A solvable quintic equation

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(Dated: Jul. 22nd, 2023)

This article presents a solvable quintic equation under the conditions that several coefficients of a quintic equation are restricted to become dependent on the other coefficients. We can solve a quintic equation by restricting two coefficients among total four coefficients available. If a quintic equation has a quadratic factor $(x^2 + b_1 x + b_0)$, then we get a two simultaneous equations, which can be solved by using a sextic equation under restriction.

A. De Moivre's Quintic Equation

A monic general quintic equation form is

$$x^5 + d_4 x^4 + d_3 x^3 + d_2 x^2 + d_1 x + d_0 = 0. \tag{1}$$

The process of solving a quintic equation is very complicated. So, we consider a reduced quintic form derived from the above equation (1) in which x is substituted with $x + \frac{d_4}{5}$, or simply $d_4 = 0$,

$$x^5 + c_3 x^3 + c_2 x^2 + c_1 x + c_0 = 0. \tag{2}$$

A solvable quintic equation is given from the de Moivre's quintic. De Moivre's theorem is the only formula that can solve a quintic equation by using its coefficients. A solution of the de Moivre's quintic equation can be easily derived as follows.

If $x = \sqrt[5]{\alpha} - \frac{s}{5\sqrt[5]{\alpha}}$, we have

$$\begin{aligned} & x^5 + s x^3 + \frac{s^2}{5} x + t \\ & = \alpha - \frac{s^5}{3125\alpha} + t \\ & = 0. \end{aligned} \tag{3}$$

where t is the coefficient of constant term.

From the above, we get a solution of the quadratic equation with respect to α,

$$\alpha = -\frac{t}{2} \pm \sqrt{\frac{t^2}{4} + \left(\frac{s}{5}\right)^5}, \tag{4}$$

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which provides a solution of the de Moivre's quintic

$$x = \sqrt[5]{-\frac{t}{2} - \sqrt{\frac{t^2}{4} + \left(\frac{s}{5}\right)^5}} + \sqrt[5]{-\frac{t}{2} + \sqrt{\frac{t^2}{4} + \left(\frac{s}{5}\right)^5}}. \tag{5}$$

B. Derivation of a solvable Quintic Equation

A monic reduced form of a quintic equation is read as follows

$$x^5 + c_3x^3 + c_2x^2 + c_1x + c_0. \tag{6}$$

If this quintic equation is factorable into a cubic and a quadratic equation as follows,

$$\begin{aligned} & x^5 + c_3x^3 + c_2x^2 + c_1x + c_0 \\ & = (x^3 - b_1x^2 + a_1x + a_0)(x^2 + b_1x + b_0). \end{aligned} \tag{7}$$

After eliminating the coefficients a_i, we get two simultaneous equations with respect to b_1 and b_0. If we get b_1 and b_0 by solving the simultaneous equations, we can get solutions of the equation (6).

To do so, developing the latter into a cubic and quadratic equation, and comparing each coefficient of nth degree of x, which is equal to each other, we can factor the quintic equation to get a solution of a quintic equation.
After developing the parentheses, and comparing to each other, we get

$$\begin{aligned} a_1 &= c_3 - b_0 + {b_1}^2, \\ a_0 &= c_2 + 2b_0b_1 - b_1c_3 - {b_1}^3, \end{aligned} \tag{8}$$

which provides with

$$\begin{aligned} & x^5 + c_3x^3 + c_2x^2 + c_1x + c_0 \\ & = \left(x^3 - b_1x^2 + (-b_0 + c_3 + {b_1}^2)x + 2b_0b_1 - b_1c_3 - {b_1}^3\right)(x^2 + b_1x + b_0), \end{aligned} \tag{9}$$

where we get two simultaneous equations with respect to b_0,

$${b_0}^2 - (c_3 + 3{b_1}^2)b_0 + c_1 - b_1c_2 + {b_1}^4 + {b_1}^2c_3 = 0, \tag{10}$$

$$2b_1{b_0}^2 + (c_2 - b_1c_3 - {b_1}^3)b_0 - c_0 = 0. \tag{11}$$

To solve these simultaneous equations (10) and (11), the resultant can be used. However, if we try to find b_1 or b_0 using the resultant, we would face more difficulties as it provides a 10th degree equation. However, the equation (6) can be solved if certain conditions are given, which has the fifth root of a quintic equation as follows.

From the equation (11), we get

$$b_0 = \frac{1}{4}c_3 + \frac{1}{4}{b_1}^2 - \frac{c_2}{4b_1} \pm \frac{\sqrt{D_1}}{4b_1}, \quad b_1 \neq 0, \tag{12}$$

where D_1 is given as the discriminant as

$$D_1 = {b_1}^6 + 2c_3{b_1}^4 - 2c_2{b_1}^3 + {c_3}^2{b_1}^2 + (8c_0 - 2c_2c_3)b_1 + {c_2}^2. \tag{13}$$

b_1 of the above can be determined by solving the sextic equation if possible.

C. A sextic equation to solve a quintic equation

A reduced sextic equation is read as;

$$x^6 + d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0 = 0. \tag{14}$$

If this equation has factors both a quartic equation and a quadratic equation, we have

$$(x^4 - v_1x^3 + u_2x^2 + u_1x + u_0)(x^2 + v_1x + v_0) = 0. \tag{15}$$

Comparing the two equations (14) and (15) after eliminating the coefficients u_i, we have

$${v_1}^5 + (-4v_0 + d_4){v_1}^3 - d_3{v_1}^2 + (d_2 - 2v_0d_4 + 3{v_0}^2)v_1 - d_1 + v_0d_3 = 0, \tag{16}$$

$$v_0{v_1}^4 + (v_0d_4 - 3{v_0}^2){v_1}^2 - v_0d_3v_1 + v_0d_2 + {v_0}^3 - {v_0}^2d_4 = d_0. \tag{17}$$

If the equation (16) becomes a de Moivre quintic, we may get v_1. Therefore, if the coefficient $d_3 = 0$ of ${v_1}^2$ term, and the square of the coefficient of ${v_1}^3$ term is equal to 5 times of the coefficient of v_1 term, we get

$$\begin{aligned}&(-4v_0 + d_4)^2 - 5(d_2 - 2v_0d_4 + 3{v_0}^2) \\ &= {v_0}^2 + 2d_4v_0 + {d_4}^2 - 5d_2 \\ &= 0.\end{aligned} \tag{18}$$

From the above, we have

$$v_0 = -d_4 \pm \sqrt{5d_2}. \tag{19}$$

One of the equation (16) provides

$$v_1{}^5 + (5d_4 - 4\sqrt{5d_2})v_1{}^3 + (16d_2 + 5d_4{}^2 - 8d_4\sqrt{5d_2})v_1 - d_1 = 0. \tag{20}$$

The above provides a solution

$$v_1 = \sqrt[5]{\frac{1}{2}d_1 - \sqrt{D_2}} + \sqrt[5]{\frac{1}{2}d_1 + \sqrt{D_2}}, \tag{21}$$

where

$$D_2 = \frac{1}{4}d_1{}^2 + \left(d_4 - \frac{4\sqrt{5d_2}}{5}\right)^5. \tag{22}$$

And d_0 is given from (17), which is dependent on the preceding coefficients

$$d_0 = (-d_4 + \sqrt{5d_2})v_1{}^4 + (-15d_2 - 4d_4{}^2 + 7d_4\sqrt{5d_2})v_1{}^2 - 21d_2d_4 - 2d_4{}^3 + 5d_4{}^2\sqrt{5d_2} + 6\sqrt{5d_2{}^3}. \tag{23}$$

Then, we have two roots of the following quadratic factor of (15),

$$x^2 + v_1x + v_0 = 0, \tag{24}$$

which provides two roots of the sextic equation (14),

$$\begin{aligned} x &= -\frac{1}{2}v_1 \pm \sqrt{\frac{1}{4}v_1{}^2 - v_0} \\ &= -\frac{1}{2}\left(\sqrt[5]{\frac{1}{2}d_1 - \sqrt{D_2}} + \sqrt[5]{\frac{1}{2}d_1 + \sqrt{D_2}}\right) \\ &\quad \pm\sqrt{d_4 - \sqrt{5d_2} + \frac{1}{4}\left(\sqrt[5]{\frac{1}{2}d_1 - \sqrt{D_2}} + \sqrt[5]{\frac{1}{2}d_1 + \sqrt{D_2}}\right)^2}. \end{aligned} \tag{25}$$

These are two roots of a sextic equation that can be factored into a quartic and a quadratic equation under restrictions.

D. Solution of a solvable quintic Equation

By using the above conditional solution of a sextic equation, we can get a restricted solution of the equation (13),

$$D_1 = b_1{}^6 + 2c_3b_1{}^4 - 2c_2b_1{}^3 + c_3{}^2b_1{}^2 + (8c_0 - 2c_2c_3)b_1 + c_2{}^2. \tag{26}$$

If this sextic equation has a quadratic factor,

$$ {b_1}^2 + v_1 b_1 + v_0 = 0. \tag{27} $$

Two unknown coefficients v_1 and v_0 are given as the simultaneous equations as follows

$$ {v_1}^5 + (2c_3 - 4v_0){v_1}^3 + 2c_2{v_1}^2 + (-4c_3v_0 + {c_3}^2 + 3{v_0}^2)v_1 - 8c_0 + 2c_2c_3 - 2c_2v_0 = 0, \tag{28} $$

$$ v_0{v_1}^4 + (2c_3v_0 - 3{v_0}^2){v_1}^2 + 2c_2v_0v_1 - 2c_3{v_0}^2 - {c_2}^2 + {v_0}^3 + {c_3}^2v_0 = 0. \tag{29} $$

Now, we have the following results from the equation (28)

$$ \begin{aligned} c_2 &= 0, \\ v_0 &= (-2 \pm \sqrt{5})c_3, \\ v_1 &= \sqrt[5]{4c_0 - \sqrt{D_3}} + \sqrt[5]{4c_0 + \sqrt{D_3}}, \\ D_3 &= 16{c_0}^2 + {c_3}^5\left(2 - \frac{4\sqrt{5}}{5}\right)^5. \end{aligned} \tag{30} $$

And we get two roots of the equation (27),

$$ \begin{aligned} b_1 &= -\frac{1}{2}v_1 \pm \sqrt{\frac{1}{4}{v_1}^2 - v_0} \\ &= -\frac{1}{2}\left(\sqrt[5]{4c_0 - \sqrt{D_3}} + \sqrt[5]{4c_0 + \sqrt{D_3}}\right) \\ &\quad \pm\sqrt{(2-\sqrt{5})c_3 + \frac{1}{4}\left(\sqrt[5]{4c_0 - \sqrt{D_3}} + \sqrt[5]{4c_0 + \sqrt{D_3}}\right)^2}. \end{aligned} \tag{31} $$

And b_0 from the equation (12) becomes

$$ b_0 = \frac{1}{4}c_3 + \frac{1}{4}{b_1}^2 \pm \frac{\sqrt{D_1}}{4b_1}, \quad b_1 \neq 0. \tag{32} $$

We can have two roots from the quadratic factor $(x^2 + b_1x + b_0)$ of the equation (7),

$$ x = -\frac{1}{2}b_1 + \sqrt{\frac{1}{4}{b_1}^2 - b_0}, \tag{33} $$

where b_1 of (31) with D_3 of (30), and b_0 of (32) with D_1 of (26) respectively.

E. Summary and Examples

A general quintic equation can be written as

$$x^5 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0.$$

If a quintic equation is factorable with a factor of $(x^2 + b_1x + b_0)$, then the quintic equation has two roots that shares two roots of the quadratic equation.
However, in the process of obtaining b_1 and b_0, which are the coefficients of the quadratic equation, we encounter the difficulty of solving the 10th degree equation(decic equation). It is therefore clear that there is no general way to solve a quintic equation normally. However, if certain conditions are given, b_1 and b_0 can be obtained, and thus the solution of a quintic equation can be obtained. Nevertheless, in the process of obtaining b_1 and b_0, it is difficult to solve the sextic equation, but we can find that the sextic equation can also be solved using the de Moivre quintic equation in the process of solving the sextic equation with a quadratic equation $(x^2 + v_1x + v_0)$ as a factor. Here, if v_1 and v_0 are obtained, then b_1 and b_0 can be obtained from them, then the roots of the quintic equation can be obtained from $(x^2 + b_1x + b_0)$. To get a solvable quintic equation, the coefficient c_2 of x^2 term equals to zero, then we get b_1 from (31)

$$\begin{aligned} b_1 &= -\frac{1}{2}\left(\sqrt[5]{4c_0 - \sqrt{D_3}} + \sqrt[5]{4c_0 + \sqrt{D_3}}\right) \\ &\pm\sqrt{(2-\sqrt{5})c_3 + \frac{1}{4}\left(\sqrt[5]{4c_0 - \sqrt{D_3}} + \sqrt[5]{4c_0 + \sqrt{D_3}}\right)^2}, \end{aligned}$$

where D_3 is given from (30) as

$$D_3 = 16{c_0}^2 + {c_3}^5\left(2 - \frac{4\sqrt{5}}{5}\right)^5.$$

And b_0 from (32) as

$$b_0 = \frac{1}{4}c_3 + \frac{1}{4}{b_1}^2 \pm \frac{\sqrt{D_1}}{4b_1}, \quad b_1 \neq 0,$$

where D_1 is given from (26)

$$D_1 = {b_1}^6 + 2c_3{b_1}^4 - 2c_2{b_1}^3 + {c_3}^2{b_1}^2 + (8c_0 - 2c_2c_3)b_1 + {c_2}^2.$$

And the coefficient c_1 of x term is given from (10),

$$c_1 = -{b_0}^2 + (c_3 + 3{b_1}^2)b_0 + b_1c_2 - {b_1}^4 - {b_1}^2c_3.$$

These five steps bring us solutions of a solvable quintic equation by a quadratic factor

$$x^2 + b_1x + b_0 = 0.$$

This quadratic equation provides two roots of a solvable quintic equation

$$x = -\frac{1}{2}b_1 \pm \sqrt{\frac{1}{4}{b_1}^2 - b_0}.$$

Writing down a solution of a quintic equation in a row is too lengthy, so it's much easier to just plug in each step one by one and get the result.

Since the real values obtained from arbitrary c_3 and c_0 are very complex. So for an easy example, let $b_1 = 2$, $b_0 = 2$, and $c_3 = -4$, $c_0 = 16$, then we have $c_1 = 12$. In this case, we get the following quintic equation, which is factored into a quadratic factor

$$x^5 - 4x^3 + 12x + 16 = (x^2 + 2x + 2)(x^3 - 2x^2 - 2x + 8).$$

For another case, let $b_1 = 2$, $c_3 = 0$, $c_0 = -3$, then we get $D_1 = 16$, $b_0 = \frac{3}{2}$ and $c_1 = -\frac{1}{4}$. The factoring is given as

$$x^5 - \frac{1}{4}x - 3 = (x^2 + 2x + \frac{3}{2})(x^3 - 2x^2 + \frac{5}{2}x - 2).$$

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