 \quad (n)$$

It is of course possible that $n$ is not the one and only prime divisor of $x^2+xy+y^2$. Having chosen $n$ greater than 3, we also recall our hypotheses 2.2 put forward in page 23, which leads us to write

$$x^2 + xy + y^2 > n$$

It is obvious that $x + y \not\equiv 0 \ (n)$, otherwise, $x \equiv 0 \ (n)$ and therefore $y \equiv 0 \ (n)$, which we ruled out (see the conclusions first point 2.3 en page 23).

Let us first assume that $n$ is the only prime divisor of $x^2 + xy + y^2$ and let us put

$$x^2 + xy + y^2 \equiv 0 \quad (n^k)$$

wherein $k \in \mathbb{N}^*$ and $k > 1$ is the largest possible exponent. The relation 2.3 stated on page 24 allows us to write

$$\begin{aligned}(x+y)^n - (x^n + y^n) &= (x+y)^n - z^n \\ &= ((x+y) - z) \sum_{j=0}^{n-1} (x+y)^{n-1-j} z^j \\ &\equiv 0 \quad (n^{k+1})\end{aligned}$$

Suppose

$$(x+y) - z \equiv 0 \quad (n^{k+1-r})$$

wherein$k + 1 - r$ is there as well the largest possible exponent and wherein

$$(r \in \mathbb{N}^*) \, (0 < r < k+1)$$

Let $\rho$ be the smaller of the two numbers $k + 1 - r$ and $r$. Then

$$\sum_{j=0}^{n-1} (x+y)^{n-1-j} z^j \equiv 0 \Longleftrightarrow n (x+y)^{n-1} \equiv n z^{n-1} \equiv 0 \quad (n^\rho)$$

But, in agreement with the equality 2.7 on page 25

$$\begin{aligned}(x+y) \not\equiv 0 \quad (n) &\Longrightarrow (x+y) \not\equiv 0 \quad (n^\rho) \\ &\Longrightarrow (x+y)^{n-1} \not\equiv 0 \quad (n^\rho)\end{aligned}$$

and likewise (see equality 2.8 page 25)

$$\begin{aligned} z \not\equiv 0 \quad (n) &\Longrightarrow z \not\equiv 0 \quad (n^\rho) \\ &\Longrightarrow z^{n-1} \not\equiv 0 \quad (n^\rho) \end{aligned}$$

and necessarily $\rho = r = 1$. And so

$$(x+y) - z \equiv 0 \quad (n^k)$$

There therefore exists a non-zero natural integer $\lambda_1$, not a multiple of $n$, such that

$$(x+y) - z = \lambda_1 n^k$$

and we must have

$$x^2 + xy + y^2 \leq (x+y) - z$$

If $n$ is not the only prime divisor of $x^2 + xy + y^2$, then there exists at least one non-zero prime integer $p$ such that

$$x^2 + xy + y^2 \equiv 0 \quad (p)$$

But, we have just established in section 2.6.1 page 26, that no prime divisor $p$ dividing $x^2+xy+y^2$ and distinct from $n$ can divide $\sum_{j=0}^{n-1} (x+y)^{n-1-j} z^j$, and therefore only $(x+y) - z \equiv 0 \ (p)$, and there again

$$x^2 + xy + y^2 \leq (x+y) - z$$

but, whatever positive $x$, $y$ and $z$ are greater than 1

$$(x+y) - z > 0 \Longrightarrow (x+y) - z < x^2 + xy + y^2$$

and we end up with an impossibilty in both cases.

The conjecture is thus proved for every prime integer $n > 3$.**QED**.