SEARCH FOR CONGRUENT NUMBERS

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Abstract

By parametrizing the Pythagorean equation with hyperbolic functions you can obtain an algebraic equation of the 3^{rd} degree that describes congruent numbers. In some cases this equation may facilitate the search for these numbers.

Keywords: congruent numbers, algebraic equation of the third degree, hyperbolic functions.

Introductory remarks:

This article uses a parametrization of the Pythagorean equation in the form of:

$$\text{ch}\,\alpha = x / y \rightarrow \mathbf{4\,y^4\,ch^2\,\alpha + y^4\,sh^4\,\alpha = y^4\,(1+ ch^2\,\alpha)^2}) \qquad (1)$$

or slightly modified: $$\mathbf{4(1+ a)^2/ (1- a)^2 + 16a^2/(1- a)^4 = 4((1+ a^2)^2/(1- a)^4} \qquad (2)$$

where a= (xy)/(x+y) = (ch α-1)/(ch α+1).

Also to be noted is that the Pythagorean equation is formed by two pairs of numbers (x , y) and [(x + y)/√2,(x - y)/√2)].

The parametrization of equations (1) and (2) does not need to be proved, since it is verified by direct substitution. In the first case we get the identity $sh^2\alpha = sh^2\alpha$, and in the second case the identity $(1+ a^2) = (1+ a^2)$.

Thus, based on equation (1) we can express the area of a right triangle as follows:

$$\mathbf{n = y^4 \cdot ch\,\alpha \cdot sh^2\,\alpha}\,. \qquad (3)$$

After squaring both sides and replacing $ch^2\alpha$ with $sh^2\alpha$, we get the equation $\mathbf{n^2= y^8(sh^6\alpha + sh^4\alpha)}$, and replacing $sh^2\alpha$ with **x** yields

$$\mathbf{n^2= y^8(x^3+ x^2)} \qquad (4)$$

Quite possibly, this equation describes all congruent numbers. But most likely this statement requires proof. Compared to other methods, this method does not provide any particular advantages, since it also requires to select the coefficient $\mathbf{y^8}$**.** But it is possible to facilitate the selection of this coefficient.

To do this, you need to change the requirements for the task. We need to search not for the number **n**, which is the area of a right triangle, but for the number that can form a right triangle, in this case **chα.** Having set a certain value for **chα**, we substitute this number into equation (1). As a result, we get the Pythagorean equation with an unknown coefficient $\mathbf{y^4}$. Obviously, if we take **chα** as an integer, then $\mathbf{y^4}$ will be equal to 1, and if we take it as a rational fraction, then the problem will be more complicated. But in each individual case, if **chα** is given, it is possible to find the coefficient $\mathbf{y^4}$.

Let us show this with two examples.

Let **chα =** 9. Then $\mathbf{sh^2\alpha}$ =80. Substituting these values into equation (1) yields**:**

$\mathbf{y^4}\cdot 4\cdot 81 + \mathbf{y^4}\cdot 80^2 = \mathbf{y^4}\cdot 82^2$.

Since $\mathbf{y^4}$ =1, let's reduce the equation to a primitive form: $9^2+40^2=41^2$.

On the other hand, substituting this value into equation (4) $\mathbf{sh^2\alpha}$=80, with $\mathbf{y^4}$ =1, we get $720^2=80^3+80^2$. Next, we can calculate all possible values of $\mathbf{y^4}$ which are obtained by decomposing the number 720 into factors:

$720^2=2^8\cdot3^4\cdot5^2 \rightarrow (2^8\cdot3^4 = 144^2)\cdot 5^2= 80^3/144^2 + 80^2/144^2$ ($\mathbf{y^8}=1/144^2$)
$6^2= 80^3/120^2 + 80^2/120^2$ ($\mathbf{y^8}=1/120^2$)
$10^2= 80^3/72^2 + 80^2/72^2$ ($\mathbf{y^8}=1/72^2$)
$15^2= 80^3/48^2 + 80^2/48^2$ ($\mathbf{y^8}=1/48^2$)

This way we have 4 potential numbers that can be taken as the areas of right triangles: 5, 6, 10, 15 which are formed from one number $\mathbf{ch\alpha}$ =9. Now, if we substitute the obtained coefficients ($\mathbf{y^4}=1/12^2$, $\mathbf{y^4}=1/120$, $\mathbf{y^4}=1/72$, $\mathbf{y^4}=1/48$) into equation (1), we get

for $\mathbf{y^4}=1/12^2 \rightarrow 4\cdot9^2/12^2 + 80^2/12^2=82^2/12^2$,

and reducing by a common factor we get:

$(9/6)^2+(20/3)^2= (41/6)^2$.

For $\mathbf{y^4}=1/120 \rightarrow 4\cdot9^2/120 + 80^2/120=82^2/120$ after transformations we get:

$4(\sqrt{10/3})^2+3(\sqrt{3/10})^2=41^2/\sqrt{30} \rightarrow 40^2+9^2=41^2$.

When $\mathbf{y^4}$ =1/72 and $\mathbf{y^4}$ =1/48 we also come to the equation $40^2+9^2=41^2$.

Thus, from the number $\mathbf{ch\alpha}$ =9 we have obtained two primitive Pythagorean equations:

$40^2+9^2=41^2$ and $(9/6)^2+(20/3)^2=(41/6)^2$ with the area of triangles **n =** 180 and 5.

Second example:

Let chα =2.125 → 2125/1000 → 17/8. sh ^{2} α=(15/8) 2.

Substitution of the values $\mathbf{ch\alpha}$ =17/8 and $sh^2\alpha=(15/8)^2$ into the equation (1) gives us

$4\cdot y^4\cdot(17/8)^2+y^4\cdot (15/8)^2=y^4\cdot[(1+(17/8)^2]^2$.

Reducing this equation by y^4 and multiplying it by 8^4 we get the first primitive Pythagorean equation

$272^2+225^2=353^2$ with area n = 30600.

On the other hand,

$$\mathbf{n^2= y^8\cdot ch\alpha^2 \cdot sh^4\alpha = y^8(x^3+ x^2).}$$

The values of chα and shα are known, so reduction by y ^{8} and calculation of the value $\mathbf{ch\alpha^2 \cdot sh^4\alpha,}$ reducing it to a rational form, yields a fraction equal to

$7650^2/1024^2 =[2^2\cdot3^4\cdot5^4\cdot17^2]/2^{20} = (x^3+ x^2), \rightarrow 2^2\cdot17^2=[2^{20}/(3\cdot5)^4](x^3+ x^2)$.

Also, do not forget that x = sh^2 α. Thus, the number 34 is probably the area of the Pythagorean triangle, but to make sure of this, we need to build this triangle. The coefficient $\mathbf{y^8}$ is known, it's equal to $2^{20}/(3\cdot5)^4 = 32^4/15^4$, so $\mathbf{y^4}$ is equal to $32^2/15^2$. We substitute this coefficient into the equation

$4\cdot y^4\cdot(17/8)^2+y^4\cdot (15/8)^2=y^4\cdot[(1+(17/8)^2]^2$,

which we have obtained earlier, and after transformations we get the triple (136/15), (15/2), (353/30). So in this case taking chα = 2.125 we've got two primitive Pythagorean triangles: $272^2+225^2=353^2$ and $(136/15)^2+(15/2)^2=(353/30)^2$.

The rest is to check the following pairs for the possibility of obtaining new congruent numbers, which can be obtained from the expansion $(2^2\cdot3^4\cdot5^4\cdot17^2)/2^{20}$. These will be pairs $(2^2\cdot3^2)$, $(2^2\cdot5^2)$, $(3^2\cdot5^2)$, $(3^2\cdot17^2)$, $(5^2\cdot17^2)$. As in the first case, you can write down all the coefficients $\mathbf{y^8}$ for each of these pairs and substitute them into the equation

$4\cdot y^4\cdot(17/8)^2+y^4\cdot(15/8)^2=y^4\cdot[(1+(17/8)^2]^2$.

In all cases we get the primitive equation $272^2+225^2=353^2$**.**

Here are a few equations, explicitly obtained by this method:

$5^2=x^3/144^2+x^2/144^2$	(x=80)
$6^2=x^3+x^2$	($x=sh^2\alpha=3$)
$7^2=[27^4/20^4]\cdot x^3+[27^4/20^4]\cdot x^2$	(x=175/81)
$30^2=256\cdot x^3+256\cdot x^2$	(x=1.25)
$34^2=32^4/15^4\cdot x^3+32^4/15^4\cdot x^2$	(x=3,515625)
$60^2=x^3+x^2$	(x=15)
$84^2=81^2\cdot x^3+81^2\cdot x^2$	(x=7/9)
$180^2=16^4\cdot x^3+16^4\cdot x^2$	($x=0{,}75^2$)
$210^2=256\cdot x^3+256\cdot x^2$	(x=5,25)
$210^2=x^3+x^2$	(x=35)

Another way to use the formula n= $\mathbf{y^4\cdot ch\alpha\cdot sh^2\alpha}$ $\to 2\cdot y^4\cdot[ch\alpha\cdot sh\alpha\cdot sh\alpha]/2\to$

$[(ch3\alpha-ch\alpha)\cdot y^4]/4\to[4ch^3\alpha-3ch\alpha-ch\alpha]\cdot y^4/4\to ch^3\alpha-ch\alpha-n/y^4=0.$ $ch\alpha=x$ $\to$ $\mathbf{x^3-x-n/y^4=0.}$ (5)

We have obtained the reduced equation of the 3rd degree. In this case, we proceed from the fact that we know the number **n** and we want to determine whether this number is the area of a right triangle or not. Accordingly, in this case **x** and **y** are unknown quantities. For **y** =1 the equation is solved by classical algebraic methods using the Cardano's formula and the Vieta's theorem, taking into account that we consider **x** and **y** to be rational and **n** to be a whole number.

Further, from the Cardano's formula we can determine the conditions for the search for $\mathbf{y^4}$**.** Since according to the Cardano's formula, in our case,

if $\sqrt{q^2/4+p^3/27}>0$, where $q^2=n^2/4\cdot y^8$, and p= -1,

then $n^2/4y^8>1/27\to y^8<27\cdot n^2/4$,

taking into account that $\mathbf{y^4}$ can be less than or greater than one. The case of $\sqrt{q^2/4+p^3/27}<0$ was not considered.

Thus, we have described two options for finding congruent numbers. In the first case, we were looking not for the congruent number **n** itself, but for **chα** - the number with which you can get a primitive Pythagorean triangle, and after that we calculated the area of the triangle. As it turned out, for one number you can get in some cases one primitive equation, and in some cases a pair. In the second case, we were looking directly for the congruent number **n** itself. This case winds up into solving the reduced cubic equation. The main difficulty for solving it is the selection of the coefficient $\mathbf{y^8}$.

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