Study on Classical Electrodynamics Spin

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Abstract. There are two mutually exclusive concepts of the electrodynamics spin. According to the widespread concept, the spin density is proportional to the gradient of the electromagnetic energy density. Therefore, an unlimited plane wave of circular polarization does not contain spin, and a real wave, limited in space, carries all spin at its boundary, separately from energy. In contrast, according to the original concept, the spin density is proportional to the energy density, and the spin of plane waves is not related to the existence of the boundaries. Within the framework of this concept, we calculate the spin fluxes of plane waves in various situations and the previously unnoticed spin flux in the dipole radiation. The reason for the transition from this initial concept to the concept of a spin proportional to energy density gradient is discussed.
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1. Introduction. Spin and moment of a linear momentum.

The idea of the classical spin of electromagnetic radiation, in fact, dates back to the 19th century and belongs to Sadowsky [1] and Poynting [2]. Poynting wrote, referring to circularly polarized electromagnetic waves: "If we put *E* for the energy in unit volume and *G* for the torque per unit area, we have $G = E\lambda/2\pi$".

This statement means, in particular, that a plane wave of circular polarization with intensity $I = cE$ [J/ m^{2}s], which propagates along the z-axis and is absorbed by the xy-plane, acts on this plane with a *distributed* torque so that the infinitesimal area da_z of the xy-plane receives a flux of angular momentum, i.e. torque

$$d\tau_z = (I/\omega)da_z .$$

In this case, there is no linear momentum directed in the xy-plane, which could create angular momentum along the z-axis. This statement means the existence of a *density* of angular momentum, which is not a moment of a linear momentum and is generally independent of a linear momentum.

This is the situation described by Weyssenhoff [3], giving the definition of a *spin liquid*: "By spin-fluid we mean a fluid each element of which possesses besides energy and linear momentum also a certain amount of angular momentum, proportional – just as energy and the linear momentum – to the volume of the element". Thus, a circularly polarized electromagnetic radiation should be regarded as a spin liquid.

At present, the torque $d\tau_z = (I/\omega)da_z$ can be easily explained using the concept of photon spin. The flux density of photons is $I/h\nu$, and each photon carries the spin angular momentum $\hbar$. Therefore, the spin flux density is just equal to I/ω.

After the work of Noether [4], this local angular momentum is mathematically expressed as a tensor spin density (in short, the spin tensor). The spin tensor is obtained by varying the action using one or the other Lagrangian. We use the letter upsilon as the root letter to denote the spin tensor: $\Upsilon^{\lambda\mu\nu}$. The meaning of the spin tensor is given by the formula for the infinitesimal 4-spin

$$dS^{\lambda\mu} = \Upsilon^{\lambda\mu\nu}dV_\nu , \quad \lambda,\mu,\nu... \to x, y, z, t . \qquad (1.1)$$

This means that the 4-volume dV_ν contains a spin angular 4-momentum $dS^{\lambda\mu}$. For example, $dS^{xy} = \Upsilon^{xyt}dV$ is the z-component of the spin in the volume dV. Weyssenhoff writes just about this amount of angular momentum.

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$dS^{xy} = \Upsilon^{xyz} da_z dt$ is the z-component of the spin that has passed through the area da_z in time dt, i.e. $d\tau^{xy} = \Upsilon^{xyz} da_z$ is the torque acting on the area da_z, and Υ^{xyz} is the surface density of the torque, similar to pressure, that is, the density of the spin flux. This quantity is denoted by Poynting *G* (here *G* is used to denote momentum volume density).

The spin tensor $\Upsilon^{\lambda\mu\nu}$ is similar to the energy-momentum tensor $T^{\mu\nu}$. The meaning of the energy-momentum tensor is given by the formula for the infinitesimal 4-momentum

$$dp^{\mu} = T^{\mu\nu} dV_{\nu} . \tag{1.2}$$

This means that the 4-volume dV_{ν} contains the 4-momentum dp^{μ}. For example, $dp^{z} = T^{zt} dV$ is a z-component of the momentum in the volume dV. That is, $T^{zt} = G^{z}$ is the z-component of the momentum volume density.

$dp^{z} = T^{zz} da_z dt$ is the z-component of the momentum that has passed through the area da_z in time dt, that is, $d\mathsf{F}^{z} = T^{zz} da_z$ is the force acting on the area da_z, and T^{zz} is the surface density of the normal force, that is, pressure $\mathsf{P} = T^{zz}$.

$dp^{t} = T^{tz} da_z dt$ is the mass-energy that has passed through the area da_z in time dt, that is, T^{tz} is the mass-energy flux density or the Poynting vector. We denote the Poynting vector I because the letter S is occupied by spin.

Using the radius vector $x^{i} = \{x, y, z\}$ allows to enter a moment of a linear momentum relative to the origin $dL^{ij} = x^{i} dp^{j} - x^{j} dp^{i}$. In 4-space, the moment of momentum looks like this

$$dL^{\lambda\mu} = x^{\lambda} dp^{\mu} - x^{\mu} dp^{\lambda} \quad \text{или} \quad dL^{\lambda\mu} = 2x^{[\lambda} dp^{\mu]} \tag{1.3}$$

(we use square brackets for antisymmetrization). The moment of momentum of a body is obtained by the integration:

$$L^{\lambda\mu} = \int 2x^{[\lambda} dp^{\mu]} = \int 2x^{[\lambda} T^{\mu]\nu} dV_{\nu} . \tag{1.4}$$

This moment of momentum is the orbital angular momentum If a body or an atom revolves only around its center of mass, the moment of momentum $L^{\lambda\mu}$ is independent of the computation point used. The total angular momentum of a body or of radiation is equal to the sum of the moment of momentum and spin

$$J^{\lambda\mu} = L^{\lambda\mu} + S^{\lambda\mu} = \int 2x^{[\lambda} T^{\mu]\nu} dV_{\nu} + \int \Upsilon^{\lambda\mu\nu} dV_{\nu} . \tag{1.5}$$

A specific generally accepted expression for the spin tensor was obtained within the framework of the Lagrangian formalism using the canonical Lagrangian [5-7] $\mathsf{L} = -F_{\mu\nu} F^{\mu\nu} / 4$:

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda} \delta_{\alpha}^{\mu]} \frac{\partial \mathsf{L}}{\partial(\partial_{\nu} A_{\alpha})} = -A^{\lambda} F^{\mu\nu} + A^{\mu} F^{\lambda\nu} = -2A^{[\lambda} F^{\mu]\nu} , \tag{1.6}$$

where A^{λ} and $F_{\mu\nu}$ are the magnetic vector potential and the field-strength tensor of the electromagnetic field, respectively. The component expressing the volume density of the spin is

$$\underset{c}{\Upsilon}^{ijt} = -2A^{[i} F^{j]t} = -2A^{[i} D^{j]} = \varepsilon_0 \mathbf{E} \times \mathbf{A} . \tag{1.7}$$

As an example, we consider the use of the canonical spin tensor (1.7) in a circularly polarized plane electromagnetic wave propagating along the z axis, as Soper do it [6]. The standard expression for such a wave looks like

$$\mathbf{E} = \exp(ikz - i\omega t)(\mathbf{x} + i\mathbf{y}) E \ \text{[V/m]}, \quad \mathbf{B} = -ik\mathbf{E}/\omega \ \text{[Vs/m}^2\text{]}, \tag{1.8}$$

here $\mathbf{x}$ and $\mathbf{y}$ are the unit coordinate vectors, and *E* is the wave amplitude. Using the temporal gauge of the vector potential (the scalar potential is zero, $\varphi = 0$), we find the vector potential

$$\mathbf{A} = -\int \mathbf{E} dt = \exp(ikz - i\omega t)(-i\mathbf{x} + \mathbf{y}) E / \omega , \tag{1.9}$$

and then find the z-component of the volume spin density:

$$\underset{c}{\Upsilon}^{xyt} = \varepsilon_0 \Re\{\bar{E}^{[x} A^{y]}\} = \varepsilon_0 E^2 / \omega \tag{1.10}$$

(here the bar means complex conjugation). Taking into account that the energy volume density in a circularly polarized wave is $\varepsilon_0 E^2$, we see that the ratio of the energy density to the spin density in the wave, ω, is the same as the ratio of the energy of photon $\hbar\omega$ to its spin $\hbar$.

The use of the spin tensor makes it possible, in particular, to check the conservation law of angular momentum, to detect the spin radiation of a rotating dipole and the transfer of spin to a mirror, and to explain the result of the classical Beth's experiment.

2. Spin conservation when reflected from a moving mirror

2.1. Formulation of the situation

We consider the reflection of a plane wave of circular polarization at normal incidence on a moving mirror in order to demonstrate the law of conservation of spin, along with the laws of conservation of momentum, energy, and the number of photons. For definiteness, a receding mirror is considered. It is shown that the number of returning photons is less than the number of incident photons by the number of photons that fill the space vacated by the receding mirror. In this case, the energy of the returning photons $\hbar\omega$ turns out to be less due to a decrease in frequency due to the Doppler effect, while the photon spin $\hbar$ remains unchanged. Therefore, the energy in the returning wave decreases more significantly than the spin decreases. This corresponds to the fact that a moving mirror, when the wave is reflected, receives energy, but does not receive spin. The results were presented in [8].

The Maxwell tensor in Minkowski space [9 (12.113)],

$$T^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} / 4, \tag{2.1}$$

is used for the calculation of momentum and energy

We consider the incident plane wave of circular polarization (1.8)

$$\mathbf{E}_1 = \exp(ik_1 z - i\omega_1 t)(\mathbf{x} + i\mathbf{y}) E_1 \text{ [V/m]}, \quad \mathbf{H}_1 = -i\varepsilon_0 c \mathbf{E}_1 \text{ [A/m]}, \quad ck_1 = \omega_1 \tag{2.2}$$

and, respectively, the reflected wave

$$\mathbf{E}_2 = \exp(-ik_2 z - i\omega_2 t)(\mathbf{x} + i\mathbf{y}) E_2, \quad \mathbf{H}_2 = i\varepsilon_0 c \mathbf{E}_2. \quad ck_2 = \omega_2 \tag{2.3}$$

As is well known [10], the frequency ratio of the reflected and incident waves coincides with the ratio of the amplitudes of these waves and is given by the formula

$$\frac{\omega_2}{\omega_1} = \frac{E_2}{E_1} = \frac{1-\beta}{1+\beta}, \tag{2.4}$$

where $\beta = v/c$, and v is the speed of the mirror.

2.2. Momentum flux density, i.e. pressure P

The wave, which impinges on the moving mirror, has the frequency related to the mirror, according to the Doppler effect [11, § 48],

$$\omega_0 = \omega_1 \sqrt{\frac{1-\beta}{1+\beta}} \tag{2.5}$$

and, respectively, has the amplitude

$$E_0 = E_1 \sqrt{\frac{1-\beta}{1+\beta}}. \tag{2.6}$$

We consider a superconducting mirror, thus the magnetic field doubles on the mirror, and the electric field is zero:

$$\mathbf{H}_0 = 2\varepsilon_0 c E_0 (\mathbf{x} + i\mathbf{y}) \exp(-i\omega_0 t). \tag{2.7}$$

Therefore the pressure on the mirror is defined by the formula $\mathsf{P}_0 = T_0^{zz} = \mu_0 H^2 / 2$ and turns out to be equal to

$$\mathsf{P}_0 = T_0^{zz} = \mu_0 \Re\{H_x \bar{H}_x + H_y \bar{H}_y\} / 4 = 2\varepsilon_0 E_0^2 = 2\varepsilon_0 E_1^2 \frac{1-\beta}{1+\beta} \text{ [N/m}^2\text{]}. \tag{2.8}$$

In addition to the momentum flux, which gives pressure on the mirror, there is a filling of the space vacated by the moving mirror by momentum. The volume density of the filling, $G^z = T_1^{zt} + T_2^{zt}$, consists of two parts, belonging to the incident and to the reflected waves:

$$T_1^{zt} + T_2^{zt} = g^{zz}(F_{1zx}F_1^{xt} + F_{1zy}F_1^{yt} + F_{2zx}F_2^{xt} + F_{2zy}F_2^{yt}) \quad (2.9)$$

$$G^z = T_1^{zt} + T_2^{zt} = -\Re(-B_{1zx}\bar{D}_1^{xt} - B_{1zy}\bar{D}_1^{yt} - B_{2zx}\bar{D}_2^{xt} - B_{2zy}\bar{D}_2^{yt})/2$$
$$= \frac{\varepsilon_0}{c}(E_1^2 - E_2^2) = \frac{\varepsilon_0 E_1^2}{c}(1 - \frac{E_2^2}{E_1^2}) = \frac{\varepsilon_0 E_1^2 4\beta}{c(1+\beta)^2} \text{ [Ns/m}^3\text{].} \quad (2.10)$$

This filling requires the momentum flux density $G^z v$, which we call $\tilde{\mathsf{P}}$:

$$\tilde{\mathsf{P}} = G^z v = \frac{\varepsilon_0 E_1^2 4\beta^2}{(1+\beta)^2} \text{ [N/m}^2\text{].} \quad (2.11)$$

The total flux density is equal to:

$$\mathsf{P} = \mathsf{P}_0 + \tilde{\mathsf{P}} = 2\varepsilon_0 E_1^2\left[\frac{1-\beta}{1+\beta} + \frac{2\beta^2}{(1+\beta)^2}\right] = 2\varepsilon_0 E_1^2 \frac{1+\beta^2}{(1+\beta)^2} \quad (2.12)$$

This total flux density is provided by the oncoming flux density $\mathsf{P} = T_1^{zz} + T_2^{zz}$. Really, in accordance with the formula (2.1), we have expressions such as

$$T^{zz} = g^{zz}(F_{zt}F^{tz} + F_{zx}F^{xz} + F_{zy}F^{yz} + F_{xt}F^{xt} + F_{yx}F^{yx} + F_{yt}F^{yt})/2$$
$$= -(B_{zx}H^{xz} + B_{zy}H^{yz} - E_x D^x - E_y D^y)/2, \quad (2.13)$$

$$T^{zz} = \mu_0(H_y^2 + H_x^2)/4 + \varepsilon_0(E_y^2 + E_x^2)/4 = \varepsilon_0 E^2 \quad (2.14)$$

for the incident or reflected waves. Thus the total momentum flux density,

$$\mathsf{P} = T_1^{zz} + T_2^{zz} = \varepsilon_0(E_1^2 + E_2^2) = \varepsilon_0 E_1^2(1 + \frac{E_2^2}{E_1^2}) = \varepsilon_0 E_1^2\left[1 + \frac{(1-\beta)^2}{(1+\beta)^2}\right] = 2\varepsilon_0 E_1^2 \frac{1+\beta^2}{(1+\beta)^2}, \quad (2.15)$$

coincides with expression (2.12).

2.3. Energy conservation law

The pressure on the mirror P_0 (2.8) produces a work because of the movement of the mirror. The corresponding mass-energy flux density is equal to:

$$I_0 = \frac{\mathsf{P}_0 v}{c^2} = \frac{2\varepsilon_0 E_1^2}{c}\frac{1-\beta}{1+\beta}\beta \left[\frac{kg}{m^2 s}\right] \quad (2.16)$$

In addition, there is a filling of the space vacated by the moving mirror by mass-energy. The volume density of this filling, $u = \langle T_1^{tt} + T_2^{tt} \rangle$, consists of two parts, belonging to the incident and to the reflected waves. Taking into account formula (2.1), we have expressions such as

$$T^{tt} = g^{tt}(F_{tx}F^{xt} + F_{ty}F^{yt} + F_{tz}F^{zt} + F_{xy}F^{xy} + F_{xz}F^{xz} + F_{yz}F^{yz})/2$$
$$= (E_x D^x + E_y D^y + B_{xz}H^{xz} + B_{yz}H^{yz})/(2c^2), \quad (2.17)$$

$$\langle T^{tt} \rangle = \varepsilon_0(E_x^2 + E_y^2)/(4c^2) + \mu_0(H_y^2 + H_x^2)/(4c^2) = \varepsilon_0 E^2/c^2 \text{ [kg/m}^3\text{].} \quad (2.18)$$

for the incident or reflected waves. Thus the total mass-energy volume density equals

$$u = \langle T_1^{tt} + T_2^{tt} \rangle = \varepsilon_0(E_1^2 + E_2^2)/c^2 = \frac{\varepsilon_0 E_1^2}{c^2}(1 + \frac{E_2^2}{E_1^2}) = \frac{\varepsilon_0 E_1^2}{c^2}\left[1 + \frac{(1-\beta)^2}{(1+\beta)^2}\right] = \frac{2\varepsilon_0 E_1^2}{c^2}\frac{1+\beta^2}{(1+\beta)^2}. \quad (2.19)$$

This filling requires the mass-energy flux density, which we call $\tilde{I} = uv$,

$$\tilde{I} = uv = \frac{2\varepsilon_0 E_1^2}{c}\frac{1+\beta^2}{(1+\beta)^2}\beta. \quad (2.20)$$

The total mass-energy flux density,

$$I_0+\tilde{I}=\frac{2\varepsilon_0 E_1^2}{c}\left[\frac{1-\beta}{1+\beta}+\frac{1+\beta^2}{(1+\beta)^2}\right]\beta=\frac{4\varepsilon_0 E_1^2\beta}{c(1+\beta)^2}\ \left[\frac{kg}{m^2 s}\right] \qquad (2.21)$$

is provided by the Poynting vector $I=<T_1^{tz}+T_2^{tz}>$. Really,

$$T_1^{tz}+T_2^{tz}=g^{zz}(F_{1zx}F_1^{xt}+F_{1zy}F_1^{yt}+F_{2zx}F_2^{xt}+F_{2zx}F_2^{xt})=-(-B_{1zx}D_1^x-B_{1zy}D_1^y-B_{2zx}D_1^x-B_{2zy}D_1^y),$$

$$=\mu_0\varepsilon_0(H_{1y}E_{1x}-H_{1x}E_{1y}+H_{2y}E_{2x}-H_{2x}E_{2y}), \qquad (2.22)$$

$$I=<T_1^{tz}+T_2^{tz}>=\frac{\varepsilon_0}{c}(E_1^2-E_2^2)=\frac{\varepsilon_0 E_1^2}{c}(1-\frac{E_2^2}{E_1^2})=\frac{\varepsilon_0 E_1^2}{c}\left[1-\frac{(1-\beta)^2}{(1+\beta)^2}\right]=\frac{4\varepsilon_0 E_1^2\beta}{c(1+\beta)^2}. \qquad (2.23)$$

The value (2.23) coincides with (2.21).

2.4. Conservation of the number of photons

The volume density of photons, n, in the space, vacated by the moving mirror, is obtained by dividing the portions of the energy density (2.19) by the energy of a single photon, i.e. by $\hbar\omega_1$ or by $\hbar\omega_2$

$$n=\varepsilon_0(\frac{E_1^2}{\hbar\omega_1}+\frac{E_2^2}{\hbar\omega_2})=\frac{\varepsilon_0 E_1^2}{\hbar\omega_1}(1+\frac{\omega_2}{\omega_1})=\frac{\varepsilon_0 E_1^2}{\hbar\omega_1}\left[1+\frac{1-\beta}{1+\beta}\right]=\frac{2\varepsilon_0 E_1^2}{\hbar\omega_1(1+\beta)}\ [1/\mathrm{m}^3]. \qquad (2.24)$$

Due to the motion of the mirror the number of the photons increases. This requires the photon number flux density

$$nv=\frac{2\varepsilon_0 E_1^2 v}{\hbar\omega_1(1+\beta)}\ [1/\mathrm{m}^2\mathrm{s}]. \qquad (2.25)$$

This flux density is provided by the difference of Poynting vectors from formula (3.8)

$$<\frac{T_1^{tz}}{\hbar\omega_1}+\frac{T_2^{tz}}{\hbar\omega_2}>c^2=\varepsilon_0 c(\frac{E_1^2}{\hbar\omega_1}-\frac{E_2^2}{\hbar\omega_2})=\frac{\varepsilon_0 E_1^2 c}{\hbar\omega_1}(1-\frac{\omega_2}{\omega_1})=\frac{\varepsilon_0 E_1^2 c}{\hbar\omega_1}\left[1-\frac{1-\beta}{1+\beta}\right]=\frac{2\varepsilon_0 E_1^2 v}{\hbar\omega_1(1+\beta)}. \qquad (2.26)$$

Photon number flux density (2.26) coincides with the flux density (2.25).

2.5. Conservation of spin

The number of photons can be calculated not only on the basis of wave energy, but also on the basis of wave spin. The volume density of wave spin is given by the component of the canonical spin tensor (1.7)

$$\Upsilon^{xyt}=-2A^{[x}F^{y]t}=-A_x D_y+A_y D_x\ [\mathrm{Js/m}^3], \qquad (2.27)$$

and the spin flux density is given by the component

$$\Upsilon^{xyz}=-2A^{[x}F^{y]z}=A_x H_x+A_y H_y\ [\mathrm{J/m}^2]. \qquad (2.28)$$

Note that the lowering of the spatial index of the vector potential is related to the change of the sign in view of the metric signature $(+---)$.

Since for a monochromatic field $A_k=-\int E_k dt=-iE_k/\omega$, densities (2.27), (2.28) can be expressed through the electromagnetic field:

$$\Upsilon^{xyt}=(iE_x D_y-iE_y D_x)/\omega,\quad \Upsilon^{xyz}=(-iE_x H_x-iE_y H_y)/\omega. \qquad (2.29)$$

In our case of reflection from a moving mirror, according to (2.2), (2.3), volume density of the spin is equal to:

$$\Upsilon^{xyt}=\Re\{(iE_{1x}\bar{D}_{1y}-iE_{1y}\bar{D}_{1x})/\omega_1+(iE_{2x}\bar{D}_{2y}-iE_{2y}\bar{D}_{2x})/\omega_2\}/2$$

$$=\varepsilon_0(\frac{E_1^2}{\omega_1}+\frac{E_2^2}{\omega_2})=\frac{\varepsilon_0 E_1^2}{\omega_1}(1+\frac{\omega_2}{\omega_1})=\frac{\varepsilon_0 E_1^2}{\omega_1}\left[1+\frac{1-\beta}{1+\beta}\right]=\frac{2\varepsilon_0 E_1^2}{\omega_1(1+\beta)}, \qquad (2.30)$$

and the photon volume density is given by dividing by $\hbar$ and coincides with the value (2.24). The spin flux density is equal to:

$$\Upsilon^{xyz}=\Re\{(-iE_{1x}\bar{H}_{1x}-iE_{1y}\bar{H}_{1y})/\omega_1+(-iE_{2x}\bar{H}_{2x}-iE_{2y}\bar{H}_{2y})/\omega_2\}/2$$

$$= \varepsilon_0 c\left(\frac{E_1^2}{\omega_1} - \frac{E_2^2}{\omega_2}\right) = \frac{\varepsilon_0 E_1^2 c}{\omega_1}\left(1 - \frac{\omega_2}{\omega_1}\right) = \frac{\varepsilon_0 E_1^2 c}{\omega_1}\left[1 - \frac{1-\beta}{1+\beta}\right] = \frac{2\varepsilon_0 E_1^2 v}{\omega_1(1+\beta)}, \qquad (2.31)$$

and the photon flux density is given by dividing by $\hbar$ and coincides with the value (2.26). Naturally, the increase in the amount of spin is provided by the spin flux:

$$\Upsilon^{xyt} v = \Upsilon^{xyz}. \qquad (2.32)$$

3. Spin absorption by a moving absorber

3.1. A symmetric absorber

In Chapter 2, the spin tensor (1.6) and the energy-momentum tensor (2.1) are used for calculations fluxes of energy, momentum and spin when a plane circularly polarized electromagnetic wave (2.2) reflects from a moving mirror. But these calculations concern no absorption. In this Chapter, we consider such a wave, which falls on a moving "symmetric absorber". We demonstrate the transfer of momentum, energy, and spin from a plane circularly polarized electromagnetic wave into the absorber. Lorentz transformations are used for these flux densities because our absorber moves. The given calculations confirm that spin is the same natural property of a plane electromagnetic wave, as energy and momentum. The results were presented in [12].

We call "symmetric absorber" a medium, which is both dielectric and magnetic with $\varepsilon = \mu$. Such a medium does not require generating a reflected wave; this simplifies formulas.

So, let a plane monochromatic circularly polarized electromagnetic wave (1.8)

$$\mathbf{E} = \exp(ikz - i\omega t)(\mathbf{x} + i\mathbf{y})E \ \text{[V/m]}, \quad \mathbf{H} = -i\varepsilon_0 c\mathbf{E} \ \text{[A/m]}, \quad ck = \omega \qquad (3.1)$$

impinges normally on a flat x,y-surface of the absorber, which is characterized by complex permittivity and permeability $\varepsilon = \mu$ and moves along the z axis with a speed v.

As is well known, the wave (3.1) carries the volume density of mass-energy u, the flux density of mass-energy (the Poynting vector) I, the volume density of momentum G, and flux density of momentum (pressure) P, as described by the formulas (2.1)

$$T^{tt} = u = \frac{\varepsilon_0 E^2}{c^2}\left[\frac{\text{kg}}{\text{m}^3}\right], \quad T^{tz} = T^{zt} = I = G = \frac{\varepsilon_0 E^2}{c}\left[\frac{\text{kg}}{\text{m}^2 s}\right], \quad T^{zz} = \mathsf{P} = \varepsilon_0 E^2\left[\frac{\text{kg}}{\text{m}s^2} = \frac{\text{N}}{\text{m}^2}\right], \qquad (3.2)$$

Besides, according to (1.6), the wave carries the volume density and the flux density of spin

$$\Upsilon^{xyt} = -2A^{[x}F^{y]t} = \Re\{-\bar{A}_x D_y + \bar{A}_y D_x\}/2 = \varepsilon_0 E^2/\omega = uc^2/\omega \ \text{[Js/m}^3\text{]}, \qquad (3.3)$$

$$\Upsilon^{xyz} = -2A^{[x}F^{y]z} = \Re\{\bar{A}_x H_x + \bar{A}_y H_y\}/2 = \varepsilon_0 c E^2/\omega = Ic^2/\omega \ \text{[J/m}^2\text{]} \qquad (3.4)$$

($A_k = -\int E_k dt = -iE_k/\omega$, $F^{kt} = D_k = \varepsilon_0 E_k$, $F^{yz} = H_x$, $F^{xz} = -H_y$, $H_k = -i\varepsilon_0 c E_k$ are used).

But because of Doppler Effect [11 § 48], our wave has lesser frequency and, according to [10], has lesser amplitude *relative to the moving absorber*

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}, \qquad E' = E\sqrt{\frac{1-\beta}{1+\beta}} \qquad (3.5)$$

where $\beta = v/c$. So, relative to the absorber, the impinging wave is expressed by the formulas

$$\mathbf{E}' = \exp(ik'z - i\omega' t)(\mathbf{x} + i\mathbf{y})E', \quad \mathbf{H}' = -i\varepsilon_0 c\mathbf{E}', \quad ck' = \omega' \qquad (3.6)$$

Accordingly, the flux densities prove to be lesser relative to the moving surface

$$u' = \frac{\varepsilon_0 E'^2}{c^2} = u\frac{1-\beta}{1+\beta}, \quad I' = \frac{\varepsilon_0 E'^2}{c} = I\frac{1-\beta}{1+\beta}, \quad \mathsf{P}' = \varepsilon_0 E'^2 = \mathsf{P}\frac{1-\beta}{1+\beta}. \qquad (3.7)$$

$$\Upsilon'^{xyt} = \frac{\varepsilon_0 E'^2}{\omega'} = \Upsilon^{xyt}\sqrt{\frac{1-\beta}{1+\beta}}, \quad \Upsilon'^{xyz} = \frac{\varepsilon_0 c E'^2}{\omega'} = \Upsilon^{xyz}\sqrt{\frac{1-\beta}{1+\beta}}. \qquad (3.8)$$

3.2. The Lorentz transformations of flux densities of mass-energy and momentum

However, from the viewpoint of an observer at rest, these latter quantities, i.e. mass-energy and momentum flux densities through the surface, have other values. These values must be found by the Lorentz transformations for coordinates of a 4-point and for components of 4-momentum

$$t = \frac{t' + vz'/c^2}{\sqrt{1-\beta^2}}, \quad z = \frac{z' + vt'}{\sqrt{1-\beta^2}}, \quad m = \frac{m' + vp'/c^2}{\sqrt{1-\beta^2}}, \quad p = \frac{p' + vm'}{\sqrt{1-\beta^2}}. \qquad (3.9)$$

We denote these flux densities by I_0, P_0. Taking into account that densities satisfy the equations,

$$I_0 = m/at, \; \mathsf{P}_0 = p/at, \;\; I' = m'/at', \; \mathsf{P}' = p'/at', \qquad (3.10)$$

where a is an area, which is not being transformed, and substituting values (3.9), when $z' = 0$, into expression (3.10), we get Lorentz transformations for the flux densities

$$I_0 = I' + v\mathsf{P}'/c^2, \;\; \mathsf{P}_0 = \mathsf{P}' + I'v. \qquad (3.11)$$

So, from the viewpoint of the observer at rest, the flux density of mass-energy, which enters into the absorber, equals

$$I_0 = I' + \frac{v\mathsf{P}'}{c^2} = \frac{\varepsilon_0 E^2}{c}\frac{1-\beta}{1+\beta} + \frac{v}{c^2}\varepsilon_0 E^2 \frac{1-\beta}{1+\beta} = \frac{\varepsilon_0 E^2}{c}(1-\beta) = I(1-\beta) = I'(1+\beta) \qquad (3.12)$$

Note, the pressure is Lorentz invariant when reflected, $I' = 0$, and $I_0 = v\mathsf{P}_0/c^2$ (2.16)!

3.3. The Lorentz transformations of spin flux densities

Spin transforms differently. In order to transform it to the laboratory at rest, we must take into account that the angular momentum flux density satisfies the identities

$$\Upsilon_0 = J/at, \quad \Upsilon' = J'/at', \qquad (3.13)$$

where $J = J'$ is an angular momentum relative to the axis z, which is not being transformed. Taking into account (3.9), equations (3.13) yield the spin flux density that enters the absorber from the viewpoint of the observer at rest:

$$\Upsilon_0^{xyz} = \Upsilon'^{xyz} t'/t = \frac{\varepsilon_0 c E^2}{\omega}\sqrt{\frac{1-\beta}{1+\beta}}\sqrt{1-\beta^2} = \Upsilon^{xyz}(1-\beta) = \Upsilon'^{xyz}\sqrt{1-\beta^2}. \qquad (3.14)$$

3.4. Filling of the space with mass and spin

Flux density I_0 (3.12) is lesser than flux density I (3.2), which is brought by the incident wave. The difference between the mass fluxes (3.2) and (3.12) is spent on filling of the space that is vacated by the moving absorber. This filling requires a mass flux density, which we denote $\tilde{I}$,

$$\tilde{I} = uv = I\beta. \qquad (3.15)$$

As a result, we obtain the simple equality

$$I = \tilde{I} + I_0. \qquad (3.16)$$

The absorbed spin Υ_0^{xyz} (3.14) is lesser than the incident spin Υ^{xyz} (3.4). The difference is the spin that fills the space vacated by the plane. Spin volume density is given by the component of the spin tensor $\Upsilon^{xyt} = \Upsilon^{xyz}/c$ (3.3). So, the filling of the space requires

$$\tilde{\Upsilon}^{xyz} = \Upsilon^{xyt} v = \Upsilon^{xyz}\beta.$$

As a result, we obtain a simple equality

$$\Upsilon^{xyz} = \tilde{\Upsilon}^{xyz} + \Upsilon^{xyz}{}_0,$$

which is similar to (3.16).

But it is desirable to demonstrate the mechanism of the absorption of mass and spin flux densities, I' (3.7) and Υ'^{xyz} (3.8), in the symmetric absorber. See next section.

3.5. Absorption of energy and spin

According to (3.6), the wave propagated in the absorber is described by the formulas

$$\mathbf{E}' = \exp(ik'\breve{k}z - i\omega' t')(\mathbf{x}+i\mathbf{y})\,E', \quad \mathbf{H}' = -i\varepsilon_0 c\mathbf{E}', \quad ck' = \omega' \quad \breve{k} = \sqrt{\varepsilon\mu} = \varepsilon = \mu = k_1 + ik_2 \qquad (3.17)$$

The mechanism of the absorption in dielectric was explained by Feynman [13]. According to the explanation, the rotating electric field $\mathbf{E}' = \exp(-i\omega' t)(\mathbf{x}+i\mathbf{y})\,E'$ exerts a torque $\tau = \mathbf{d}\times\mathbf{E}'$ on the rotating dipole moments of molecules $\mathbf{d}$ of the polarized dielectric and makes a work. The power volume density of this work is

$$w_e = |\mathbf{P}_e \times \mathbf{E}'|\,\omega' \ \ [\mathrm{J/m^3 s}], \quad \mathbf{P}_e = (\breve{\varepsilon}-1)\varepsilon_0\mathbf{E}', \qquad (3.18)$$

$\mathbf{P}_e$ is the polarization vector, and

$$\mathbf{P}_e \times \mathbf{E}' \ [\mathrm{J/m^3}]$$

is the *torque volume density*[2]. The calculation gives

$$w_e = \frac{\omega'}{2}\Re\{P_{ex}\bar{E}'_y - P_{ey}\bar{E}'_x\} = \frac{\omega'\varepsilon_0}{2}\Re\{(\varepsilon-1)(E'_x\bar{E}'_y - E'_y\bar{E}'_x)\} = \frac{\omega'\varepsilon_0}{2}\exp(-2k'k_2 z)\Re\{(\varepsilon-1)(-i-i)\}E'^2$$

$$= \omega'\varepsilon_0\exp(-2k'k_2 z)\Im(\varepsilon-1)E'^2 = \omega'\varepsilon_0\exp(-2k'k_2 z)k_2 E'^2\,. \qquad (3.19)$$

Naturally, the rotating magnetic field of electromagnetic wave (3.17) makes the same work over rotating magnetic dipoles in the absorber.

$$w_m = |\mathbf{P}_m \times \mathbf{H}'|\,\mu_0\,\omega' \ \ [\mathrm{J/m^3 s}], \quad \mathbf{P}_m = (\mu-1)\mathbf{H}', \qquad (3.20)$$

$$w_m = \omega'\Re\{P_{mx}\bar{H}'_y - P_{my}\bar{H}'_x\}\mu_0/2 = \omega'\mu_0\Re\{(\mu-1)(H'_x\bar{H}'_y - H'_y\bar{H}'_x)\}/2\,. \qquad (3.21)$$

Substituting value (3.17) for the magnetic field into (3.21), we see that the work of the magnetic field is equal to the work of the electric field

$$w_m = \omega'\varepsilon_0\Re\{(\varepsilon-1)(E'_x\bar{E}'_y - E'_y\bar{E}'_x)\}/2 = w_e\,. \qquad (3.22)$$

The energy flux density, $I'c^2$ (not mass flux density I'), which is carried to the surface of the absorber by the wave, can be obtained by the integration of the total power volume density,

$$w = w_e + w_m = 2w_e = 2\omega'\varepsilon_0\exp(-2k'k_2 z)k_2 E'^2, \qquad (3.23)$$

over z

$$I'c^2 = \int_0^\infty 2w_e dz = 2\omega'\varepsilon_0\int_0^\infty \exp(-2k'k_2 z)k_2 E'^2 dz = \frac{\omega'\varepsilon_0}{k'}E'^2 = \varepsilon_0 c E'^2\left[\frac{\mathrm{J}}{\mathrm{m}^2 s}\right]. \qquad (3.24)$$

This coincides with $I'c^2$ (3.7).

But we must recognize that the torque volume density[3]

$$\tau_{\sim} = w/\omega' = \mathbf{P}_e\times\mathbf{E}' + \mathbf{P}_m\times\mathbf{H}'\mu_0 = 2\varepsilon_0\exp(-2k'k_2 z)k_2 E'^2, \qquad (3.25)$$

which brings energy into the absorber, is also a volume density of the *angular momentum flux*, i.e. a volume density of the spin flux, which enters into the absorber. The torque volume density $\tau_{\sim}$ produces specific mechanical stresses in the dielectric [14]. And the torque volume density requires spin flux density, which is brought onto the surface of the absorber by the wave. We get this spin flux density by integrating the torque volume density $\tau_{\sim}$ over z.

$$\Upsilon'^{xyz} = \int_0^\infty |\mathbf{P}_e\times\mathbf{E}' + \mathbf{P}_m\times\mathbf{H}'\mu_0|\,dz = \frac{1}{\omega'}\int_0^\infty (w_e + w_m)dz = \frac{I'c^2}{\omega'} = \frac{\varepsilon_0 c}{\omega'}E'^2 \ \left[\frac{\mathrm{J}}{\mathrm{m}^2}\right]. \qquad (3.26)$$

This coincides with (3.8).

The results of this Section concerning the absorption of energy and spin in dielectric were first published in [15].

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[2] Do you remember? Poynting's *G* is a torque *surface* density!

[3] We mark pseudo densities by index *tilda*. The torque volume density $\tau_{\sim}$ is a pseudo *density*, as opposed to the torque τ.

3.6. The use of the energy-momentum and spin tensors

In this Section, the same results, (3.23) and (3.25), are obtained directly using the energy-momentum and spin tensors (see also [16]).

So let the wave propagated in the absorber is described by the formulas (3.17) (without stroke)

$$F_{\alpha\beta} = \{E_x = F_{tx} = 1,\, E_y = F_{ty} = i,\, B^x = F_{zy} = -i\varepsilon / c,\, B^y = F_{xz} = \varepsilon / c\}\exp(ik\breve{k}z - i\omega t)E\,,$$

$$ck = \omega, \quad \breve{k} = \sqrt{\varepsilon\mu} = \varepsilon = \mu = k_1 + ik_2\,, \tag{3.27}$$

$$F^{\mu\nu} = \{D^x = F^{xt} = \varepsilon\varepsilon_0,\, D^y = F^{yt} = i\varepsilon\varepsilon_0,\, H_x = F^{zy} = -ic\varepsilon_0,\, H_y = F^{xz} = c\varepsilon_0\}\exp(ik\breve{k}z - i\omega t)E\,. \tag{3.28}$$

Using the Maxwell tensor (2.1) yields the Poynting vector in the absorber

$$c^2 T^{tz} = \Re\{\bar{F}_{tx}F^{xz} + \bar{F}_{ty}F^{yz}\}/2 = \mathbf{E}\times\mathbf{H} = c\varepsilon_0 \exp(-2kk_2 z)E^2 \tag{3.29}$$

Power volume density of the released energy in the absorber is

$$w = -\partial_z(\mathbf{E}\times\mathbf{H}) = 2k_2\omega\varepsilon_0 \exp(-2kk_2 z)E^2\,. \tag{3.30}$$

This is (3.23).

Using the spin tensor (1.6) and $A_i = -\int E_i dt = -iE_i / \omega$ yields the spin flux density in the absorber

$$\underset{c}{\Upsilon}^{xyz} = \Re\{-\bar{A}^x F^{yz} + \bar{A}^y F^{xz}\}/2 = \Re\{\bar{A}_x H_x + \bar{A}_y H_y\}/2 = c\varepsilon_0 \exp(-2kk_2 z)E^2 / \omega\,. \tag{3.31}$$

The torque volume density from the absorbed spin in the absorber is

$$\tau_{\sim} = -\partial_z \underset{c}{\Upsilon}^{xyz} = 2k_2\varepsilon_0 \exp(-2kk_2 z)E^2 \ \ [\text{J/m}^3]. \tag{3.32}$$

This is (3.25).

4. Absorption of energy and spin by a conducting medium

In Chapter 3, the energy-momentum and spin tensors were used to calculate the energy absorption and spin in a non-conductive symmetric absorber. In this chapter, the same tensors are used to calculate absorption in an electrically conductive medium [17,18].

Let

$$\mathbf{E} = \exp[i(\breve{k}z - t)](\mathbf{x} + i\mathbf{y}), \quad \mathbf{B} = \exp[i(\breve{k}z - t)]\breve{k}(-i\mathbf{x} + \mathbf{y}), \quad \breve{k} = k_1 + ik_2 \tag{4.1}$$

is a damping plane circularly polarized electromagnetic wave, which is propagated in a conducting medium for $z > 0$. We set $\varepsilon_0 = \mu_0 = c = \omega = 1$, and we indicate complex numbers by the *breve* mark: $\breve{k}$. The equations (4.1) mean that

$$F_{\alpha\beta} = \{E_x = F_{tx} = 1,\, E_y = F_{ty} = i,\, B^x = F_{zy} = -i\breve{k},\, B^y = F_{xz} = \breve{k}\}\exp(i\breve{k}z - it)\,, \tag{4.2}$$

$$F^{\mu\nu} = \{F^{tx} = -1,\, F^{ty} = -i,\, F^{zy} = -i\breve{k},\, F^{xz} = \breve{k}\}\exp(i\breve{k}z - it)\,. \tag{4.3}$$

Уравнения Максвелла $\xi_{\gamma\alpha\beta} = 3\partial_{[\gamma}F_{\alpha\beta]}$ для магнитных токов $\xi_{\gamma\alpha\beta}$ показывают, что магнитные токи отсутствуют, например

$$\xi_{yzt} = \partial_z F_{ty} + \partial_t F_{yz} = (i\breve{k}i - ii\breve{k})\exp(i\breve{k}z - it) = 0 \tag{4.4}$$

Уравнения Максвелла $j^\mu = \partial_\lambda F^{\lambda\mu}$ для электрических токов дают

$$j^x = \partial_t F^{tx} + \partial_z F^{zx} = (i - i\breve{k}^2)\exp(i\breve{k}z - it), \quad j^y = \partial_t F^{ty} + \partial_z F^{zy} = (-1 + \breve{k}^2)\exp(i\breve{k}z - it) \tag{4.5}$$

Закон Ома, $j^x = \gamma E^x$, $j^y = \gamma E^y$, определяет $\breve{k}$:

$$\breve{k}^2 = 1 + i\gamma, \quad \gamma = 2k_1 k_2 \tag{4.6}$$

где γ есть вещественная проводимость. Но это не существенно для нас.

Волна (4.1) создается падающей и отраженной волнами, распространяющимися при $z < 0$:

$$\mathbf{E}_1 = \exp[i(z - t)](1 + \breve{k})(\mathbf{x} + i\mathbf{y})/2, \quad \mathbf{B}_1 = \exp[i(z - t)](1 + \breve{k})(-i\mathbf{x} + \mathbf{y})/2\,, \tag{4.7}$$

$$\mathbf{E}_2 = \exp[i(-z - t)](1 - \breve{k})(\mathbf{x} + i\mathbf{y})/2, \quad \mathbf{B}_2 = \exp[i(-z - t)](1 - \breve{k})(i\mathbf{x} - \mathbf{y})/2 \tag{4.8}$$

are the incident and reflected waves for $z < 0$, respectively.

Vector potential waves can be written by the formula $\mathbf{A} = -\int \mathbf{E}dt = -i\mathbf{E}$

$$\mathbf{A} = \exp(i\breve{k}z - it)(-i\mathbf{x} + \mathbf{y}), \tag{4.9}$$

$$\mathbf{A}_1 = \exp(iz - it)(1 + \breve{k})(-i\mathbf{x} + \mathbf{y})/2, \tag{4.10}$$

$$\mathbf{A}_2 = \exp(-iz - it)(1 - \breve{k})(-i\mathbf{x} + \mathbf{y})/2. \tag{4.11}$$

The use of the Maxwell energy-momentum tensor gives the Poynting vectors of the waves:

$$T^{tz} = \Re\{-\overline{F}_{ti}F^{zi}\}/2 = \Re\{\overline{E}^xB^y - \overline{E}^yB^x\}/2 = \Re\{\overline{e}_k\breve{k}e_k - (-i\overline{e}_k)(-i\breve{k}e_k)\}/2 = k_1\exp(-2k_2z), \tag{4.12}$$

$$T_1^{tz} = \Re\{\overline{E}_1^xB_1^y - \overline{E}_1^yB_1^x\}/2 = \Re\{(1+\overline{k})\overline{e}(1+\breve{k})e - (1+\overline{k})(-i\overline{e})(1+\breve{k})(-ie)\}/8 = (1+2k_1+k^2)/4, \tag{4.13}$$

$$T_2^{tz} = \Re\{\overline{E}_2^xB_2^y - \overline{E}_2^yB_2^x\}/2 = \Re\{(1-\overline{k})\overline{e}(1-\breve{k})e - (1-\overline{k})(-i\overline{e})(1-\breve{k})(-ie)\}/8 = -(1-2k_1+k^2)/4, \tag{4.14}$$

$$T_1^{tz} + T_2^{tz} = T^{tz}\Big|_{z=0}. \tag{4.15}$$

Here we denoted to shorten the record: $e_k \equiv \exp(i\breve{k}z - it)$, $e \equiv \exp(iz - it)$, or $e \equiv \exp(-iz - it)$ and $k^2 = \overline{k}\breve{k}$

The use of the canonical spin tensor (1.6) gives the spin fluxes in the waves:

$$\underset{c}{\Upsilon}^{xyz} = \Re\{-\overline{A}^xF^{yz} + \overline{A}^yF^{xz}\}/2 = \Re\{\overline{A}^xB^x + \overline{A}^yB^y\}/2 = \Re\{i\overline{e}_k\breve{k}(-ie_k) + \overline{e}_k\breve{k}e_k\}/2 = k_1\exp(-2k_2z), \tag{4.16}$$

$$\underset{c}{\Upsilon}_1^{xyz} = \Re\{\overline{A}_1^xB_1^x + \overline{A}_1^yB_1^y\}/2 = \Re\{(1+\overline{k})i\overline{e}(1+\breve{k})(-ie) + (1+\overline{k})\overline{e}(1+\breve{k})e\}/8 = (1+2k_1+k^2)/4 \tag{4.17}$$

$$\underset{c}{\Upsilon}_2^{xyz} = \Re\{\overline{A}_2^xB_2^x + \overline{A}_2^yB_2^y\}/2 = \Re\{(1-\overline{k})i\overline{e}(1-\breve{k})(-ie) + (1-\overline{k})\overline{e}(1-\breve{k})e\}/8 = -(1-2k_1+k^2)/4 \tag{4.18}$$

$$\underset{c}{\Upsilon}_1^{xyz} + \underset{c}{\Upsilon}_2^{xyz} = \underset{c}{\Upsilon}^{xyz}\Big|_{z=0}. \tag{4.19}$$

The difference between the energy and spin fluxes in the incident and reflected waves is absorbed in the medium. The equality between the energy fluxes (4.12) – (4.14) and spin fluxes (4.16) – (4.19) is natural because energy of photon $\hbar\omega$ equals spin of photon $\hbar$ if $\omega = 1$.

It is natural that the absorbed power density satisfies $-\partial_z T^{tz} = (\mathbf{j}\cdot\mathbf{E})$. Really,

$$-\partial_z T^{tz} = 2k_1k_2\exp(-2k_2z) \quad \text{and}$$

$$(\mathbf{j}\cdot\mathbf{E}) = \Re\{j^x\overline{E}^x + j^y\overline{E}^y\}/2 = \Re\{i(1-\breve{k}^2) - (\breve{k}^2-1)(-i)\}/2 = 2k_1k_2\exp(-2k_2z) \tag{4.20}$$

It is important that absorption of the spin flux densities $-\partial_z \Upsilon^{xyz}$ equals the torque volume density $\tau_\wedge = (\mathbf{j}\times\mathbf{A})^{xy}$ [18]. Really

$$-\partial_z \underset{c}{\Upsilon}^{xyz} = 2k_1k_2\exp(-2k_2z), \text{ and}$$

$$\mathbf{j}\times\mathbf{A} = \Re\{j^x\overline{A}^y - j^y\overline{A}^x\}/2 = 2k_1k_2\exp(-2k_2z) \tag{4.21}$$

The torque volume density $\tau_\wedge = \mathbf{j}\times\mathbf{A}$ is analogical to the Lorentz force density $f_\wedge = \mathbf{j}\times\mathbf{B}$

$$-\partial_i T^{ki} = j_iF^{ik} = \mathbf{j}\times\mathbf{B}. \tag{4.22}$$

5 Radiation of spin by a rotating dipole

5.1. Emission of energy and moment of momentum according to the classical electrodynamics

As is known, a rotating electric dipole or two dipole oscillators perpendicular to each other,

$$p^x = p\exp(-i\omega t), \quad p^y = ip\exp(-i\omega t)\,, \tag{5.1}$$

radiate electromagnetic waves. (In this Chapter, p denotes electric dipole moment) The power and the angular distribution of this power (Fig. 1) are, respectively, [11 § 67, Problem 1], [19]

$$P = \omega^4 p^2 / 6\pi\varepsilon_0 c^3\,, \tag{5.2}$$

$$dP/d\Omega = \omega^4 p^2(\cos^2\theta + 1)/32\pi^2\varepsilon_0 c^3 \tag{5.3}$$

where $d\Omega = \sin\theta d\theta d\varphi$ (We use the system of units where $\mathrm{div}\mathbf{E} = \rho/\varepsilon_0$). The polarization of this radiation is circular along the axis of rotation and is linear in the plane of rotation (Fig. 3).

The electromagnetic field of a rotating electric dipole contains moment of linear momentum L_z, which flux is [11 § 72, § 75]

$$dL_z / dt = \omega^3 p^2 / 6\pi\varepsilon_0 c^3\,, \tag{5.4}$$

Tthis flux is located in the neighborhood of the plane of rotation where the polarization is near linear. The angular distribution of the moment of momentum flux, according to [20-24] (see Fig. 2) is

$$dL_z / dtd\Omega = \omega^3 p^2 \sin^2\theta / 16\pi^2\varepsilon_0 c^3\,. \tag{5.5}$$

Heitler noted [25], “The angular momentum (5.4) is not contained in the wave zone, where the field strengths are perpendicular to $\mathbf{r}$ and behave like $1/r$. In this zone L_z vanishes: L_z is proportional to E^r, and $E^r \sim 1/r^2$”. So, we must recognize that this flux is not a radiation; this is an orbital angular momentum flux, although Heitler claims that it is spin radiation: “the contributions to L_z arise from a subtle interference effect”.

The presence of the orbital angular momentum in the field of a rotating dipole is naturally. This field is a multipole field of order $(l = 1, m = 1)$. And equalities (5.3) and (5.4) are in the agreement with formula

$$dL_z / dt = mP/\omega\,. \tag{5.6}$$

This formula is from [25], [9 (9.144)].Equation (5.6) is an additional proof that the moment of linear momentum L_z is not a spin. According to (5.6), each photon has an angular momentum $L_z = m\hbar$, not $\hbar$.

5.2. Spin radiation by a rotating dipole in the frame of the electrodynamics

At the same time, the modern electrodynamics does not notice an angular momentum flux in the direction of the axis of rotation, where the radiation is intense and the polarization is circular, although it was suggested as early as 1899 by Sadowsky [1] and as 1909 by Poynting [2] that circularly polarized light carries angular momentum volume *density,* and the angular momentum density is proportional to the energy volume density.

The classical experiments [24 – 27] confirm that the angular momentum density is proportional to energy density. In these experiments, the angular momentum of the light was transferred to a half-wave plate, which rotated. So, work was performed *in any point of the plate*. This (positive or negative) amount of work reappeared as an alteration in the frequency of the light, which resulted in moving fringes *in any point* of the interference pattern in a suitable interference experiment. So, it is natural to recognize a spin radiation (5.1) along the axis of rotation. A calculation of this spin radiation is presented here.

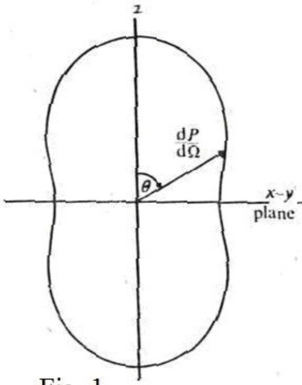

Fig. 1.

Angular distribution of the energy flux.

$dP/d\Omega \propto (\cos^2\theta+1)$

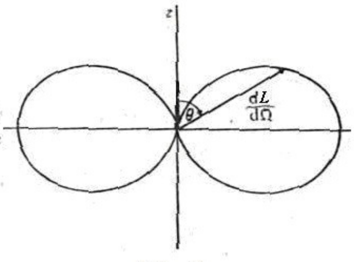

Fig. 2.

Angular distribution of z-component of the moment of momentum flux

$dL_z/dtd\Omega \propto \sin^2\theta$

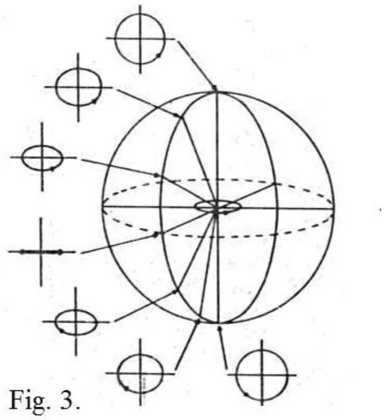
Fig. 3.

Polarization of the electric field seen by looking from different directions at a circular oscillator

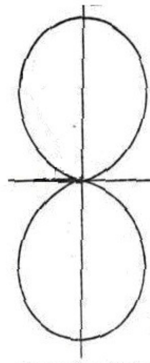
Fig. 4.

Angular distribution of z-component of the spin flux

$dS_Z/dtd\Omega \propto \cos^2\theta$

The spin volume density $\varepsilon_0\mathbf{E}\times\mathbf{A}$ (1.7) is integrated over a thin spherical layer (of thickness dr), which surrounds the source of the radiation, and then the integral is divided by dt on the assumption $dr/dt = c$. So, the formula for the spin flux is

$$dS^{xy}/dt = \int \mathbf{Y}^{xyt} r^2 d\Omega dr/dt\,, \tag{5.7}$$

The expression for *radiated* electric field [19, 30] is used

$$\mathbf{E} = \frac{\omega^2(\mathbf{p}r^2 - (\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0 c^2 r^3}\exp(ikr - i\omega t) \tag{5.8}$$

$$E_x = \frac{\omega^2 p(r^2 - x^2 - ixy)}{4\pi\varepsilon_0 c^2 r^3}\exp(ikz - i\omega t),\quad E_y = \frac{\omega^2 p(ir^2 - xy - iy^2)}{4\pi\varepsilon_0 c^2 r^3}\exp(ikr - i\omega t) \tag{5.9}$$

$$\mathbf{A} = -\int \mathbf{E}dt = -i\mathbf{E}/\omega\,. \tag{5.10}$$

Inserting (5.1), (1.7), (5.9), (5.10) into (5.7) yields the time averaged spin flux:

$$dS^{xy}/dt = \Re\int \varepsilon_0(E_x\overline{A}_y - E_y\overline{A}_x)cr^2 d\Omega/2 = \Re\int i\varepsilon_0(E_x\overline{E}_y - E_y\overline{E}_x)cr^2 d\Omega/2\omega\,. \tag{5.11}$$

Here

$$\begin{aligned}&(E_x\overline{E}_y - E_y\overline{E}_x)\\ &= \frac{\omega^4 p^2}{16\pi^2\varepsilon_0^2 c^4 r^6}[(r^2 - x^2 - iyx)(-ir^2 - xy + iy^2) - (ir^2 - xy - iy^2)(r^2 - x^2 + iyx)]\\ &= \frac{-i\omega^4 p^2 z^2}{8\pi^2\varepsilon_0^2 c^4 r^4} = \frac{-i\omega^4 p^2}{8\pi^2\varepsilon_0^2 c^4 r^2}\cos^2\theta.\end{aligned} \tag{5.12}$$

Inserting (5.12) into (5.11) yields

$$dS^{xy}/dt = \int \frac{\omega^3 p^2}{16\pi^2\varepsilon_0 c^3}\cos^2\theta d\Omega\,. \tag{5.13}$$

So, the angular distribution of the spin flux (see Fig. 4) is

$$dS_z/dtd\Omega = \omega^3 p^2\cos^2\theta/16\pi^2\varepsilon_0 c^3\,. \tag{5.14}$$

Integration of equality (5.13) gives the spin flux

$$dS_z/dt = \omega^3 p^2/12\pi\varepsilon_0 c^3\,. \tag{5.15}$$

These results were presented in [31-34].

Thus the total angular momentum flux, orbital + spin, (5.4) + (5.15), is

$$dJ_z/dt = dL_z/dt + dS_z/dt = \omega^3 p^2/6\pi\varepsilon_0 c^3 + \omega^3 p^2/12\pi\varepsilon_0 c^3 = \omega^3 p^2/4\pi\varepsilon_0 c^3\,. \tag{5.16}$$

Thus, the total angular momentum flux exceeds 1.5 times the value now recognized (5.4).

Note that for $\theta = 0$, i.e. where there is no orbital angular momentum (4.5), according to (5.3) and (5.14), the photon relation is valid:

$$(\hbar\omega \text{ energy}) = \omega(\hbar \text{ spin}), \quad dPdt = \omega dS_z = \omega(\omega^3 p^2 / 16\pi^2 \varepsilon_0 c^3) d\Omega dt\,. \tag{5.17}$$

5.3. Spin radiation by a rotating dipole in the frame of the quantum mechanics

It is remarkable that the result (5.14), $dS_z / dtd\Omega \propto \cos^2\theta$, for the angular distribution of z-component of the spin flux was obtained by Feynman beyond the standard electrodynamics. Really, the amplitudes that a RHC photon and a LHC photon are emitted in the direction θ into a certain small solid angle $d\Omega$ are [35 (18.1), (18.2)]

$$a(1+\cos\theta)/2 \quad \text{and} \quad -a(1-\cos\theta)/2\,. \tag{5.18}$$

So, in the direction θ, the spin flux density is proportional to

$$[a(1+\cos\theta)/2]^2 - [a(1-\cos\theta)/2]^2 = a^2\cos\theta\,. \tag{5.19}$$

The projection of the spin flux density on z-axis is

$$dS_z / dtd\Omega \propto a^2\cos^2\theta\,. \tag{5.20}$$

Note that the Feynman's method gives the power distribution (5.3) as well:

$$dP / d\Omega \propto [a(1+\cos\theta)/2]^2 + [a(1-\cos\theta)/2]^2 = a^2(1+\cos^2\theta)/2\,. \tag{5.21}$$

6. Radiation reaction to an emitting rotating dipole

6.1. Is there a violation of the momentum conservation law?

Chapter 5 presents an amazing result. A rotating dipole emits power (5.2)

$$P = \omega^4 p^2 / 6\pi\varepsilon_0 c^3, \tag{6.1}$$

The dipole receives this power from the source of rotation, which for this must act on the dipole with a torque $\tau = P/\omega$. In this case, the source of rotation will transmit to the dipole, in addition to the energy flux P, also the moment of momentum flux $\tau = P/\omega$. However, according to (5.16), the rotating dipole emits a flux of angular momentum, which is one and a half times greater than $\tau = P/\omega$, due to the radiation of the spin:

$$dJ_z / dt = dL_z / dt + dS_z / dt = \omega^3 p^2 / 6\pi\varepsilon_0 c^3 + \omega^3 p^2 / 12\pi\varepsilon_0 c^3 = \omega^3 p^2 / 4\pi\varepsilon_0 c^3\,. \tag{6.2}$$

Thus, a rotating dipole emits more angular momentum, $\omega^3 p^2 / 4\pi\varepsilon_0 c^3$, than it receives from the source of rotation, $\tau = P/\omega = \omega^3 p^2 / 6\pi\varepsilon_0 c^3$. A problem arises with the implementation of the law of angular momentum conservation in relation to a rotating dipole. In this regard, it is interesting to study the mechanism of energy and angular momentum transfer from a rotating dipole to an electromagnetic field. What is the field response to a rotating dipole? How does the field affect the dipole, from which the field receives energy P and angular momentum per unit time $1{,}5P/\omega$?

As a rotating electric dipole, we consider a pair of oscillating dipoles perpendicular to each other and having a quarter-period oscillation shift in time (5.1),

$$p^x = p\exp(-i\omega t), \quad p^y = ip\exp(-i\omega t)\,. \tag{6.3}$$

In the present chapter, the energy and angular momentum fluxes (6.1), (6.2) are calculated using the same type by the use of the retarded electromagnetic Jefimenko's field [9, p. 247]. It turned out that the forces acting on the dipoles and responsible for the loss of energy (6.1) differ from the forces responsible for the loss of angular momentum (6.2) [36]. So it is found that the energy taken from the dipole is equal to the recognized value of the radiated energy. At the same time, it is confirmed that the angular momentum flux exceeds the generally accepted value by the spin radiation not seen before.

6.2. Energy loss by a rotating dipole

The value (6.1) is calculated as the result of the influence of the electromagnetic field of the dipole on the dipole itself, according to the formula for the density of the resulting power

$$P_\wedge = -(\mathbf{j}\cdot\mathbf{E})\,; \tag{6.4}$$

here $\mathbf{j}$ and $\mathbf{E}$ are current density flowing along the dipole and the electric field in the dipole, respectively, and the index $\wedge$ for $P_\wedge$ means, in this case, "volume density", $dP = P_\wedge d^3x$. First, the effect of the *x*-dipole on itself is calculated.

The electric field near the dipole taking into account the retardation [9 (6.55)] is

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \left\{ \frac{\hat{\mathbf{r}}}{r^2}[\rho(\mathbf{x}',t')]_{\text{ret}} + \frac{\hat{\mathbf{r}}}{cr}[\partial_{t'}\rho(\mathbf{x}',t')]_{\text{ret}} - \frac{1}{c^2 r}[\partial_{t'}\mathbf{j}(\mathbf{x}',t')]_{\text{ret}} \right\}, \tag{6.5}$$

and an "elementary vibrator" is considered as a dipole; the current of the dipole is the same at all points, and the charges are only at the ends (see Figure 6.1)

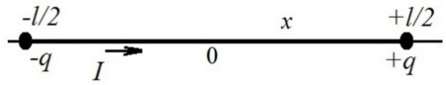

Figure 6.1. *x*-dipole

The dipole current I_x is obtained by differentiating the charge

$$p^x = ql\exp(-i\omega t)\,, \tag{6.6}$$

$$I_x = \partial_t p^x / l = -i\omega q \exp(-i\omega t)\,. \tag{6.7}$$

The first term of expression (6.5),

$$E_1^x = \frac{1}{4\pi\varepsilon_0}\int d^3x' \left\{ \frac{1}{r^2}[\rho(\mathbf{x}',t')]_{\text{ret}} \right\},$$

is simply the retarded Coulomb field at the point x. Therefore, replacing

$$d^3x'\rho \to d\breve{q}'\,,\;\; t \to t-(l/2 \pm x)/c\,,\;\; r \to l/2 \pm x$$

and taking into account the direction of the electric field, we obtain the electric field strength from both charges at the point x:

$$E_1^x(x) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-\exp[-i\omega t + i\omega(l/2+x)/c]}{(l/2+x)^2} - \frac{\exp[-i\omega t + i\omega(l/2-x)/c]}{(l/2-x)^2} \right]. \tag{6.8}$$

The corresponding contribution of this term to the power generated by the dipole is given by formula (6.4) (we replace $jd^3x \to Idx$, and the bar means complex conjugation)

$$\begin{aligned} P_1 &= \frac{-1}{2}\int_{-l/2}^{l/2} dx \Re\{\bar{I}E_1^x\} \\ &= -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx \Re\left\{ i\exp(i\omega t)\left[\frac{-\exp[-i\omega t + i\omega(l/2+x)/c]}{(l/2+x)^2} - \frac{\exp[-i\omega t + i\omega(l/2-x)/c]}{(l/2-x)^2} \right]\right\} \\ &= -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx \left[\frac{\sin[\omega(l/2+x)/c]}{(l/2+x)^2} + \frac{\sin[\omega(l/2-x)/c]}{(l/2-x)^2} \right]. \end{aligned} \tag{6.9}$$

Taking into account the small size of the dipole, we consider only two terms of the expansion of the sine in a series

$$P_1 = -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx \left[\frac{\omega}{c(l/2+x)} - \frac{\omega^3(l/2+x)}{6c^3} + \frac{\omega}{c(l/2-x)} - \frac{\omega^3(l/2-x)}{6c^3} \right]$$

$$P_1 = -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx \left[\frac{\omega}{c(l/2+x)} - \frac{\omega^3(l/2+x)}{6c^3} + \frac{\omega}{c(l/2-x)} - \frac{\omega^3(l/2-x)}{6c^3} \right]. \tag{6.10}$$

Similarly to formula (6.8), we find the electric field provided by the second term of formula (6.5)

$$E_2^x = \frac{i\omega q}{4\pi\varepsilon_0 c}\left[\frac{\exp[-i\omega t + i\omega(l/2+x)/c]}{(l/2+x)} + \frac{\exp[-i\omega t + i\omega(l/2-x)/c]}{(l/2-x)} \right]. \tag{6.11}$$

In contrast to formula (6.8), this formula contains *i*.

Formula (6.4) gives the contribution of this term, E_2^x, to the power generated by the dipole

$$P_2 = \frac{\omega^2 q^2}{8\pi\varepsilon_0 c}\int_{-l/2}^{l/2} dx\Re\left\{\exp(i\omega t)\left[\frac{\exp[-i\omega t + i\omega(l/2+x)/c]}{(l/2+x)} + \frac{\exp[-i\omega t + i\omega(l/2-x)/c]}{(l/2-x)}\right]\right\}$$

$$= \frac{\omega^2 q^2}{8\pi\varepsilon_0 c}\int_{-l/2}^{l/2} dx\left[\frac{\cos[\omega(l/2+x)/c]}{(l/2+x)} + \frac{\cos[\omega(l/2-x)/c]}{(l/2-x)}\right]. \quad (6.12)$$

Restricting ourselves to two terms of the expansion of cosine in a series, we have

$$P_2 = \frac{\omega^2 q^2}{8\pi\varepsilon_0 c}\int_{-l/2}^{l/2} dx\left[\frac{1}{(l/2+x)} - \frac{\omega^2(l/2+x)}{2c^2} + \frac{1}{(l/2-x)} - \frac{\omega^2(l/2-x)}{2c^2}\right]. \quad (6.13)$$

Surprisingly, the integrals diverging at the ends of the dipole are shortened upon the addition $P_1 + P_2$, and the remaining terms are constants. As $l/6 - l/2 = -l/3$, this part of the power is

$$P_1 + P_2 = -\frac{\omega^4 p^2}{24\pi\varepsilon_0 c^3}. \quad (6.14)$$

The third term of formula (6.5),

$$E_3^x = \frac{1}{4\pi\varepsilon_0}\int dx'\left\{-\frac{1}{c^2 r}[\partial_{t'} I(x',t']_{\text{ret}}\right\},$$

uses the derivative of the current

$$\partial_t I_x = -\omega^2 q \exp(-i\omega t). \quad (6.15)$$

To calculate the strength at the point x, we divided the region of integration into two parts by the point x

$$E_3^x = -\frac{\omega^2 q}{4\pi\varepsilon_0 c^2}\left\{\int_{-l/2}^{x} dx'\frac{-\exp[-i\omega t + i\omega(x-x')/c}{x-x'} + \int_{x}^{l/2} dx'\frac{-\exp[-i\omega t + i\omega(x'-x)/c}{x'-x}\right\}. \quad (6.16)$$

Using formula (6.4), $dP = -(\mathbf{j}\cdot\mathbf{E})d^3x = -IE^x dx$, and current (6.7), we obtain the power corresponding to the third term of formula (6.5)

$$P_3 = -\frac{\omega^3 q^2}{8\pi\varepsilon_0 c^2}\int_{-l/2}^{l/2} dx\Re\left\{i\omega\exp(i\omega t)\left[\int_{-l/2}^{x} dx'\frac{\exp[-i\omega t + i\omega(x-x')/c]}{x-x'} + \int_{x}^{l/2} dx'\frac{\exp[-i\omega t + i\omega(x'-x)/c]}{x'-x}\right]\right\}$$

$$= -\frac{\omega^3 q^2}{8\pi\varepsilon_0 c^2}\int_{-l/2}^{l/2} dx\left[\int_{-l/2}^{x} dx'\frac{-\sin[\omega(x-x')/c]}{x-x'} + \int_{x}^{l/2} dx'\frac{-\sin[\omega(x'-x)/c]}{x'-x}\right]. \quad (6.17)$$

Restricting ourselves to one term in the expansion of the sine in a series, we easily obtain

$$P_3 = \frac{\omega^4 \mathrm{d}^2}{8\pi\varepsilon_0 c^3}. \quad (6.18)$$

Thus, the power radiated by one *x*-dipole is

$$P = P_1 + P_2 + P_3 = \omega^4 p^2/(12\pi\varepsilon_0 c^3). \quad (6.19)$$

Naturally, the y-dipole, acting on itself, makes the same contribution. It will be shown below that the electric field of one oscillating dipole in the territory of another oscillating dipole is perpendicular to the current, and therefore does not produce energy. So a rotating dipole delivers power (6.1) to the radiation:

$$P = \omega^4 p^2 / 6\pi\varepsilon_0 c^3.$$

6.3. The torque experienced by the charges of a rotating dipole

We now calculate the electric field $\mathbf{E}$ created by the *y*-dipole at the location of the charge of the *x*-dipole, that is, at the point $x = l/2$ (see Fig. 6.2)

To shorten the notation, we rewrite formula (6.5) in terms of charges and currents:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{\hat{\mathbf{r}}}{r^2}[q_y]_{\text{ret}} + \frac{\hat{\mathbf{r}}}{cr}[\partial_{t'}q_y]_{\text{ret}} - \int_{-l/2}^{l/2}\frac{1}{c^2 r}[\partial_{t'}I_y]_{\text{ret}}\mathbf{dy}\right); \tag{6.20}$$

here the charge and current belong to the *y*-dipole:

$$q_y = iq\exp(-i\omega t),\ [q_y]_{\text{ret}} = iq\exp(-i\omega t + i\omega r/c),\ [\partial_{t'}q_y]_{\text{ret}} = \omega q\exp(-i\omega t + i\omega r/c) \tag{6.21}$$

$$I_y = \partial_t q_y = \omega q\exp(-i\omega t),\ [I_y]_{\text{ret}} = \omega q\exp(-i\omega t + i\omega r/c).\ [\partial_{t'}I_y]_{\text{ret}} = -i\omega^2 q\exp(-i\omega t + i\omega r/c). \tag{6.22}$$

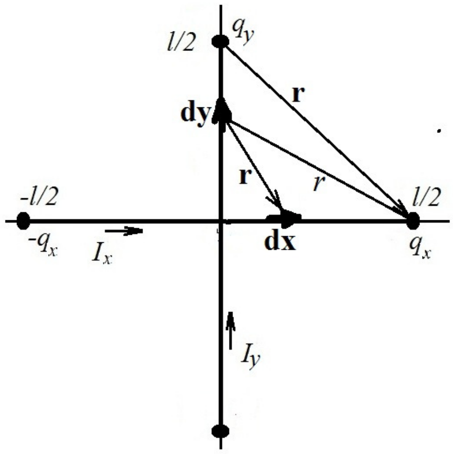

Fig. 6.2. The pair of oscillating dipoles. Current elements and radii used in the formulas are indicated.

The electric field created by the charge consists of two terms:

$$\mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{r}}([q_y]_{\text{ret}}/r^2 + [\partial_{t'}q_y]_{\text{ret}}/cr)/4\pi\varepsilon_0. \tag{6.23}$$

However, the forces created by these terms on the charge $q_x = q\exp(-i\omega t)$ are mutually eliminated when the dipole size tends to zero, although they tend to infinity.

$$\begin{aligned} F_1 + F_2 &= \Re\{(E_1 + E_2)\overline{q}_x\}/2 = \Re\{i\exp(i\omega r/c)/r^2 + \omega\exp(i\omega r/c)/cr\}q^2/8\pi\varepsilon_0 \\ &= \{-\sin(\omega r/c)/r^2 + \omega\cos(\omega r/c)/cr\}q^2/8\pi\varepsilon_0 \to (-\omega/cr + \omega/cr)q^2/8\pi\varepsilon_0 = 0. \end{aligned} \tag{6.24}$$

Similarly, the total interaction forces of other pairs of charges are zero. So the damping of the rotating dipole is provided only by the third term of the formula (6.20):

$$\mathbf{E}_3 = -\int_{-l/2}^{l/2}\frac{1}{c^2 r}[-i\omega^2 q\exp(-i\omega t + i\omega r/c)]\mathbf{dy} \tag{6.25}$$

The force acting on the charge q_x along the *y*-axis is

$$\begin{aligned} F_3 &= \Re\{E_3\overline{q}_x\}/2 = \Re\{\int_{-l/2}^{l/2}\frac{1}{c^2 r}i\omega^2 q^2\exp(i\omega r/c)dy\}/8\pi\varepsilon_0 \\ &= -\int_{-l/2}^{l/2}\frac{1}{c^2 r}\omega^2 q^2\sin(\omega r/c)dy\}/8\pi\varepsilon_0 \to -\int_{-l/2}^{l/2}\omega^3 q^2 dy/8\pi\varepsilon_0 c^3 = -\omega^3 q^2 l/8\pi\varepsilon_0 c^3. \end{aligned} \tag{6.26}$$

The torque acting on both charges of the *x*-dipole is $-\omega^3 q^2 l^2/8\pi\varepsilon_0 c^3$. Therefore, the torque acting on the rotating dipole is directed against the rotation of the dipole and is equal to (6.2)

$$\omega^3 p^2/4\pi\varepsilon_0 c^3.$$

6.4. Magnetic field torque

In addition to the electric field (6.5), Jefimenko's formulas give the magnetic field [9 (6.56)],

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{[\mathbf{j}(\mathbf{x}',t')]_{\text{ret}} \times \frac{\hat{\mathbf{r}}}{r^2} + [\partial_{t'}\mathbf{j}(\mathbf{x}',t')]_{\text{ret}} \times \frac{\hat{\mathbf{r}}}{cr}\right\}, \tag{6.27}$$

This field acts on dipoles (6.3) by the Lorentz force. Consider the magnetic field created by the *y*-dipole on the territory of the x-dipole. By analogy with (6.20), we write

$$\mathbf{B} = \frac{1}{4\pi\varepsilon_0}\int\left([I_y]_{\rm ret}/r^2 + [\partial_{t'} I_y]_{\rm ret}/cr\right)\mathbf{dy}\times\hat{\mathbf{r}}\,, \qquad (6.28)$$

However, using (6.22) and (6.7), we find that the average value of the Lorentz force acting on the x-dipole is zero:

$$\mathbf{F} = \Re\{\int \bar{I}_x \mathbf{dx}\times\mathbf{B}\}/2 = \Re\{\iint i\omega q[\omega q\exp(i\omega r/c)/r^2 - i\omega^2 q\exp(i\omega r/c)/cr]\mathbf{dx}\times(\mathbf{dy}\times\hat{\mathbf{r}})\}/8\pi\varepsilon_0$$

$$= \iint[-\sin(\omega r/c)\omega^2/r^2 + \cos(\omega r/c)\omega^3/cr]q^2\mathbf{dx}\times(\mathbf{dy}\times\hat{\mathbf{r}})/8\pi\varepsilon_0 \to 0 \quad \text{if} \quad r\to 0\,. \qquad (6.29)$$

So the total torque acting on the rotating dipole is (6.2)

6.5. Conclusion. Angular momentum without rotation

A direct calculation showed that a pair of oscillating dipoles (6.3), which creates the field of a rotating dipole, actually provides the energy flux P (6.1) and the angular momentum flux $\tau_{\rm tot} = 1{,}5P/\omega$ (6.2), which are not related to each other. In this case, there is no violation of the conservation laws, because these flows are provided by the sources of oscillations of these dipoles. These flows are generated independently of each other, because in this case ω is not a rotational speed.

In the case of a really rotating dipole, that is, in the case of really rotating electrons, the energy flux $P = \omega\tau_{\rm L}$ is created by the angular momentum flux $\tau_{\rm L} = P/\omega$, coming from the source of rotation, due to the presence of the angular frequency ω. The spin flux in the field of a rotating dipole, $\tau_{\rm S} = \tau_{\rm tot} - \tau_{\rm L} = 0{,}5P/\omega$, is not associated with energy. From this we can conclude that spin is not related to rotation. Indeed, Hehl writes, referring to the spin of an electron:

“The current density in Dirac’s theory can be split into a convective part and a polarization part. The polarization part is determinated by the spin distribution of the electron field. It should lead to *no* energy flux in the rest system of the electron because the genuine spin “motion” take place only within a region of the order of the Compton wavelength of the electron. The convective part of the current density describes the average motion of the electron field and leads to a momentum and to an energy flux”.

The spin flux in the field, $\tau_{\rm S}$, is also not associated with the flux of angular momentum $\tau_{\rm L}$ coming from the source of electron rotation. By virtue of the angular momentum conservation law, it can be concluded from this that rotating electrons accumulate in themselves a spin of the opposite orientation relative to the orientation of the emitted spin. In other words, a rotating electric dipole is magnetized by radiation. This conclusion was made in [22]

7. Modification of the canonical energy-momentum and spin tensors

7.1.Incorrectness of the canonical spin tensor

In the previous chapters, the canonical spin tensor (1.6) $\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}$ has been successfully applied. However, generally speaking, the tensor is not true. Really, consider e.g. a circularly polarized plane wave (4.1) when $\breve{k} = 1$

$$E^x = \cos(z-t),\ E^y = -\sin(z-t),\ B^x = \sin(z-t),\ B^y = \cos(z-t),$$
$$A^x = \sin(z-t),\ A^y = \cos(z-t) \qquad (7.1)$$

A calculation of components of the canonical spin tensor (1.6) yields

$$\underset{c}{\Upsilon}^{xyt} = 1,\quad \underset{c}{\Upsilon}^{xyz} = 1,\quad \underset{c}{\Upsilon}^{zxy} = -A^zF^{xy} + A^xF^{zy} = A^xB_x = \sin^2(z-t)\,,\quad \underset{c}{\Upsilon}^{yzx} = A^yB_y = \cos^2(z-t)\,. \quad (7.2)$$

This result is absurd, because, though $\underset{c}{\Upsilon}^{xyt}$ and $\underset{c}{\Upsilon}^{xyz}$ are adequate (we used them), the result means that there are spin fluxes in the directions, along x, and along y, which are transverse to the direction of the wave propagation.

Moreover, the canonical spin tensor predicts an interference of counter propagating waves, which is not actually observed. Let us consider a standing electromagnetic wave. The wave incident on a mirror and the reflected wave are supplied with indices 1 and 2, respectively, and the following expressions are used for them:

$$\mathbf{E}_1 = (\mathbf{x} + i\mathbf{y})e^{iz-it}, \quad \mathbf{B}_1 = (-i\mathbf{x} + \mathbf{y})e^{iz-it} \tag{7.3}$$

$$\mathbf{E}_2 = (-\mathbf{x} - i\mathbf{y})e^{-iz-it}, \quad \mathbf{B}_2 = (-i\mathbf{x} + \mathbf{y})e^{-iz-it} \tag{7.4}$$

Here $\mathbf{x}, \mathbf{y}$ are the unit coordinate vectors, and for the sake of simplicity $\omega = k = c = \varepsilon_0 = \mu_0 = 1$. Bearing in mind expression (1.6), we write out the components of the field-strength tensor (without an exponential factor)

$$E_{1x} = F_{1tx} = 1, \quad E_{1y} = F_{1ty} = i, \quad B_{1x} = F_{1zy} = -i, \quad B_{1y} = F_{1xz} = 1, \tag{7.5}$$

$$E_{2x} = F_{2tx} = -1, \quad E_{2y} = F_{2ty} = -i, \quad B_{2x} = F_{2zy} = -i, \quad B_{2y} = F_{2xz} = 1, \tag{7.6}$$

Raising the indices gives, by virtue of the signature $(+---)$,

$$F_1^{tx} = -1, \quad F_1^{ty} = -i, \quad F_1^{zy} = -i, \quad F_1^{xz} = 1, \tag{7.7}$$

$$F_2^{tx} = 1, \quad F_2^{ty} = i, \quad F_2^{zy} = -i, \quad F_2^{xz} = 1, \tag{7.8}$$

When calculating the magnetic vector potential, it is natural to use the Weyl gauge, $\varphi = 0$, so $F_{tk} = \partial_t A_k = -iA_k$, $\quad A_k = iF_{tk}$.

$$A_{1x} = i, \quad A_{1y} = -1, \quad A_{2x} = -i, \quad A_{2y} = 1. \tag{7.9}$$

Raising indices reverses the signs

$$A_1^x = -i, \quad A_1^y = 1, \quad A_2^x = i, \quad A_2^y = -1. \tag{7.10}$$

Let us first determine the spin density in the incident wave (the bar means complex conjugation).

$$< \underset{c}{\Upsilon}{}_1^{xyt} >= -\Re\{\bar{A}_1^x F_1^{yt} - \bar{A}_1^y F_1^{xt}\}/2 = -(ii - 1\cdot 1)/2 = 1. \tag{7.11}$$

The spin density in the reflected wave, $\underset{c}{\Upsilon}{}_2^{xyt}$, is naturally the same.

$$< \underset{c}{\Upsilon}{}_2^{xyt} >= -\Re\{\bar{A}_2^x F_2^{yt} - \bar{A}_2^y F_2^{xt}\}/2 = -\{(-i)(-i) - (-1)(-1)\}/2 = 1. \tag{7.12}$$

However, the spin density in the real field, $A^k = A_1^k + A_2^k$, $F^{kl} = F_1^{kl} + F_2^{kl}$, calculated using formula (1.6), contains the nonphysical oscillating cross term.

$$\begin{aligned} < \underset{c}{\Upsilon}{}^{xyt} > &= -\Re\{(\bar{A}_1^x + \bar{A}_2^x)(F_1^{yt} + F_2^{yt}) - (\bar{A}_1^y + \bar{A}_2^y)(F_1^{xt} + F_2^{xt})\}/2 = \\ &= < \underset{c}{\Upsilon}{}_1^{xyt} > + < \underset{c}{\Upsilon}{}_2^{xyt} > - \Re\{\bar{A}_1^x F_2^{yt} - \bar{A}_1^y F_2^{xt} + \bar{A}_2^x F_1^{yt} - \bar{A}_2^y F_1^{xt}\}/2 \end{aligned} \tag{7.13}$$

The cross term:

$$\begin{aligned} -\Re\{\bar{A}_1^x F_2^{yt} - \bar{A}_1^y F_2^{xt} + \bar{A}_2^x F_1^{yt} - \bar{A}_2^y F_1^{xt}\}/2 &= -\{[i(-i) - 1(-1)]e^{-2iz} + [(-i)i - (-1)1]e^{2iz}\}/2 = \\ &= -2\cos 2z \end{aligned} \tag{7.14}$$

$$\text{So} \quad < \underset{c}{\Upsilon}{}^{xyt} >= 2 - 2\cos 2z. \tag{7.15}$$

7.2. Incorrectness of the canonical energy-momentum tensor

To modify the canonical spin tensor we must remember that the electrodynamics starts from the canonical Lagrangian [37 (4-111)], $\underset{c}{L} = -F_{\mu\nu}F^{\mu\nu}/4$. Then, by the Lagrange formalism, the canonical energy-momentum tensor [37 (4-113)]

$$\underset{c}{T}{}^{\mu\nu} = \partial^\mu A_\alpha \frac{\partial \underset{c}{L}}{\partial(\partial_\nu A_\alpha)} - g^{\mu\nu}\underset{c}{L} = -g^{\mu\alpha}\partial_\alpha A_\beta F^{\nu\beta} + g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}/4 \tag{7.16}$$

and the canonical total angular momentum tensor [37 (4-147)]

$$\underset{c}{J}{}^{\lambda\mu\nu} = 2x^{[\lambda}\underset{c}{T}{}^{\mu]\nu} + \underset{c}{\Upsilon}{}^{\lambda\mu\nu} \tag{7.17}$$

are obtained, where

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial \underset{c}{L}}{\partial(\partial_\nu A_\alpha)} = -2A^{[\lambda}F^{\mu]\nu}, \tag{7.18}$$

is the incorrect canonical spin tensor (1.6) [37 (4-150)].

However, the canonical energy-momentum tensor (7.16) is incorrect as well [38, 39]. It gives, for example, a negative energy density $\underset{c}{T}^{tt}$ of a constant uniform electric field E_x (we assume $c = \varepsilon_0 = 1$):

$$E = F_{tx} = -F^{tx}, \quad F_{tx} = -\partial_x A_t, \quad A_t = -xE, \quad A_x = 0, \quad F_{\alpha\beta}F^{\alpha\beta}/4 = F_{tx}F^{tx}/2 = -E^2/2$$

The first term in (7.16) is zero, and the second is negative,

$$\underset{c}{T}^{tt} = -g^{tt}(\partial_t A_x)F^{tx} + g^{tt}F_{\alpha\beta}F^{\alpha\beta}/4 = -E^2/2$$

In contrast, in Maxwell's tensor (2.1)

$$T^{\mu\nu} = g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}/4 \tag{7.19}$$

the first term is E^2:

$$T^{tt} = -g^{tt}(\partial_t A_x - \partial_x A_t)F^{tx} + g^{tt}F_{\alpha\beta}F^{\alpha\beta}/4 = E^2/2.$$

Moreover, the canonical tensor (7.16) is not symmetric and has irregular divergence.

As is known, divergence of the Maxwell tensor is [11 (33.7)]

$$\partial_\nu T^{\mu\nu} = -g^{\mu\alpha}j^\beta F_{\alpha\beta} \tag{7.20}$$

where $g^{\mu\alpha}j^\beta F_{\alpha\beta} = f^\mu$ is volume density of the 4-Lorentz force, with which the electromagnetic field acts on charges and currents of matter, and $j^\beta = \partial_\nu F^{\nu\beta}$ is the current density. The canonical energy-momentum tensor differs from Maxwell's tensor by $g^{\mu\alpha}\partial_\beta A_\alpha F^{\nu\beta}$,

$$\underset{c}{T}^{\mu\nu} = T^{\mu\nu} - g^{\mu\alpha}\partial_\beta A_\alpha F^{\nu\beta}$$

Therefore, instead of (7.20), the divergence of the canonical energy-momentum tensor is equal to the inappropriate value

$$\partial_\nu \underset{c}{T}^{\mu\nu} = -g^{\mu\alpha}j^\beta(\partial_\alpha A_\beta - \partial_\beta A_\alpha) - g^{\mu\alpha}(\partial_{\nu\beta}A_\alpha F^{\nu\beta} + \partial_\beta A_\alpha j^\beta) = -g^{\mu\alpha}j^\beta\partial_\alpha A_\beta \tag{7.21}$$

7.3. The Belinfante-Rosenfeld procedure

In order to transform the canonical energy-momentum tensor (7.16) into the Maxwell tensor (7.19), the term $\partial_\beta A^\mu F^{\nu\beta}$ should be added to the canonical tensor (7.16). Instead, as a part of the Belinfante-Rosenfeld procedure, a divergence-free divergence of an antisymmetric quantity is added to the canonical tensor. We name this quantity $\Delta^{\mu\nu}$:

$$\Delta^{\mu\nu} = \partial_\beta(A^\mu F^{\nu\beta}). \tag{7.22}$$

The result is a meaningless tensor $\underset{BR}{T}^{\mu\nu}$ that differs from both the Maxwell tensor and the canonical energy-momentum tensor

$$\underset{BR}{T}^{\mu\nu} = \underset{c}{T}^{\mu\nu} + \partial_\beta(A^\mu F^{\nu\beta}) = T^{\mu\nu} - A^\mu j^\nu. \tag{7.23}$$

This tensor is never used and its existence is not a problem. However, the addition of term (7.22) to the canonical energy-momentum tensor is accompanied by the addition of a term $\Delta^{\lambda\mu\nu}$ to the canonical spin tensor (7.18).

$$\underset{BR}{\Upsilon}^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + \Delta^{\lambda\mu\nu}$$

The term $\Delta^{\lambda\mu\nu}$ is related to the term $\Delta^{\mu\nu}$ by the well-known relation [6 (9.4)]

$$\partial_\nu \Delta^{\lambda\mu\nu} = 2\Delta^{[\lambda\mu]}, \tag{7.24}$$

and the term $\Delta^{\lambda\mu\nu}$ turned out to be equal to the canonical spin tensor (7.18) with a minus sign

$$\Delta^{\lambda\mu\nu} = -\underset{c}{\Upsilon}^{\lambda\mu\nu}. \tag{7.25}$$

Really:

$$-\partial_\nu \underset{c}{\Upsilon}^{\lambda\mu\nu} = 2\partial_\nu (A^{[\lambda} F^{\mu]\nu}) = 2\Delta^{[\lambda\mu]} \tag{7.26}$$

This means that the Belinfante-Rosenfeld procedure eliminates the spin tensor of electrodynamics. Thus, the Belinfante-Rosenfeld spin tensor is zero

$$\underset{BR}{\Upsilon}^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + \Delta^{\lambda\mu\nu} = -A^{[\lambda} F^{\mu]\nu} + A^{[\lambda} F^{\mu]\nu} = 0, \tag{7.27}$$

7.4. Professor Soper's mistake

But let us return temporarily to the energy-momentum tensor. Wanting to obtain the Maxwell tensor by adding a divergence-free term to a canonical tensor, Soper changes the Lagrangian by adding the term $-A_\nu j^\nu$ [6 (8.3.3)],

$$\underset{S}{\mathsf{L}} = -F_{\mu\nu} F^{\mu\nu} / 4 - A_\lambda j^\lambda, \tag{7.28}$$

and obtains a "Soper-canonical" energy-momentum tensor [6 (8.3.7),(8.3.8)]

$$\underset{Sc}{T}^{\mu\nu} = \partial^\mu A_\alpha \frac{\partial \underset{S}{\mathsf{L}}}{\partial(\partial_\nu A_\alpha)} - g^{\mu\nu} \underset{S}{\mathsf{L}} = -g^{\mu\alpha} \partial_\alpha A_\beta F^{\nu\beta} + g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} / 4 + A^\mu j^\nu = \underset{c}{T}^{\mu\nu} + A^\mu j^\nu, \tag{7.29}$$

This tensor really differs from Maxwell's tensor by a divergence-free term, and therefore has the correct divergence (7.20). However, this tensor is not obtained from the Soper Lagrangian. Soper was mistaken in calculating his additional term in tensor (7.29). In fact, Lagrangian (7.28) obviously gives another energy-momentum tensor

$$\tilde{T}^{\mu\nu} = \underset{c}{T}^{\mu\nu} + g^{\mu\nu} A_\lambda j^\lambda, \tag{7.30}$$

which is no better than the usual canonical tensor (7.16). There is no Lagrangian giving Maxwell's tensor within the canonical formalism.

7.5. Electrodynamics' spin tensor

We have noted that the Maxwell energy-momentum tensor can be gained by adding the term

$$t^{\mu\nu} = T^{\mu\nu} - \underset{c}{T}^{\mu\nu} = \partial_\beta A^\mu F^{\nu\beta}, \tag{7.31}$$

instead of (7.22) $\Delta^{\mu\nu} = \partial_\beta (A^\mu F^{\nu\beta})$, to the canonical energy-momentum tensor $\underset{c}{T}^{\mu\nu}$. Here a question arises, what term $s^{\lambda\mu\nu}$, instead of (7.25) $\Delta^{\lambda\mu\nu}$, must be added to the canonical spin tensor $\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda} F^{\mu]\nu}$ to transform it into a required unknown electrodynamics spin tensor

$$\Upsilon^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + s^{\lambda\mu\nu} ?$$

Our answer is [39,40]: the addends $t^{\mu\nu}$, $s^{\lambda\mu\nu}$ must satisfy the relationship (7.24)

$$\partial_\nu s^{\lambda\mu\nu} = 2t^{[\lambda\mu]}, \text{ i.e. } \partial_\nu s^{\lambda\mu\nu} = 2\partial_\nu A^{[\lambda} F^{\mu]\nu}. \tag{7.32}$$

A simple expression

$$s^{\lambda\mu\nu} = 2A^{[\lambda} \partial^{\mu]} A^\nu \tag{7.33}$$

satisfies Eq. (7.32). So, the suggested electrodynamics spin tensor is

$$\Upsilon^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + s^{\lambda\mu\nu} = -2A^{[\lambda} F^{\mu]\nu} + 2A^{[\lambda} \partial^{\mu]} A^\nu = 2A^{[\lambda} \partial^{|\nu|} A^{\mu]} = 2A^{[\lambda} \partial_\kappa A^{\mu]} g^{\kappa\nu}. \tag{7.34}$$

The spin tensor (7.34) does not give spin fluxes across the direction of wave propagation as the canonical spin tensor (1.6) gives. For the wave (7.1), it turns out, for example,

$$\Upsilon^{zxy} = -A^z \partial^y A^x + A^x \partial^y A^z = 0. \tag{7.35}$$

But the spin tensor (7.34) predicts the interference of counter propagating waves as well as the canonical spin tensor predicts. For the field (7.7) - (7.10), spin tensor (7.34) gives the cross term

$$\Re\{\bar{A}_1^x\partial_t A_2^y - \bar{A}_1^y\partial_t A_2^x + \bar{A}_2^x\partial_t A_1^y - \bar{A}_2^y\partial_t A_1^x\}/2$$
$$= \{[i(-i)(-1)-(-i)i]e^{-2iz} + [(-i)(-i)-(-1)(-i)(-i)]e^{2iz}\}/2 = -2\cos 2z \tag{7.36}$$

7.6. Symmetrisation of the electrodynamics' spin tensor

The expression (7.34) was obtained heuristically. It is not final one. We saw the imperfection of the spin tensor (7.34) in the fact that it unjustifiably selects the part of the electromagnetic field, which is associated with the magnetic vector potential A and, accordingly, with the electric current j. It represents only the electric field, **E**, $\mathbf{A} = -\int \mathbf{E}dt$. The fields of this part constitute, according to [41,42], the chain

$$\underset{\circ}{j}{}^{\mu}_{\wedge}\ (\partial)\ \underset{\times}{F}{}^{\mu\nu}_{\wedge}\ \bigstar\ \underset{\circ}{F}{}_{\alpha\beta}\ (\partial)\ \underset{\times}{A}{}_{\beta}\ \bigstar\ \underset{\circ}{A}{}^{\lambda}_{\wedge}\,. \tag{7.37}$$

Here the index $\wedge$ marks tensor densities of weight +1; the five-pointed asterisk is the conjugation operator: $\bigstar = g_{\beta\lambda}/\sqrt{g}_{\wedge}$ or $\bigstar = \sqrt{g}_{\wedge} g^{\mu\alpha} g^{\nu\beta}$; symbol (∂) is a boundary operator: $(\partial)\ \underset{\times}{A}{}_{\beta} = 2\partial_{[\alpha}\ \underset{\times}{A}{}_{\beta]}$ or $(\partial)\ \underset{\times}{F}{}^{\mu\nu}_{\wedge} = \partial_\nu\ \underset{\times}{F}{}^{\mu\nu}_{\wedge}$; the symbol $\circ$ denotes closed differential forms or closed vector densities, and $\times$ denotes conjugate closed quantities.

The spin tensor (7.34) is composed of the fields of this chain. So, we denote it the electric spin tensor

$$\underset{e}{\Upsilon}{}^{\lambda\mu\nu} = 2A^{[\lambda}\partial_\kappa A^{\mu]} g^{\kappa\nu}\,. \tag{7.38}$$

However, there is an alternative chain of fields, including the electric three-vector potential V and the current density of magnetic monopoles ξ

$$\underset{\circ}{\xi}{}_{\gamma\alpha\beta}\ (\partial)\ \underset{\times}{F}{}_{\alpha\beta}\ \bigstar\ \underset{\circ}{F}{}^{\mu\nu}_{\wedge}(\partial)\ \underset{\times}{V}{}^{\mu\nu\lambda}_{\wedge}\ \bigstar\ \underset{\circ}{V}{}_{\gamma\alpha\beta}\,. \tag{7.39}$$

The corresponding spin tensor must be composed of the fields $\underset{\times}{V}{}^{\mu\nu\lambda}_{\wedge}$ of this chain. To give this spin tensor the form of (7.34), dual expressions are used, obtained using the antisymmetric pseudo density $\varepsilon^*_{\alpha\beta\gamma\delta}$. We will mark pseudo-values with the asterisk $*$:

$$V_*^{\beta\gamma\delta}\varepsilon^*_{\alpha\beta\gamma\delta} = V^*_\alpha\,. \tag{7.40}$$

This gives the magnetic spin tensor

$$\underset{m}{\Upsilon}{}^{\lambda\mu\nu} = 2V_*^{[\lambda}\partial_\kappa V_*^{\mu]} g^{\kappa\nu}\,. \tag{7.41}$$

An analogue of the Weyl gauge $\varphi = 0$ is now $V^{xyz} = 0$. Therefore, to obtain the electric potential from the formula $F^{\mu\nu} = \partial_\lambda V^{\mu\nu\lambda}$, only $F^{kl} = \partial_t V^{klt} = -iV^{klt}$ is used. So $V^{klt} = iF^{kl}$. The field (7.7), (7.8) give a contravariant electric potential in the considered standing wave situation.

$$V_1^{zyt} = 1,\quad V_1^{xzt} = i,\quad V_2^{zyt} = 1,\quad V_2^{xzt} = i\,. \tag{7.42}$$

Lowering indices does not change these values

$$V_{1zyt} = 1,\quad V_{1xzt} = i,\quad V_{2zyt} = 1,\quad V_{2xzt} = i\,. \tag{7.43}$$

After dualizing with $\varepsilon^{zytx} = \varepsilon^{xzty} = 1$, we obtain the values for composing the magnetic spin tensor in the considered situation

$$V_{*1}^x = V_{*2}^x = V_{1zyt}\varepsilon^{zytx} = 1,\quad V_{*1}^y = V_{*2}^y = V_{1xzt}\varepsilon^{xzty} = i, \tag{7.44}$$

and the magnetic spin tensor (7.41) gives the cross term of the opposite sign

$$\Re\{\bar{V}_{*1}^x\partial_t V_{*2}^y - \bar{V}_{*1}^y\partial_t V_{*2}^x + \bar{V}_{*2}^x\partial_t V_{*1}^y - \bar{V}_{*2}^y\partial_t V_{*1}^x\}/2$$
$$= \{[(-i)i-(-i)(-i)]e^{-2iz} + [(-i)i-(-i)(-i)]e^{2iz}\}/2 = 2\cos 2z \tag{7.45}$$

Since the fields of both chains are equally present in electromagnetic radiation n vacuum, it is natural to use the half-sum of the electric and magnetic tensors as the spin tensor

$$\Upsilon^{\lambda\mu\nu} = (\underset{e}{\Upsilon}{}^{\lambda\mu\nu} + \underset{m}{\Upsilon}{}^{\lambda\mu\nu})/2\,. \tag{7.46}$$

In the considered case of a standing wave, such a generalized spin tensor gives the correct result

$$\Upsilon^{\lambda\mu\nu} = 2. \tag{7.47}$$

8. Explanation of the Beth's experiment

8.1. Poynting vector in the Beth's experiment.

The well-known Beth's experiment [43] proves that circularly polarized light contains an angular momentum, as predicted by Sadowsky [1] and Poynting [3]. According to Beth's idea, a beam of circularly polarized light passes through a half-wave plate, which changes the chirality of the light and, accordingly, changes the direction of rotation of the electromagnetic vectors and the direction of the angular momentum of light to opposite directions. As a result, by virtue of the angular momentum conservation law, the half-wave plate receives twice the amount of angular momentum contained in the beam. However, as it was just proved at the conference [44], in the Beth's experiment, in reality, there is no rotation of electromagnetic mass-energy. Moreover, there is no mass-energy flow at all. The Poynting vector $\mathbf{E}\times\mathbf{H}$ and the linear momentum density $\varepsilon_0\mathbf{E}\times\mathbf{B}$ are equal to zero in the Beth's apparatus. This has also been proven earlier [39,40,45,46]. As a consequence, the electromagnetic field in the Beth's apparatus has no angular momentum, according to the existing definition of the angular momentum of an electromagnetic field [9,25,47-49]

$$\mathbf{J} = \varepsilon_0 \int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV . \tag{8.1}$$

According to definition (8.1), the angular momentum of an electromagnetic field is the moment of the linear momentum of the field, and it is equal to zero in the Beth's apparatus. So the experimentally recorded transfer of the angular momentum of light to the plate occurs in the absence of any angular momentum in the light, according to (8.1). Therefore, the receipt of the angular momentum by the half-wave plate from the electromagnetic field in the Beta experiment is inexplicable within the framework of definition (8.1).

The point is that in the Beth's experiment, light that has passed through a half-wave plate passes through it a second time after being reflected from a mirror covered with a quarter-wave plate. And such a mirror does not change the chirality of light when reflected. Therefore, the light that has passed through the half-wave plate returns to it after reflection with the same chirality. But circularly polarized beams of the same chirality, having the opposite direction, create the opposite rotation of electromagnetic vectors. Therefore, when such beams interference in the Beth's apparatus, a pulsation of the field vectors arises without rotation at any point in space around the half-wave plate.

To show this, consider a simple model of a right-handed circularly polarized light beam directed along z-axis with the plane phase front, which was proposed by Jackson [9]:

$$\mathbf{E}_1 = \exp(iz - it)[\mathbf{x} + i\mathbf{y} + \mathbf{z}(i\partial_x - \partial_y)]E_0(r), \quad r^2 = x^2 + y^2, \tag{8.2}$$

$$\mathbf{H}_1 = \exp(iz - it)[-i\mathbf{x} + \mathbf{y} + \mathbf{z}(\partial_x + i\partial_y)]E_0(r), \tag{8.3}$$

Here $\mathbf{E}_1$ and $\mathbf{H}_1$ are complex vectors of the electromagnetic field, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are unit coordinate vectors. ∂_x, ∂_y mean partial derivatives with respect to *x* and *y*. For simplicity, $\omega = k = c = \varepsilon_0 = \mu_0 = 1$. Index 1 in (8.2), (8.3) means that the formulas describe the primary beam after passing through the half-wave plate. The beam amplitude is indicated by $E_0(r)$. The function $E_0(r)$ is considered constant throughout the entire beam area, that is, under the condition $r < R$, where R is the radius of the beam. However, on the surface of the beam, where $r \approx R$, the function $E_0(r)$ quickly decreases to zero.

We mark the reflected beam incident on the plate with index 2. This beam has the same helicity as the primary beam passing through the plate (that is, it has the same mutual direction of momentum and spin). Therefore, it's formulas are obtained from formulas (8.2), (8.3) by changing the signs of z and y:

$$\mathbf{E}_2 = \exp(-iz - it)[\mathbf{x} - i\mathbf{y} - \mathbf{z}(i\partial_x + \partial_y)]\, E_0(r)\,, \qquad (8.4)$$

$$\mathbf{H}_2 = \exp(-iz - it)[-i\mathbf{x} - \mathbf{y} - \mathbf{z}(\partial_x - i\partial_y)]\, E_0(r) \qquad (8.5)$$

Adding the primary and reflected beams and writing out explicitly the real parts of the complex expressions, we obtain the components of the resulting electromagnetic field

$$E_x = \Re[\exp(iz - it) + \exp(-iz - it)]E_0 = 2E_0 \cos z \cos t\,, \qquad (8.6)$$

$$E_y = \Re[i\exp(iz - it) - i\exp(-iz - it)]E_0 = -2E_0 \sin z \cos t\,, \qquad (8.7)$$

$$\begin{aligned} E_z &= \Re[\exp(iz - it)(i\partial_x - \partial_y) + \exp(-iz - it)(-i\partial_x - \partial_y)]E_0 \\ &= -2(\sin z\partial_x + \cos z\partial_y)E_0 \cos t \end{aligned} \qquad (8.8)$$

$$H_x = \Re[-i\exp(iz - it) - i\exp(-iz - it)]E_0 = -2E_0 \cos z \sin t\,, \qquad (8.9)$$

$$H_y = \Re[\exp(iz - it) - \exp(-iz - it)]E_0 = 2E_0 \sin z \sin t\,, \qquad (8.10)$$

$$\begin{aligned} H_z &= \Re[\exp(iz - it)(\partial_x + i\partial_y) + \exp(-iz - it)(-\partial_x + i\partial_y)]E_0 \\ &= 2(\sin z\partial_x + \cos z\partial_y)E_0 \sin t \end{aligned}\,, \qquad (8.11)$$

and the resulting electromagnetic field in the vector form is

$$\mathbf{E} = 2[\mathbf{x}\cos z - \mathbf{y}\sin z) - \mathbf{z}(\sin z\partial_x + \cos z\partial_y)]E_0 \cos t\,, \qquad (8.12)$$

$$\mathbf{H} = -2[\mathbf{x}\cos z - \mathbf{y}\sin z) - \mathbf{z}(\sin z\partial_x + \cos z\partial_y)]E_0 \sin t\,. \qquad (8.13)$$

It can be seen that the electric and magnetic fields are parallel to each other everywhere. Therefore, the Poynting vector is zero everywhere. There is no movement. There is no momentum.

8.2. Spin tensor in the Beth's experiment.

Let us now calculate the spin flux in the resulting electromagnetic field (8.12), (8.13) adjacent to the Beth's half-wave plate on one side. We calculate sequentially, first the vector potential $\mathbf{A}$, and then, using formula (1.6), the component Υ^{xyz} of the spin tensor $\Upsilon^{\lambda\mu\nu}$,

$$\mathbf{A} = -\int \mathbf{E}dt = -2(\mathbf{x}\cos z - \mathbf{y}\sin z)\, E_0 \sin t\,, \qquad (8.14)$$

$$\Upsilon^{xyz} = -A^x F^{yz} + A^y F^{xz} = A^x H_x + A^y H_y = 4E_0^2 \sin^2 t\,,\ \ <\Upsilon^{xyz}>= 2E_0^2\,. \qquad (8.15)$$

A similar calculation for the other side of the plate gives the same result. Thus, as a result of the existence of the spin flows to two sides, the plate receives the resultant torque

$$\tau_{\text{tot}} = 4\pi R^2 E_0^2 = 4P\,, \qquad (8.16)$$

where P represents the power of the beam. This is consistent with the outcome of the Beth's experiment.

Thus the concept of spin of electromagnetic radiation, which goes back to Sadowsky & Poynting [1.2], according to which the angular momentum is proportional to the radiation energy, requires the spin term to be introduced into the definition of the angular momentum of electromagnetic radiation (1.5). The Beth's experiment proves this.

8.3. Illustrations

Now we illustrate the content of this article. See please. Figures 1, 2, 3 present the interference of an incident beam and the beam reflected by an ordinary mirror. Figures 5, 6, 7 present the interference in the Beth's apparatus.

In Figure 1, the left helix of the right-hand circular polarization wave moves translationally upward along the z-axis with the speed of light V. Electric field E is represented by red arrows, magnetic field H is represented by blue arrows. The right half of the figure shows a side view of the wave. The crosses in circles represent the tails of the arrows. The dots within the circles represent the noses of the arrows. The left half of the figure shows cross sections of the wave by xy-planes at three different locations. The direction of rotation of the E-H pair of vectors observed at these locations is shown. The spins of the photons are directed along the z-axis; we say that they have +z-spin. The direction of the spins coincides with the direction of the wave velocity. Therefore, the spin flux is positive. At the same time, this means the existence of a downward –z-spin flux. The spin flux situation is similar to the momentum flux situation, i.e. to pressure situation. Positive pressure in a vertical cylinder means that the +z-directed momentum passes through the upper end of the cylinder and, at the same time, the –z-momentum passes down through the lower end of the cylinder. But we do not know how to depict a flow in the picture.

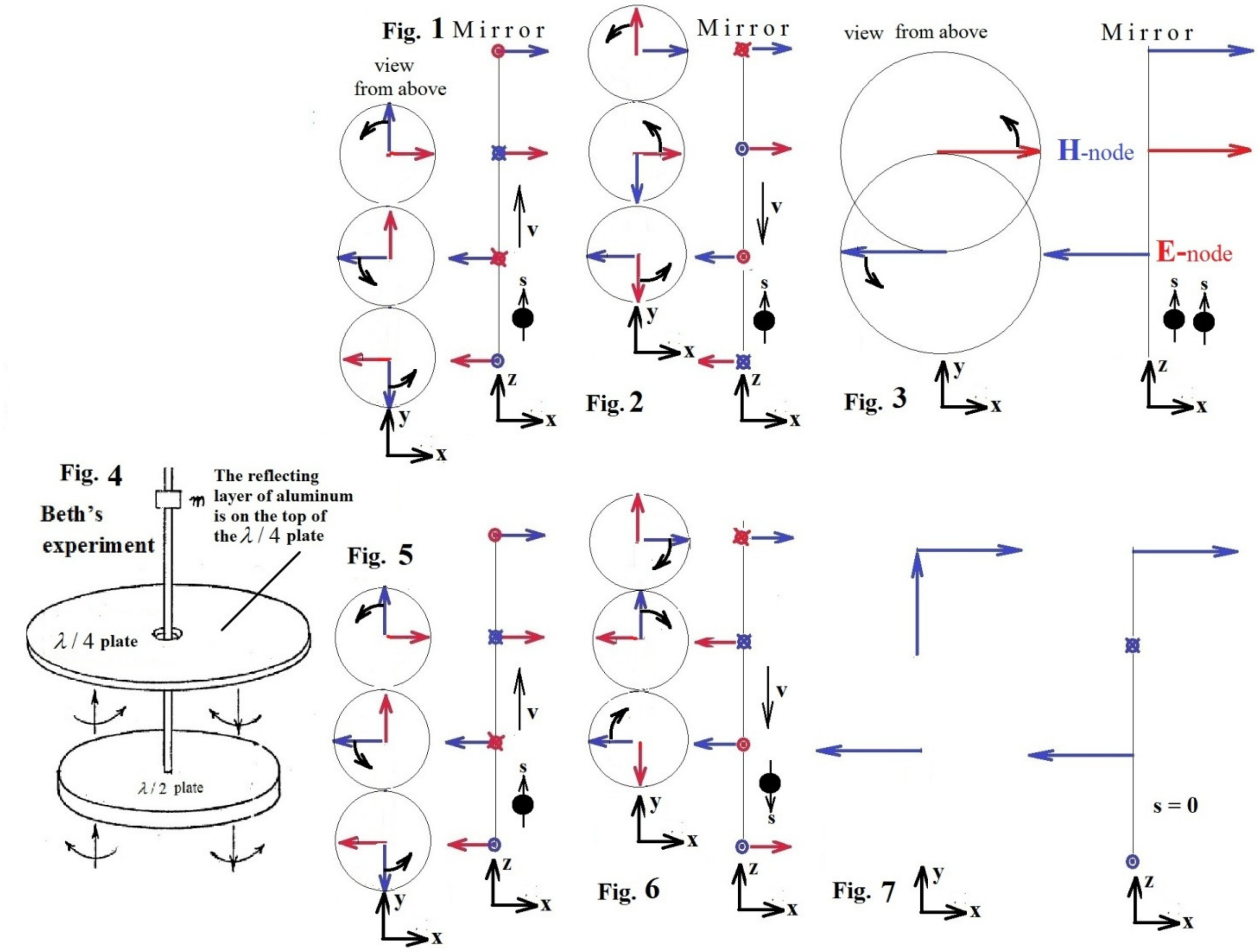

Figure 2 shows the same wave after reflection from an ordinary mirror. It moves in the opposite +z-direction. But the pair of vectors E-H rotates in the same way as in Figure 1. Accordingly, the direction of the photon spins remains + z. However, the speed changes direction to the opposite direction. Therefore, the spin flux is negative. This is analogous to negative pressure. This wave has left-hand circular polarization.

Figure 3 shows the resulting standing wave of circular polarization. The total vectors E and H rotate in the same way as in Figures 1 and 2. However, now the E and H-fields have nodes. In some places there is no electric field, and the magnetic field is doubled, in other places there is no magnetic field, and the electric field is doubled. The volume density of the spin is doubled and still has the +z direction. But the spin flux is zero. Spin is without spin flux! This is natural, because the spin flux onto the mirror is zero. The average speed of movement of the electromagnetic mass-energy is zero.

Figure 4 shows a portion of the Beth's apparatus. We are considering the space between the half-wave plate and the quarter-wave plate with mirror sputtering.

Figure 5 shows the same wave of right-hand circular polarization as in Fig. 1. The direction of the photon spins coincides with the direction of the velocity and with the direction of the +z-spin flux. So, we have a -z-spin flux down again. But now this wave is used in the Beth's experiment. It emerges from the half-wave plate and is directed at the mirror covered with a quarter-wave plate.

Figure 6 shows the wave of Figure 5 after reflection from the mirror covered with the quarter-wave plate. When reflected from such a mirror, the wave passes through the quarter-wave plate twice. Therefore, the quarter wave plate plays the role of a half wave plate. But a half-wave plate changes the chirality of the transmitted wave to the opposite one. Therefore, in contrast to the ordinary reflection as in Figure 2, in the Beth's experiment, the reflected wave retains the right-hand circular polarization. Its speed is downward. Spin of photons is directed downward as well. It is the –z-spin. We have a -z-spin flux down. Thus, the total flux of the –z-spin down to the half-wave plate is doubled. The plate experiences a torque corresponding to this doubled flux from the considering space. This torque is directed against the z-axis.

The standing electromagnetic wave arising in the Beth's experiment (Figure 7) differs significantly from the usual standing wave shown in Figure 3. There are no nodes in such a wave. For example, at the depicted time moment, there is a doubled magnetic field in all space. Over time, this magnetic field, without changing its direction, is replaced by an electric field, because the vectors of both fields are obtained by adding the vectors of the primary and reflected waves, which have opposite rotation. In this case, the vectors E and H of the fields always and everywhere coincide in direction. This means that the Poynting vector is identically zero. There is no rotation, and even no movement of the electromagnetic mass-energy. All fluxes are equal to zero, except for the spin flux. In this case, the volume density of the spin is equal to zero due to the fact that the spins of the primary and reflected waves have the opposite direction.

8.4. Conclusion

The concept of spin of electromagnetic radiation, which goes back to Sadowsky & Poynting [2,3], according to which the angular momentum is proportional to the electromagnetic energy, requires the spin term to be present in the definition of the angular momentum of electromagnetic radiation (14). Definition (1) is not correct. The Beth's experiment proves this.

I am eternally grateful to Robert Romer for courageously publication the question, "Does a plane wave really carry no spin?" [50].

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Biography of author(s)

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Radi Khrapko graduated from the Moscow University. Radi Khrapko's diploma work "Towards the theory of an inhomogeneous anisotropic universe" won a prize at the competition of student scientific papers. At the same time, inventions were made: "Pulse amplifier" (together with I. Khrapko) as a consequence of amateur radio, and "Underwater intercom" (together with N. Rasskazikhina) as a result of diving. Radi Khrapko's all scientific and pedagogical activities are associated with the Moscow Aviation Institute. Some works on the theory of gravity was carried out, the interior of the "black hole" was investigated, and the thesis "Application of complex coordinate values in the general theory of relativity" was defended. When analyzing the problem of gravitational energy, the popular "Papapetrou equations" were criticized, and it was shown that Einstein's pseudotensor of the energy-momentum of the gravitational field is an erroneous construction. Radi Khrapko's book "Visible representation of exterior differential forms" (Lambert 2011) shows that Maxwell's equations are fragments of a "chain of fields", and presents results concerning harmonic fields, Helmholtz expansion, Hodge operator. However, the main result is the introduction of spin in electrodynamics; spin is absent in the Maxwell's theory. The incorrectness of the modern concept of the electrodynamics angular momentum has been shown. Doubts about this concept were first reported in Amer. J. Phys. 69 405 (2001).