Proof of the Riemann Hypothesis

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Abstract

We prove the Riemann Hypothesis by studying the behavior of a holomorphic function $\hat{f}(s)$ which has the same non-trivial zeros as the Riemann zeta function $\zeta(s)$. This function is given by $g(s) \equiv \hat{f}(x+iy) = \int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t}+1} e^{iyt} dt$ and is for an assigned $x > 0$, the Fourier transform of $f(x,t) = \frac{e^{xt}}{e^{e^t}+1}$.

1 The Riemann zeta function $\zeta(s)$

1.1 Dirichlet series

As is well known, the *Riemann zeta function* is defined by:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \qquad s = x + iy \tag{1}$$

The Dirichlet series (1) is convergent for $\operatorname{Re} s > 1$, and uniformly convergent in any finite region in which $\operatorname{Re} s \geq 1 + \delta$, $\delta > 0$. It therefore definis an holomorphic function $\zeta(s)$ for $\operatorname{Re} s > 1$ [1].

1.2 The functional equation and the non-trivial zeros

Riemann found the analytic extension (or *holomorphic extension*) of the sum of the Dirichlet series (1) over all $\mathbb{C}$ except the point $z = 1$, which turns out to be a simple pole with residue 1.

The aforesaid analytical extension is represented by the following functional equation [1]:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \tag{2}$$

where $\Gamma(s)$ is the Eulerian gamma function. The *non-trivial zeros* of $\zeta(s)$ fall in the *critical strip* [1]-[2] of the complex plane defined by

$$A = \{s \in \mathbb{C} \mid 0 \leq \operatorname{Re} s \leq 1, \ -\infty < \operatorname{Im} s < +\infty\} \tag{3}$$

More precisely, there are no zeros for $\operatorname{Re} s = 0$, $\operatorname{Re} s = 1$ so we should refer to the open strip:

$$\{s \in \mathbb{C} \mid 0 < \operatorname{Re} s < 1, \ -\infty < \operatorname{Im} s < +\infty\} \tag{4}$$

In the following, we will denote the geometric locus (4) by A.

The Eulerian gamma function has no zeros [3], so

$$s_0 \in A \mid \zeta(s_0) = 0 \Longleftrightarrow \zeta(1 - s_0) = 0 \tag{5}$$

1.3 Symmetries

1.3.1 Complex conjugation

Let $f(s)$ be a complex function defined in a field $T \subseteq \mathbb{C}$. Denoting with s^* the complex conjugate of $s = x + iy$ i.e. $s^* = x - iy$, we plan to study the behavior of $f(s)$ with respect to the complex conjugation $s \to s^*$. To do this, we separate the real and imaginary parts of $f(s)$:

$$f(s) = u(x, y) + iv(x, y)$$

The following special cases are of interest:

1. $u(x, y) \equiv u(x, -y)$, $v(x, y) \equiv v(x, -y)$, i.e. u and v are even functions with respect to the variable y. It follows

   $$f(s^*) = u(x, -y) + iv(x, -y) \equiv u(x, y) + iv(x, y) \Longrightarrow f(s^*) \equiv f(s)$$

   so $f(s)$ is invariant under the transformation $s \to s^*$.

2. $u(x, y) \equiv u(x, -y)$, $v(x, y) \equiv -v(x, -y)$, i.e. u is an even function while v is odd with respect to the variable y. It follows

   $$f(s^*) = u(x, -y) + iv(x, -y) \equiv u(x, y) - iv(x, y) \Longrightarrow f(s^*) \equiv f(s)^*$$

Example 1 *Let's consider the function $f(s) = e^s = e^x(\cos y + i \sin y)$, for which*

$$u(x, y) = e^x \cos y, \ v(x, y) = e^x \sin y$$

So we are in case 2: $e^{s^*} = (e^s)^*$.

For the function $\zeta(s)$ the following property holds:

Proposition 2 *(Property of complex conjugation)*

$$\zeta(s^*) = \zeta(s)^*, \quad \forall s \in \mathbb{C} \setminus \{1\} \tag{6}$$

Proof. It is sufficient to prove the (6) for $\mathrm{Re}\, s > 1$, using the representation through the Dirichlet series (1) since the property is conserved in the holomorphic extension.

$$\zeta(s) = \sum_{n=1}^{+\infty} n^{-x} n^{-iy} = \sum_{n=1}^{+\infty} n^{-x} e^{-iy \ln n} = \sum_{n=1}^{+\infty} n^{-x} \left[\cos(y \ln n) - i \sin(y \ln n)\right]$$

Separating the real part from the imaginary part:

$$\zeta(s) = \sum_{n=1}^{+\infty} n^{-x} \cos(y \ln n) - i \sum_{n=1}^{+\infty} \sin(y \ln n)$$

$$\Longrightarrow \zeta(s^*) = \sum_{n=1}^{+\infty} n^{-x} \cos(y \ln n) + i \sum_{n=1}^{+\infty} \sin(y \ln n)$$

from which

$$\zeta(s^*) = \zeta(s)^* \tag{7}$$

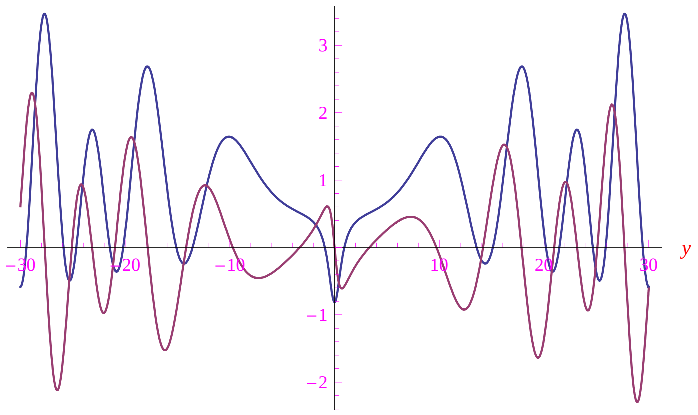

Figure 1: Trend of $\operatorname{Re}\zeta\left(\frac{1}{4}+iy\right)$, $\operatorname{Im}\zeta\left(\frac{1}{4}+iy\right)$.

■

From this it follows that $\operatorname{Re}\zeta\left(x+iy\right)$ is an even function with respect to the variable y, while $\operatorname{Im}\zeta\left(x+iy\right)$ is an odd function. is an odd function. This is evident in the graph of fig. 1.

The proposition 2 implies that the non-trivial zeros are symmetric about the real axis (fig. 2). In fact, if s_0 is a non-trivial zero, it must still occur

$$\zeta\left(s_0^*\right)=\zeta\left(s_0\right)^* \tag{8}$$

But $\zeta\left(s_0\right)=0 \Longrightarrow \zeta\left(s_0\right)^*=0 \Longrightarrow \zeta\left(s_0^*\right)=0$. Stated another way, the nontrivial zeros are distributed for complex conjugate pairs.

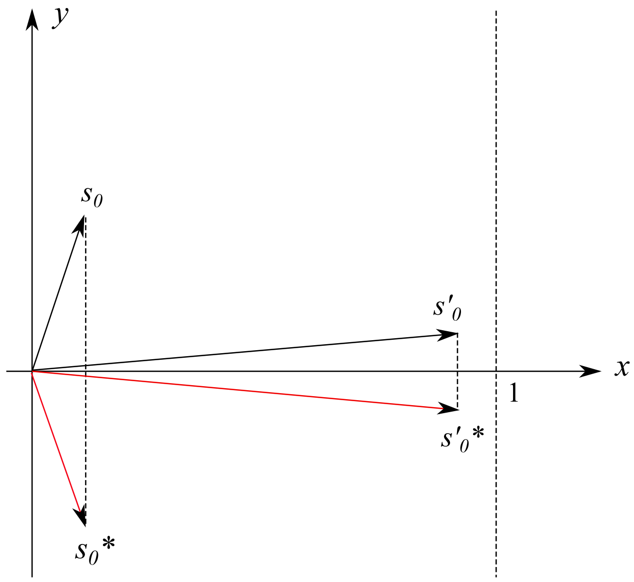

Figure 2: Symmetry of the distribution of zeros with respect to the real axis.

1.3.2 Symmetry about the point $\left(\frac{1}{2},0\right)$

The (5) has an immediate geometric interpretation illustrated in fig. 3 from which we see that the zeros s_0 and $1-s_0$ are symmetrical with respect to the point $\left(\frac{1}{2},0\right)$.

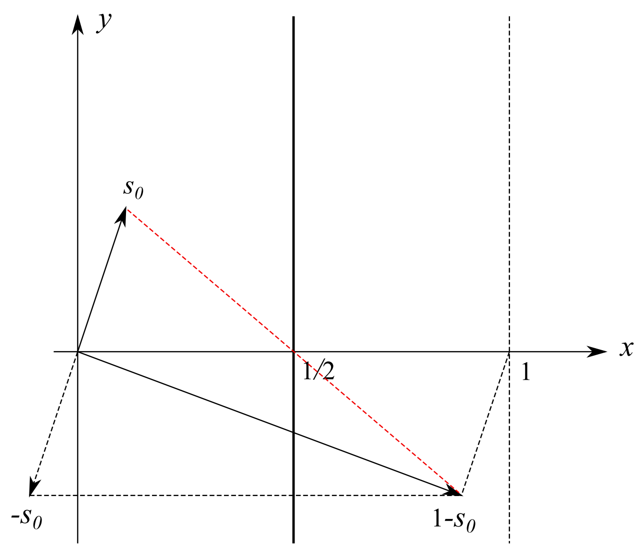

Figure 3: Symmetry of the distribution of zeros with respect to point $\left(\frac{1}{2}, 0\right)$.

The symmetries just examined imply that the non-trivial zeros are symmetric about the line $\operatorname{Re} s = 1/2$ and the real axis (fig.(4)).

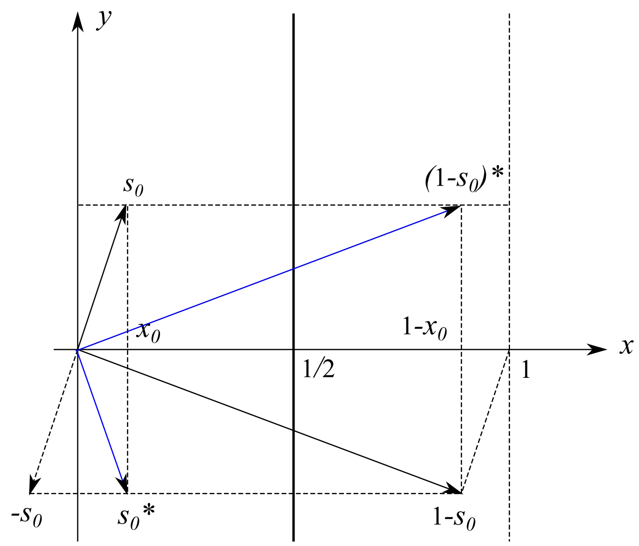

Figure 4: Symmetry of the distribution of non trivial zeros.

2 A remarkable integral representation

In Quantum Statistical Mechanics [5] the following generalized integrals which are not elementary expressible often appear

$$\int_0^{+\infty} \frac{t^{x-1} dt}{e^t \pm 1} \tag{9}$$

From known results:

$$\int_0^{+\infty} \frac{t^{x-1}dt}{e^t+1} = \left(1-2^{1-x}\right)\Gamma(x)\zeta(x), \quad \forall x \in (0,+\infty) \tag{10}$$
$$\int_0^{+\infty} \frac{t^{x-1}dt}{e^t-1} = \Gamma(x)\zeta(x), \quad \forall x \in (1,+\infty)$$

where $\zeta(x)$ is the Riemann zeta function $\zeta(s)$ evaluated for $\mathrm{Im}\, s = 0$. We rewrite the first of (10) for $\mathrm{Im}\, s \neq 0$:

$$\int_0^{+\infty} \frac{t^{s-1}dt}{e^t+1} = \left(1-2^{1-s}\right)\Gamma(s)\zeta(s), \qquad \mathrm{Re}\, s > 0 \tag{11}$$

In the integral we perform the change of variable $t = e^{t'}$, so

$$\int_0^{+\infty} \frac{t^{s-1}dt}{e^t+1} = \int_0^{+\infty} \frac{t^{x-1}t^{iyt}dt}{e^t+1} = \int_{-\infty}^{+\infty} \frac{e^{xt'}e^{-t'}e^{iyt'}e^{t'}}{e^{e^{t'}}+1}dt' = \int_{-\infty}^{+\infty} \frac{e^{xt'}}{e^{e^{t'}}+1}e^{iyt'}dt'$$

Redefining the variable $t' \equiv t$:

$$\int_0^{+\infty} \frac{t^{s-1}dt}{e^t+1} = \int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t}+1}e^{iyt}dt$$

so (11) becomes

$$\int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t}+1}e^{iyt}dt = \left(1-2^{1-s}\right)\Gamma(s)\zeta(s), \qquad \mathrm{Re}\, s > 0 \tag{12}$$

We define

$$f(x,t) = \frac{e^{xt}}{e^{e^t}+1}, \quad \begin{cases} x \in (0,1) & \text{parameter} \\ t \in (-\infty,+\infty) & \text{independent variable} \end{cases} \tag{13}$$

It follows (from (12)):

$$\hat{f}(s) \equiv \hat{f}(x+iy) = \int_{-\infty}^{+\infty} f(x,t)\, e^{iyt}dt = \left(1-2^{1-s}\right)\Gamma(s)\zeta(s), \quad \mathrm{Re}\, s > 0 \tag{14}$$

Notation 3 *The correct notation is*

$$\hat{f}(x,y) = \int_{-\infty}^{+\infty} f(x,t)\, e^{iyt}dt, \quad g(s) = \hat{f}(x+iy), \quad g: A \to \mathbb{C}$$

However to avoid a proliferation of symbols, we use the notation (14). So the symbols $\hat{f}(x,y)$, $\hat{f}(x+iy)$, $\hat{f}(s)$ *denote the same function.*

Proposition 4 *The function* $\hat{f}(s)$ *is holomorphic in the region*

$$A = \{s \in \mathbb{C} \mid 0 < \mathrm{Re}\, s < 1, \ -\infty < \mathrm{Im}\, s < +\infty\}$$

Proof. It immediately follows from the holomorphy of $\left(1-2^{1-s}\right)\Gamma(s)$ in the region A. ■

Lemma 5

$$\left|\left(1-2^{1-s}\right)\Gamma(s)\right| > 0, \quad \forall s \in A \tag{15}$$

Proof. The inequality (15) derives from the fact that the gamma function has no zeros [3], while $1-2^{1-s}$ is manifestly zero-free in A. ■

Theorem 6 $\hat{f}(s)$ *and* $\zeta(s)$ *have the same (non-trivial) zeros.*

Proof. It follows from the lemma 5. ■

The line $\mathrm{Re}\, s = 1/2$ is called *critical line*. G. H. Hardy [4] proved that infinitely many zeros fall on this line.

3 Riemann Hypothesis. Fourier Transform

From (14) we see that for a given $\bar{x} \in (0,1)$ the function $\hat{f}(y) \equiv (\bar{x}+iy)$ is the Fourier transform of (13). By a known property [6] $\hat{f}(y)$ is uniformly continuous in $(-\infty,+\infty)$. Also, by the inversion formula [7]:

$$\frac{e^{\bar{x}t}}{e^{e^t}+1} = \frac{1}{2\pi}\lim_{\delta\to+\infty}\int_{-\delta}^{\delta}\left(1-\frac{|y|}{\delta}\right)\hat{f}(y)\,e^{-iyt}dy \tag{16}$$

Conjecture 7 *(***Riemann Hypothesis – RH***)*
The non-trivial zeros of the Riemann zeta-function have real part $x = 1/2$.

From proposition 5 follows that the non-trivial zeros of the function

$$\hat{f}(x+iy) = \int_{-\infty}^{+\infty} f(x,t)\,e^{iyt}dt, \quad \text{with} \quad f(x,t) = \frac{e^{xt}}{e^{e^t}+1} \tag{17}$$

have real part $x = 1/2$.

Let us first study the behavior of the function

$$f(x,t) = \frac{e^{xt}}{e^{e^t}+1} \tag{18}$$

which for each value of the parameter $x \in (0,1)$ is defined in $(-\infty,+\infty)$.

Sign and intersections with the axes

It turns out $f(x,t) > 0$, $\forall t \in (-\infty,+\infty)$ for which the graph of f lies in the semi-plane of the positive ordinates. It does not intersect the abscissa axis, while it does intersect the ordinate axis at $\left(0,(e+1)^{-1}\right)$.

Behavior at extremes

After calculations:

$$\lim_{t\to+\infty} f(x,t) = 0^+, \quad \forall x \in (0,1)$$

The order of infinitesimal:

$$\lim_{t\to+\infty} t^\alpha f(x,t) = 0^+, \quad \forall\alpha > 0 \qquad \text{(infinitesimal of infinitely large order)} \tag{19}$$

$$\lim_{t\to-\infty} f(x,t) = \begin{cases} \frac{1}{2}^-, & \text{if } x=0 \\ 0^+, & \text{if } x>0 \end{cases} \tag{20}$$

Precisely:

$$\lim_{t\to-\infty} t^\alpha f(x>0,t) = 0^+, \quad \forall\alpha>0 \tag{21}$$

Conclusion: for $|t| \to +\infty$ the function $f(x>0,t)$ is an infinitesimal of order infinitely large, provided that it is $x > 0$.

First derivative

$$f'(x,t) \equiv \frac{\partial}{\partial t} f(x,t) = \frac{e^{xt}\left[x\left(e^{e^t}+1\right)-e^{t+e^t}\right]}{\left(e^{e^t}+1\right)^2}$$

For $x = 0$

$$f'(0,t) = -\frac{e^{t+e^t}}{\left(e^{e^t}+1\right)^2} < 0, \quad \forall t \in (-\infty,+\infty)$$

so the function is strictly decreasing.

For $x>0$

$$f'(x,t)=0 \Longleftrightarrow x\left(e^{e^t}+1\right)-e^{t+e^t}=0 \tag{22}$$

The roots of the transcendental equation (22) depend parametrically on x, so let's denote them by $t_*(x)$. For $x=1$:

$$t_*(1)\simeq 0.246$$

From (20):

$$\lim_{x\to 0^+} t_*(x)=-\infty$$

so that

$$0<x<1 \Longrightarrow -\infty<t_*(x)\lesssim 0.246 \tag{23}$$

$t_*(x)$ is a continuous function, so by the theorem of zeros:

$$\exists \xi\in(0,1)\mid t_*(\xi)=0$$

Numerically: $\xi\simeq 0.731$. Some values for assigned $x\in(0,1)$:

$$t_*\left(\frac{1}{5}\right)\simeq -1.07$$
$$t_*\left(\frac{1}{4}\right)\simeq -0.88$$
$$t_*\left(\frac{1}{2}\right)\simeq -0.30$$
$$t_*\left(\frac{2}{3}\right)\simeq -0.07$$
$$t_*\left(\frac{3}{4}\right)\simeq 0.02$$

The sign is

$$-\infty<t<t_*(x)\Longrightarrow f'(x,t)>0$$
$$t_*(x)<t<+\infty\Longrightarrow f'(x,t)<0$$

Hence the function is strictly increasing in $(-\infty,t_*(x))$ and it is strictly decreasing in $(t_*(x),+\infty)$. So $t_*(x)$ is a point of relative maximum for the function.

Second derivative

$$f''(x,t)=\frac{e^{xt}\left[e^{2\left(e^t+t\right)}-e^{e^t+2t}+x^2\left(1+e^{e^t}\right)^2-(2x+1)\left(e^{t+e^t}+e^{2e^t+t}\right)\right]}{\left(1+e^{e^t}\right)^3} \tag{24}$$

For $x=0$

$$f''(0,t)=\frac{e^{2\left(e^t+t\right)}-e^{e^t+2t}-\left(e^{t+e^t}+e^{2e^t+t}\right)}{\left(1+e^{e^t}\right)^3}$$

which has a zero in $t_*'(x=0)\simeq 0.43$. The sign is

$$-\infty<t<t_*'(x=0)\Longrightarrow f''(0,t)<0$$
$$t_*'(x=0)<t<+\infty\Longrightarrow f''(0,t)>0$$

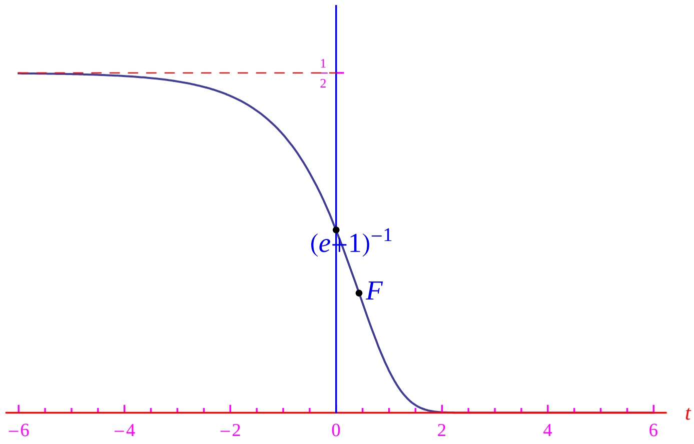

Figure 5: Trend of $f(0,t)$.

It follows that the graph of $f(0,t)$ is convex in $(-\infty, t'_*(x=0))$ and concave in $(t'_*(x=0), +\infty)$. So $(0.43, 0.18)$ is an inflection point with an oblique tangent. In fig. 5 we report the graph of $f(0,t)$.

For $x>0$ we perform a qualitative analysis. The parameter x decisively controls the slope of the graph of $f(t)$ in $(-\infty,0)$ since

$$\frac{\partial}{\partial t}e^{xt} = xe^{xt}$$

For $t \in (0,+\infty)$ the slope is controlled by e^{e^t} in denominator. This implies that the effects of the parameter x are felt for $t \in (-\infty,0)$, while in $(0,+\infty)$ the trend is practically independent of this parameter. Fig. 6 plots $f(x,t)$ for increasing values of the parameter x starting from $x=0$.

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By a known property of the Fourier transform [6], for a given value of x, the real function $\left|\hat{f}(x,y)\right|$ is limited. In fact, from (17):

$$\left|\hat{f}(x,y)\right| \leq \int_{-\infty}^{+\infty}\left|\frac{e^{xt}}{e^{e^t}+1}\right|dt = \int_{-\infty}^{+\infty}\frac{e^{xt}}{e^{e^t}+1}dt \overset{def}{=} F(x)$$

It follows

$$F(x) = \int_{-\infty}^{0}\frac{e^{xt}}{e^{e^t}+1}dt + \underbrace{\int_{0}^{+\infty}\frac{e^{xt}}{e^{e^t}+1}dt}_{\text{converges } \forall x\in\mathbb{R}}$$

For $x=0$

$$f(0,t) = \frac{1}{e^{e^t}+1} \underset{t\to-\infty}{\longrightarrow} \frac{1}{2} \Longrightarrow \int_{-\infty}^{0}\frac{dt}{e^{e^t}+1} = +\infty \Longrightarrow \lim_{x\to0^+}F(x) = +\infty$$

For $x>0$ the trend in $t\in(-\infty,0)$ is dominated by e^{xt}

$$\frac{e^{xt}}{e^{e^t}+1} \underset{t\to-\infty}{\longrightarrow} e^{xt}$$

so the integral converges. As x increases in $(0,1)$ the slope increases, and this favors the convergence of the integral[1], simultaneously decreases the area of the rectangleoid and therefore the value of $F(x)$. This shows that $F(x)$ is strictly decreasing, as confirmed by the graph fig. 7.

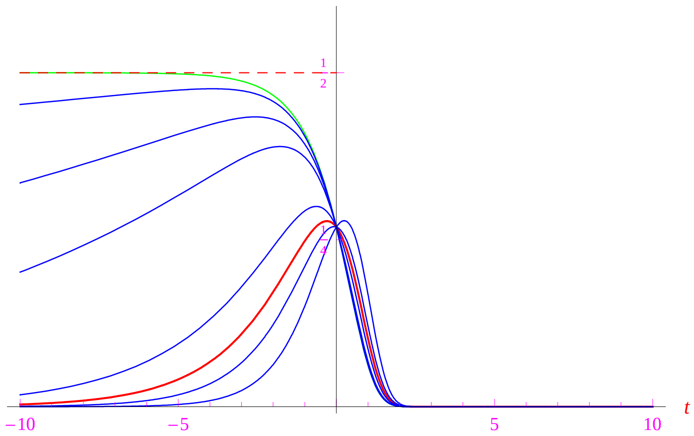

Figure 6: Trend of $f(x,t)$ for different values of x. Curve in green: $x=0$. The flattest curve towards the ordinate axis is for $x=1$.

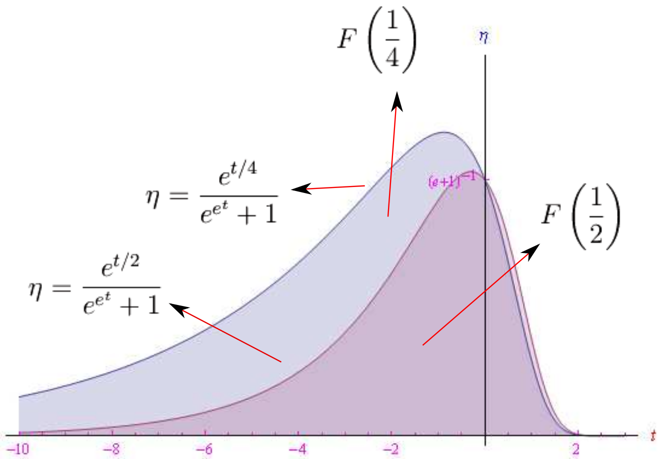

Figure 7: Geometric interpretation of $F(x)$ for $x=\frac{1}{4},\frac{1}{2}$. Note the decreasing trend.

A more quantitative analysis can be performed by numerically calculating the integral $F(x)=\int_{-\infty}^{+\infty}\frac{e^{xt}}{e^{e^t}+1}dt$ for an array of x values, or using the *Mathematica* built-in function `Zeta[x+iy]` for $y=0$ and taking into account the (12) for $y=\operatorname{Im}s=0$:

$$F(x)=\left(1-2^{1-x}\right)\Gamma(x)\,\zeta(x)$$

In other words, we graph with *Mathematica* the second member of (12). The result is in fig. 8.

[1]The parameter x therefore controls the speed of convergence of the integral in the interval $(-\infty,0)$.

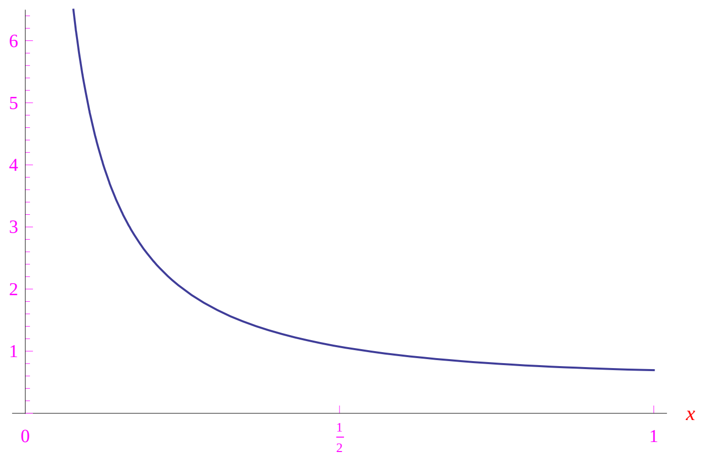

Figure 8: Trend of $F(x)$.

4 Zeros of the Fourier Transform

4.1 Introduction

The integral (17) can be seen as:

- complex function of the real variables (x, y) i.e. $\hat{f}(x, y)$;
- complex function of the complex variable $x + iy$;

Due to the symmetry property established in the number 1.2, we can limit the search for zeros in the region:

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, \ \ -\infty < y < +\infty\right\} \tag{25}$$

Search for zeros:

$$\hat{f}(x, y) = 0 \Longleftrightarrow \underbrace{\int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t} + 1} e^{iyt} dt}_{I_-(x,y)} + \underbrace{\int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t} + 1} e^{iyt} dt}_{I_+(x,y)} = 0 \tag{26}$$

As established in § 3

$$\begin{aligned} I_-(x, y) &= \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t} + 1} e^{iyt} dt \quad \text{converges if and only if } x > 0, \\ I_+(x, y) &= \int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t} + 1} e^{iyt} dt \ \ \text{converges for each } x \in \mathbb{R} \end{aligned} \tag{27}$$

We express the complex quantities $I_\pm(x, y)$ in polar representation:

$$\begin{aligned} I_\pm(x, y) &= F_\pm(x, y)\, e^{i\varphi_\pm(x,y)} \\ F_\pm(x, y) &= |I_\pm(x, y)| \, ; \ \ \varphi_\pm(x, y) = \arg I_\pm(x, y) \quad (0 \leq \varphi_\pm(x, y) < 2\pi) \end{aligned} \tag{28}$$

From (26):

$$\hat{f}(x, y) = 0 \Longleftrightarrow I_-(x, y) = -I_+(x, y) \tag{29}$$

Taking into account the (28):

$$\begin{cases} F_-(x,y) = F_+(x,y) \\ \varphi_-(x,y) = \pi + \varphi_+(x,y) \end{cases} \tag{30}$$

So if $s_0 = x_0 + iy_0$ is a zero of $\zeta(s)$, the ordered pair $(x_0, y_0) \in \mathbb{R}^2$ solves the system (30). It follows that the equality of the modules

$$F_-(x_0, y_0) = F_+(x_0, y_0) \tag{31}$$

expresses a necessary (but not sufficient) condition for $s_0 = x_0 + iy_0$ to be a zero of $\zeta(s)$.

4.2 Remarkable properties of $F_\pm(x,y)$

Promemoria:

$$F_-(x,y) = \left| \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} e^{iyt} dt \right|, \quad F_+(x,y) = \left| \int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t}+1} e^{iyt} dt \right| \tag{32}$$

Proposition 8 *The functions (32) are even with respect to the variable y.*

Proof. It follows immediately by expressing the exponential e^{iyt} with Euler's formula. ■

Notation 9 *Parity* $(+1)$ *is a general property of the modulus of a Fourier transform:*

$$\hat{f}(y) = \int_{-\infty}^{+\infty} f(t) e^{iyt} dt \Longrightarrow \left|\hat{f}(-y)\right| \equiv \left|\hat{f}(y)\right|$$

Proposition 10

$$\lim_{y \to \pm\infty} F_\pm(x,y) = 0 \tag{33}$$

Proof. It follows from a well-known property of Fourier transforms [6]:

$$\lim_{y \to \pm\infty} \left|\hat{f}(x,y)\right| = 0 \tag{34}$$

Alternatively: for an assigned $x_0 \in (0,1)$

$$\begin{aligned} \left| \int_{-\infty}^{0} \frac{e^{x_0 t}}{e^{e^t}+1} e^{iyt} dt \right| &= \left| \int_{-\infty}^{0} \frac{e^{x_0 t}}{e^{e^t}+1} \cos(yt)\, dt + i \int_{-\infty}^{0} \frac{e^{x_0 t}}{e^{e^t}+1} \sin(yt)\, dt \right| \\ &= \left|\tilde{g}_1(y) + \tilde{g}_2(y)\right| \end{aligned}$$

where

$$\begin{aligned} \tilde{g}_1(y) &= \int_{-\infty}^{0} \frac{e^{x_0 t}}{e^{e^t}+1} \cos(yt)\, dt \\ \tilde{g}_2(y) &= \int_{-\infty}^{0} \frac{e^{x_0 t}}{e^{e^t}+1} \sin(yt)\, dt \end{aligned} \tag{35}$$

It suffices to prove $\lim_{y\to\pm\infty} \tilde{g}_1(y) = \lim_{y\to\pm\infty} \tilde{g}_2(y) = 0$. Furthermore, taking into account the proposition 8, it suffices to refer to the case $y \to +\infty$. For this purpose we arbitrarily take $\varepsilon > 0$,then we impose

$$\tilde{g}_1(y) = \varepsilon$$

which determines $\delta_\varepsilon > 0$

$$\tilde{g}_1(\delta_\varepsilon) = \varepsilon \Longleftrightarrow \int_{-\infty}^{0} \frac{e^{x_0 t}}{e^{e^t}+1} \cos(\delta_\varepsilon t)\, dt = \varepsilon$$

We have to show that

$$y > \delta_\varepsilon \Longrightarrow \tilde{g}_1(y) = \left| \int_{-\infty}^{0} \frac{e^{x_0 t}}{e^{e^t}+1} \cos(yt)\, dt \right| < \varepsilon$$

For this purpose we consider the integrand function

$$\psi(y,t) = \frac{e^{x_0 t}}{e^{e^t}+1} \cos(yt)\, dt \tag{36}$$

which for a given y is a cosine oscillation between the curves of equation $\eta = \pm \frac{e^{x_0 t}}{e^{e^t}+1}$ as can be seen in fig. 9. As y increases, the «density» of the number of oscillations increases as we can see from the graph in fig. 10.

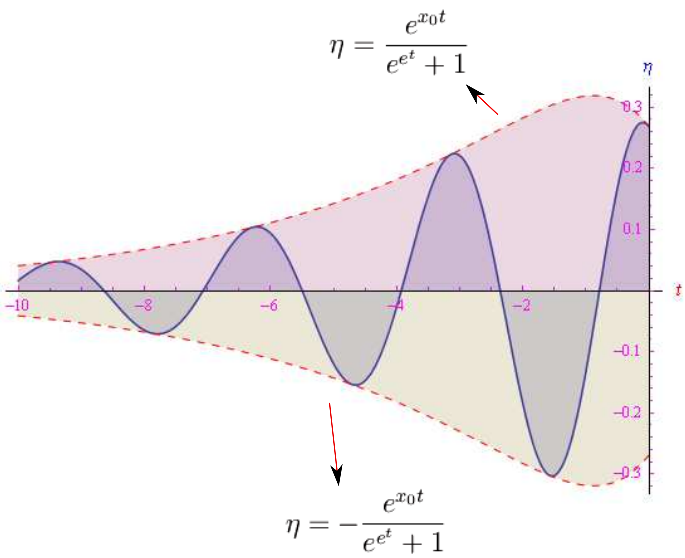

Figure 9: Trend of $\psi(y,t) = \frac{e^{x_0 t}}{e^{e^t}+1} \cos(yt)\, dt$ for $x_0 = \frac{1}{4}$, $y = 2$.

It follows a reduction of the area of the rectangleoid and therefore of $\tilde{g}_1(y)$. For $y \to +\infty$ the predicted density diverges positively and the area of the rectangleoid tends to zero. So:

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0 \mid y > \delta_\varepsilon \Longrightarrow \tilde{g}_1(y) < \varepsilon$$

i.e.

$$\lim_{y \to +\infty} \tilde{g}_1(y) = 0$$

In a similar way we arrive at $\lim_{y \to +\infty} \tilde{g}_2(y) = 0$. ■

Proposition 11 *The functions $F_\pm(x,y)$ are analytic in A.*

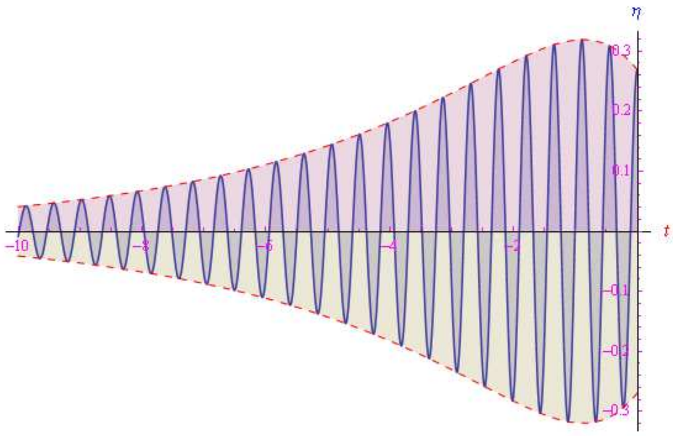

Figure 10: Trend of $\psi(y,t) = \frac{e^{x_0 t}}{e^{e^t}+1}\cos(yt)\,dt$ for $x_0 = \frac{1}{4}$, $y = 14$.

Proof. From the holomorphy of the function

$$\hat{f}(x+iy) = \int_{-\infty}^{+\infty} \frac{e^{xt}}{e^{e^t}+1} e^{iyt} dt$$

follows the analyticity of real functions $u(x,y) = \mathrm{Re}\,\hat{f}$, $v(x,y) = \mathrm{Im}\,\hat{f}$ [9]. Dalla (26):

$$\left|\hat{f}(x+iy)\right|^2 = \left|I_-(x,y)^2 + I_+(x,y)\right|^2$$

After some algebra:

$$u(x,y)^2 + v(x,y)^2 = F_-(x,y)^2 + F_+(x,y)^2 + 2J(x,y) \tag{37}$$

where

$$J(x,y) = [\mathrm{Re}\,I_-(x,y)]\,[\mathrm{Re}\,I_+(x,y)] + [\mathrm{Im}\,I_-(x,y)]\,[\mathrm{Im}\,I_+(x,y)]$$

For the above, the first member function of (37) is analytic, hence the analyticity of the sum $F_-(x,y)^2 + F_+(x,y)^2 + 2J(x,y)$ and therefore, some $F_\pm(x,y)$. ■

Proposition 12 *For a given $y \in \mathbb{R}$, the function $F_-(x,y)$ is monotonically decreasing in $(0,1)$.*

Proof. Given arbitrarily $y_0 \in \mathbb{R}$, let's say:

$$f_-(x) = F_-(x,y_0) = \left|\int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} e^{iy_0 t} dt\right| \tag{38}$$

If $y_0 = 0$

$$f_-(x) = \left|\int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} dt\right| = \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} dt$$

Derivating with respect to x and taking into account the uniform convergence of the integral:

$$f_-'(x) = \int_{-\infty}^{0} \frac{te^{xt}}{e^{e^t}+1} dt < 0, \quad \forall x \in (0,1)$$

so $F_-(x,0)$ is monotonically decreasing in $(0,1)$.

For $y_0 \neq 0$, expanding the imaginary exponential we have::

$$f_-(x) = |g_1(x) + ig_2(x)|$$

where

$$g_1(x) = \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} \cos(y_0 t)\, dt \tag{39}$$
$$g_2(x) = \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} \sin(y_0 t)\, dt$$

So

$$f_-(x) = +\sqrt{g_1(x)^2 + g_2(x)^2}$$

It suffices to show that $g_1(x)$ and $g_2(x)$ are monotonically decreasing in $(0,1)$. Precisely, for an assigned $t<0$, however we take $x', x'' \in (0,1)$ with $x'' > x'$, we have:

$$e^{x''t} < e^{x't} \Longrightarrow \frac{e^{x''t}}{e^{e^t}+1} < \frac{e^{x''t}}{e^{e^t}+1} \Longrightarrow \int_{-\infty}^{0} \frac{e^{x''t}}{e^{e^t}+1} \cos(y_0 t)\, dt < \int_{-\infty}^{0} \frac{e^{x't}}{e^{e^t}+1} \cos(y_0 t)\, dt$$

so $g_1(x)$ is monotonically decreasing. This conclusion is corroborated by the graph of fig. 11.

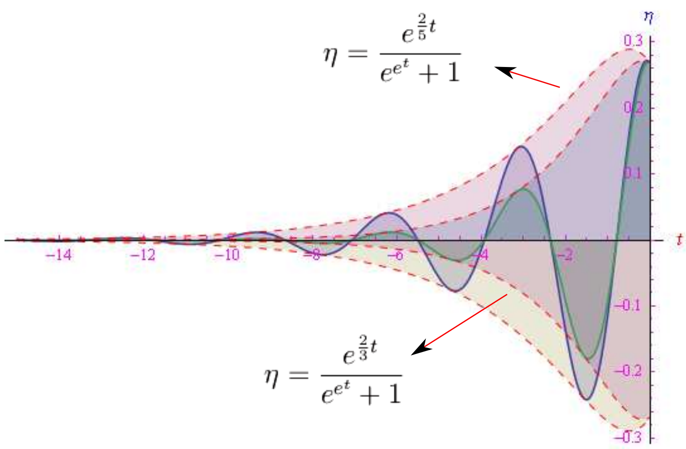

Figure 11: Trend of the integrand function of $g_1(x)$ respectively for $x = 2/5$ and $x = 2/3$, and for $y_0 = 2$. The value assumed by $g_1(x)$ for these values of x, is the area of the rectangleoid related to the sinusoidal oscillations. As x increases, these oscillations reduce in amplitude so that the area decreases.

We proceed in a similar way for $g_2(x)$. ■

Proposition 13 *For a given $y \in \mathbb{R}$, the function $F_+(x,y)$ is monotonically increasing in $(0,1)$.*

Proof. Given arbitrarily $y_0 \in \mathbb{R}$, let's say:

$$f_+(x) = F_+(x, y_0) = \left| \int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t}+1} e^{iy_0 t} dt \right| \tag{40}$$

If $y_0 = 0$

$$f_+(x) = \left| \int_0^{+\infty} \frac{e^{xt}}{e^{e^t}+1} dt \right| = \int_0^{+\infty} \frac{e^{xt}}{e^{e^t}+1} dt$$

Derivating with respect to x and taking into account the uniform convergence of the integral:

$$f'_-(x) = \int_0^{+\infty} \frac{te^{xt}}{e^{e^t}+1} dt > 0, \quad \forall x \in (0,1)$$

so $F_+(x,0)$ is monotonically increasing in $(0,1)$.

For $y_0 \neq 0$

$$f_+(x) = |h_1(x) + ih_2(x)|$$

where

$$h_1(x) = \int_0^{+\infty} \frac{e^{xt}}{e^{e^t}+1} \cos(y_0 t)\, dt \qquad (41)$$
$$h_2(x) = \int_0^{+\infty} \frac{e^{xt}}{e^{e^t}+1} \sin(y_0 t)\, dt$$

So

$$f_+(x) = +\sqrt{h_1(x)^2 + h_2(x)^2}$$

It suffices to show that $h_1(x)$ and $h_2(x)$ are monotonically increasing in $(0,1)$. Precisely, for an assigned $t > 0$, however we take $x', x'' \in (0,1)$ with $x'' > x'$, we have:

$$e^{x''t} > e^{x't} \Longrightarrow \frac{e^{x''t}}{e^{e^t}+1} > \frac{e^{x''t}}{e^{e^t}+1} \Longrightarrow \int_0^{+\infty} \frac{e^{x''t}}{e^{e^t}+1} \cos(y_0 t)\, dt > \int_0^{+\infty} \frac{e^{x't}}{e^{e^t}+1} \cos(y_0 t)\, dt$$

so $h_1(x)$ is monotonically increasing. This conclusion is corroborated by the graph of fig. 12.

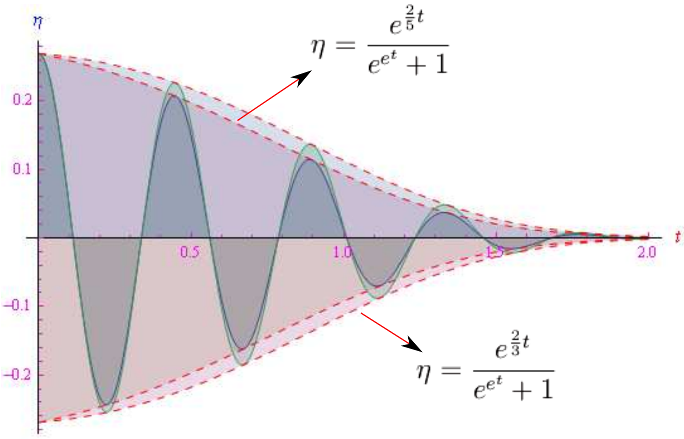

Figure 12: Trend of the integrand function of $h_1(x)$ respectively for $x = 2/5$ e $x = 2/3$, and for $y_0 = 14$. The value assumed by $h_1(x)$ for these values of x, is the area of therectangleoid related to the sinusoidal oscillations. As x increases, these oscillations grow in amplitude so that the area increases.

We proceed in a similar way for $h_2(x)$. ■

Proposition 14 *The functions $F_{\pm}(x,y)$ have no zeros.*

Proof. Proceeding by absurdity

$$\exists\,(\xi,\eta)\in A \mid F_{-}(\xi,\eta)=0 \tag{42}$$

Since $F_{-}(x,\eta)$ is monotonically decreasing for $x\in(0,1)$ (proposition 12), the (42) implies

$$\begin{aligned} F_{-}(x,\eta) > 0 \quad &\text{for} \quad 0<x<\xi \\ F_{-}(x,\eta) < 0 \quad &\text{for} \quad \xi<x<1 \end{aligned}$$

The second is absurd since F_{-} is nonnegative. We proceed in a similar way for F_{+}. The absurd proves the assertion. ■

From the proposition just proved it follows that the only zeros of $F_{\pm}(x,y)$ are at infinity in the y coordinate (proposition 10).

4.3 Study of surfaces $S_\pm$

As established in the section 4.1, for the search for the zeros of $\hat{f}(x,y)$ we must impose

$$F_-(x,y) = F_+(x,y) \Longleftrightarrow \left|\int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} e^{iyt} dt\right| = \left|\int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t}+1} e^{iyt} dt\right| \tag{43}$$

From the impossibility of solving the equation (43) follows the need to force its solutions by examining the intersection of the two open surfaces S_- and S_+, of cartesian representation:

$$S_\pm : z = F_\pm(x,y), \quad (x,y) \in A \tag{44}$$

having assigned an orthogonal cartesian reference$\mathcal{R}(Oxyz)$. An obvious parametrization of $S_\pm$ is

$$x = u, \quad y = v, \quad z = F_\pm(u,v), \quad (u,v) \in A$$

whose Jacobian matrix is:

$$J_\pm(u,v) = \begin{pmatrix} 1 & 0 & \frac{\partial F_\pm}{\partial u} \\ 0 & 1 & \frac{\partial F_\pm}{\partial v} \end{pmatrix} \Longrightarrow rank\left(J_\pm(u,v)\right) = 2, \ \ \forall (u,v) \in A \tag{45}$$

From the proposition 11 the functions $F_\pm(u,v)$ are analytic, so taking into account (45) we have that the surfaces $S_\pm$ they are regular analytics[2].

From the proposition 8 follows the symmetry of $S_\pm$ with respect to the y axis. Furthermore, $S_\pm$ are plotted in the half-space $z > 0$. Inequality in the strict sense is a consequence of the proposition 14.

From the proposition 10 it follows that for $y \to \pm\infty$ the surfaces $S_\pm$«recline» on the coordinate plane xy.

An obvious implicit representation of $S_\pm$ is

$$G_\pm(x,y,z) = 0 \tag{46}$$

being $G_\pm(x,y,z) = F_\pm(x,y) - z$ defined in $B = A \times [0,+\infty)$. From the regularity of $S_\pm$ [8]

$$\boldsymbol{\nabla} G_\pm(x,y,z) \neq \mathbf{0}, \quad \forall (x,y,z) \in B$$

Thus the normal unit vector fields for both surfaces are uniquely determined.

$$\mathbf{n}_1^{(\pm)}(x,y,z) = \frac{\boldsymbol{\nabla} G_\pm(x,y,z)}{|\boldsymbol{\nabla} G_\pm(x,y,z)|}, \ \ \mathbf{n}_2^{(\pm)}(x,y,z) = -\frac{\boldsymbol{\nabla} G_\pm(x,y,z)}{|\boldsymbol{\nabla} G_\pm(x,y,z)|}, \quad \forall (x,y,z) \in B$$

hence the adjustability of $S_\pm$.

Given this, we study the sections of these surfaces or the intersections with planes parallel to the coordinate planes. We start with the intersection of S_- with a plane π_0 parallel to the coordinate plane xz, so its equation is $y = y_0$ with y_0 assigned arbitrarily. Let γ_- be the orthogonal projection of this intersection on the xz plane. By varying y_0 we obtain the family of plane curves with one parameter:

$$\mathcal{F}_- = \{\gamma_- : z = F_-(x,y)\} \tag{47}$$

[2]A surface is called regular analytic if however we take its regular parametric representation $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $(u,v) \in \mathcal{B} \subseteq \mathbb{R}$, the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are analytic in $\mathcal{B}$.

where the single curves have in common the asymptote $x = 0$, since

$$\lim_{x \to 0^+} F_-(x, y) = +\infty, \qquad \forall y \in \mathbb{R} \tag{48}$$

From the first of (32):

$$F_-(x,y) \leq \int_{-\infty}^{0} \left| \frac{e^{xt}}{e^{e^t}+1} e^{iyt} \right| dt = \int_{-\infty}^{0} \left| \frac{e^{xt}}{e^{e^t}+1} \right| \underbrace{\left| e^{iyt} \right|}_{=1} dt = \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} dt = F_-(x,0)$$

Furthermore

$$\sup_{\mathbb{R}} \left(\frac{1}{e^{e^t}+1} \right) = \frac{1}{2} \Longrightarrow \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} dt < \frac{1}{2} \int_{-\infty}^{0} e^{xt} dt = \frac{1}{2x}$$

So

$$0 < F_-(x,y) \leq F_-(x,0) < \frac{1}{2x}, \quad \forall x \in (0,1) \tag{49}$$

From (49) it follows that the curves γ_- are contained in the internally connected domain which is identified with the rectangleoid related to $F_-(x,0)$ of basis $(0,1)$:

$$\mathcal{D}_- = \left\{ (x,y) \in \mathbb{R}^2 \mid 0 < x < 1, \ 0 \leq y \leq F_-(x,0) \right\} \tag{50}$$

The curve $z = F_-(x,0)$ is manifestly the intersection of S_- with the coordinate plane xz. Note that $F_-(x,0)$ can be evaluated exactly for $x = 1$. In fact, by means of an elementary substitution we arrive at:

$$F_-(1,0) = \int_{-\infty}^{0} \frac{e^t dt}{e^{e^t}+1} = 1 + \ln 2 - \ln(1+e) \simeq 0.38 \tag{51}$$

For each $x \in (0,1)$ the function can only be determined numerically, obtaining the trend plotted in fig. 13.

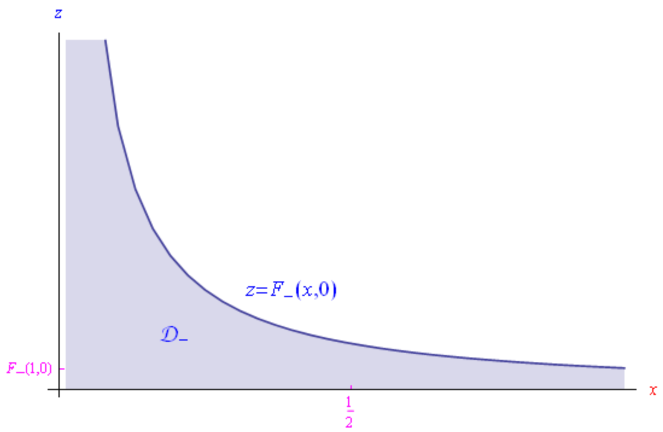

Figure 13: Trend of $F_-(x,0)$. The curvers γ_- curves are plotted in the domain $\mathcal{D}_-$.

We denote by γ_+ the orthogonal projections of the intersections of S_+ with π_0. It follows that whatever the value of the parameter y, we have:

$$\gamma_+ : z = F_+(x,y)$$

and therefore constituting a family $\mathcal{F}_+$ of plane curves with one parameter. It turns out:

$$F_+(x,y) = \left| \int_0^{+\infty} \frac{e^{xt}}{e^{e^t}+1} e^{iyt} dt \right| \leq F_+(x,0) \tag{52}$$

From (52) it follows that the curves γ_+ are contained in the internally connected domain which is identified with the rectangleoid related to $F_+(x,0)$ of basis $(0,1)$:

$$\mathcal{D}_+ = \left\{ (x,y) \in \mathbb{R}^2 \mid 0 < x < 1, \ 0 \leq y \leq F_+(x,0) \right\} \tag{53}$$

where the curve $z = F_+(x,0)$ is the intersection of S_+ with the xz plane. $F_+(x,0)$ can also be evaluated exactly for $x = 1$:

$$F_+(1,0) = \int_0^{+\infty} \frac{e^t dt}{e^{e^t}+1} = -1 + \ln(1+e) \simeq 0.31 \tag{54}$$

For each $x \in (0,1)$ the function can be determined only numerically, obtaining the trend plotted in fig. 14. In fig. 15 we report the trend of both curves.

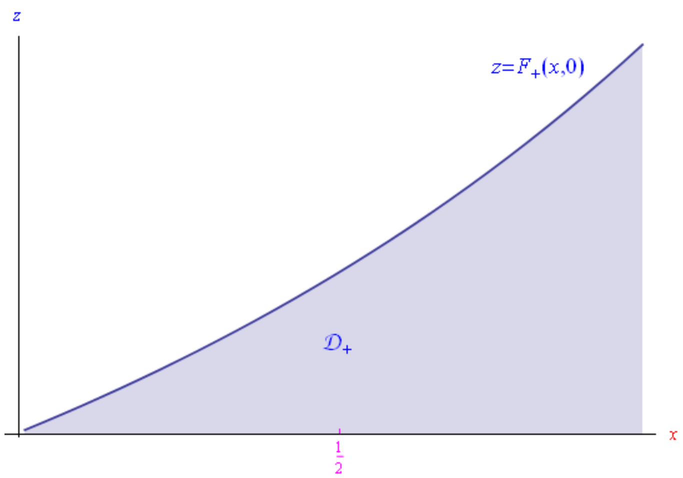

Figure 14: Trend of $F_+(x,0)$. The curvers γ_+ curves are plotted in the domain $\mathcal{D}_-$. $\mathcal{D}_+$.

Since $S_\pm$ are symmetrical with respect to the xz plane, y and $-y$ identify the same curve:

$$\gamma_\pm : z = F_\pm(x,y) \equiv F_\pm(x,-y)$$

Furthermore, taking into account the propositions 12-13, we have:

$$0 < F_+(x,y) \leq F_+(x,0) < F_+(1,0) < F_-(1,0) < F_-(x,0), \quad 0 < x < 1$$

from which

$$F_+(x,0) < F_-(x,0), \quad \forall x \in (0,1)$$

It follows the non-existence of trivial zeros in the critical strip. This confirms a known result [1][2].

We arrive at the same conclusion by looking for the solutions of the equation in x

$$F_-(x,0) = F_+(x,0) \Longleftrightarrow \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} dt = \int_0^{+\infty} \frac{e^{xt}}{e^{e^t}+1} dt$$

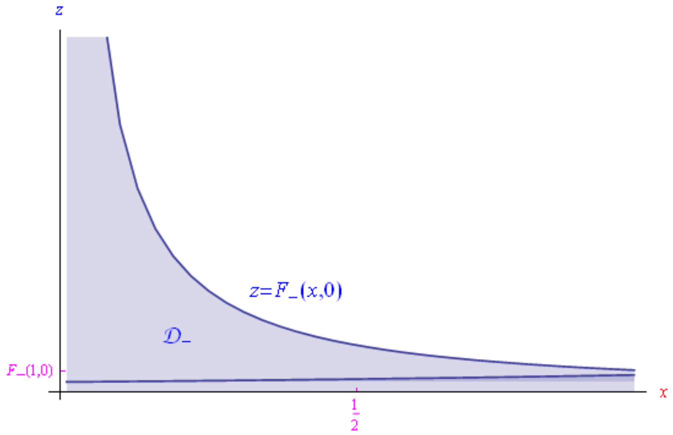

Figure 15: Trend of $F_{\pm}(x,0)$. The curve below has the equation $z = F_{+}(x,0)$ and the rectangoloid related to this function is $\mathcal{D}_{+}$ (in fig.it is not indicated for graphical reasons).

Turns out

$$\nexists x \in (0,1) \mid \int_{-\infty}^{0} \frac{e^{xt}}{e^{e^t}+1} dt = \int_{0}^{+\infty} \frac{e^{xt}}{e^{e^t}+1} dt$$

since $mis\,(\mathcal{R}_{-}) > mis\,(\mathcal{R}_{+})$ where

$$\begin{aligned} \mathcal{R}_{-} &= \left\{ (t,\eta) \in \mathbb{R} \mid -\infty < t \leq 0,\ \ 0 \leq \eta \leq \frac{e^{xt}}{e^{e^t}+1} \right\} \\ \mathcal{R}_{+} &= \left\{ (t,\eta) \in \mathbb{R} \mid 0 \leq t < +\infty,\ \ 0 \leq \eta \leq \frac{e^{xt}}{e^{e^t}+1} \right\} \end{aligned} \tag{55}$$

as we see in fig. 16.

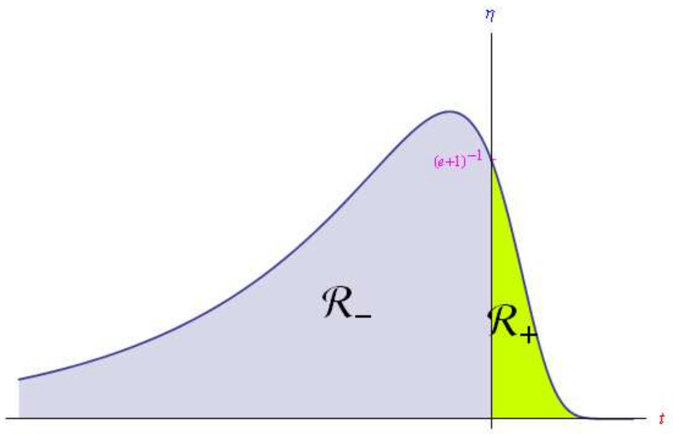

Figure 16: The regions (55) for an assigned $x \in (0,1)$.

We conclude the study of the intersections of $S_{\pm}$ with planes parallel to the coordinate plane xz, observing that the monotonicity of the corresponding functions is exactly what we expect, since the assumed values are correlated to the absolute maximum of the module $\left|\hat{f}(x,y)\right|$ of the Fourier transform as a function of x. Recalling that the «dominant» integral is the one relating to the interval $(-\infty,0)$ it follows that this amplitude diverges as $x \to 0^{+}$ to then become monotonically decreasing as $0 < x < 1$.

***

As regards the intersections of $S_\pm$ with planes parallel to the yz coordinate plane, we limit ourselves to observing that qualitatively we have the typical oscillating behavior of a Fourier transform.

Finally, the study of the intersections of $S_\pm$ with planes parallel to the xy plane (contour lines) is impractical due to the implicit representation of these geometric loci:

$$\left|\int_{-\infty}^{0}\frac{e^{xt}}{e^{e^t}+1}e^{iyt}dt\right| = C_-, \quad \left|\int_{0}^{+\infty}\frac{e^{xt}}{e^{e^t}+1}e^{iyt}dt\right| = C_+$$

dove $C_\pm > 0$.

5 Proof of RH

Lemma 15 *Given arbitrarily* $\bar{y} \in \mathbb{R}$, *the set of points of intersection of the curves*

$$\bar{\gamma}_- : z = F_-(x, \bar{y})$$
$$\bar{\gamma}_+ : z = F_+(x, \bar{y}),$$

either it is the empty set or it reduces to a single point.

Proof. It follows immediately from the monotonicity property proved in the propositions 12-13. Graphically in fig. 17. ■

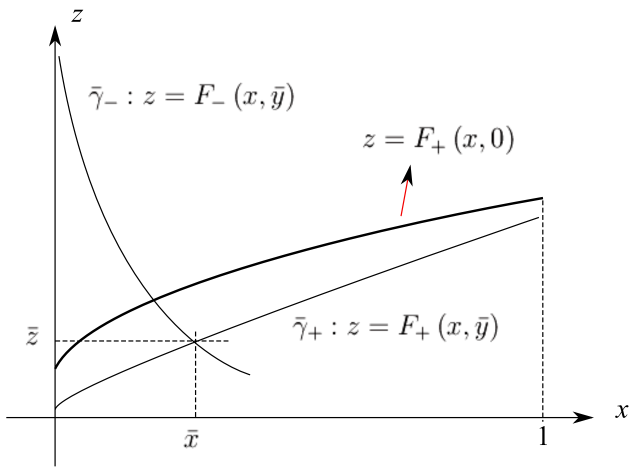

Figure 17: Proof of lemma 15:if the intersection exists, it is unique.

It is known that $\zeta(s)$ has infinitely many non-trivial zeros in A. It follows that $\hat{f}(s)$ has infinitely many non-trivial zeros in A.

Theorem 16 *The non-trivial zeros of* $\hat{f}(s)$ *have real part* $1/2$.

Proof. Absurdly: (x_0, y_0) is a non-trivial zero with $x_0 \neq 1/2$. This implies

$$F_-(x_0, y_0) = F_+(x_0, y_0)$$

The curves

$$\gamma_-^{(0)} : z = F_-(x, y_0) \tag{56}$$
$$\gamma_+^{(0)} : z = F_+(x, y_0).$$

intersect[3] at (x_0, z_0), where $z_0 = F_-(x_0, y_0) = F_+(x_0, y_0) > 0$. By the previous lemma the point (x_0, z_0) is unique. But this contradicts the symmetry property of the distribution of zeros according to which $(1 - x_0, y_0)$ is still a zero and therefore $(1 - x_0, z_0)$ is a second point of intersection of the curves (56), hence the assertion.

Graphically in figg. 18-19-20. ■

[3] By the proposition 14 the functions $F_\pm$ are finitely zero-free, so $z_0 > 0$.

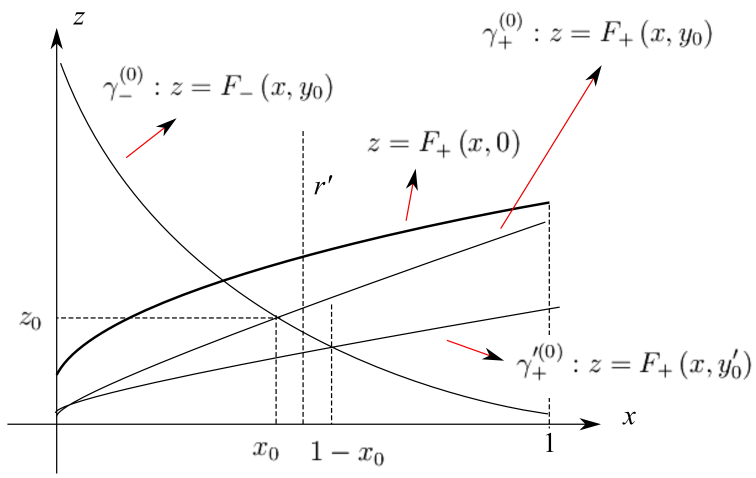

Figure 18: Proof of the theorem 18. The intersection between the curves (56) occurs at the single point (x_0, z_0). But for the symmetry of the zeros, there must be a second point of intersection $(1 - x_0, z_0)$ between the same curves. If such an intersection exists, it necessarily occurs with another curve identified by $y_0' \neq y_0$.

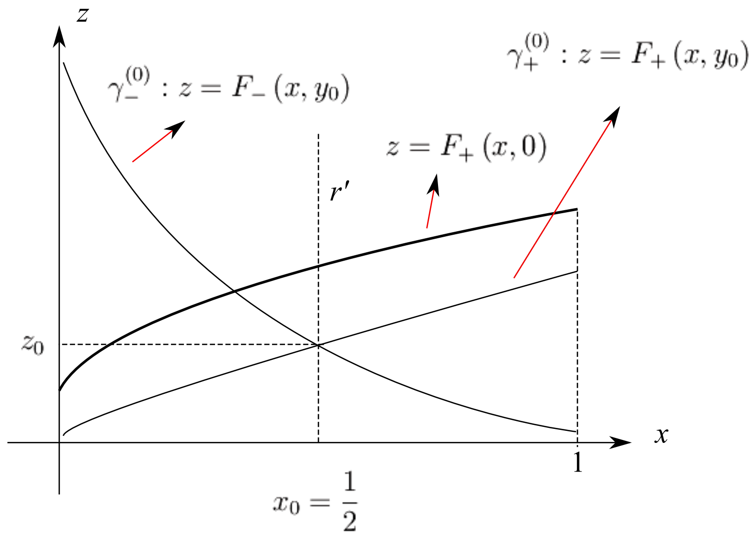

Figure 19: Proof of the theorem 18. As established in fig. 18, the only intersection to which a zero of $\hat{f}(x, y)$ corresponds can only occur on the straight line $r' : x = 1/2$. Precisely, on the segment $\left\{(x, z) \in \mathbb{R}^2 \mid x = \frac{1}{2},\, 0 < z < F_+\left(\frac{1}{2}, 0\right)\right\}$.

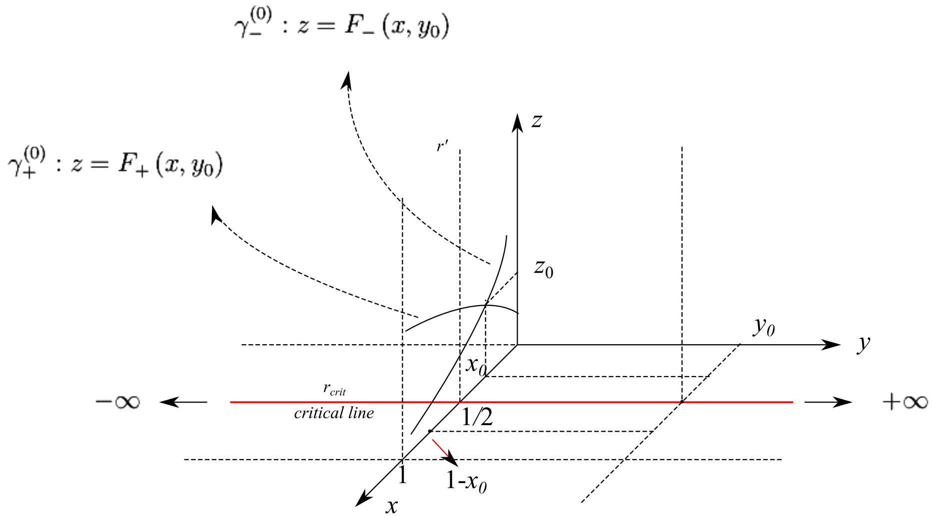

Figure 20: Proof of the theorem 18. Three-dimensional representation.

6 Conclusion

The conclusion of our work is graphically interpreted in fig. 21.

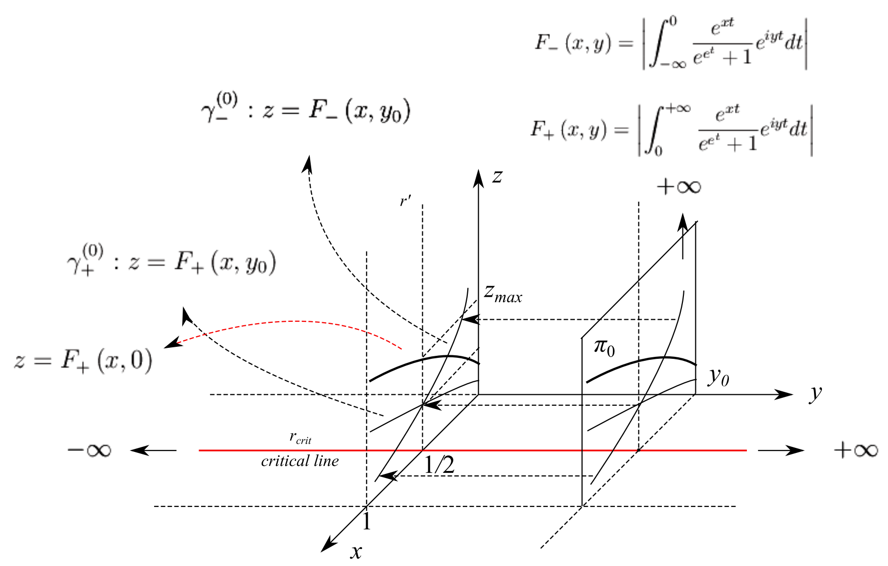

Figure 21: Graphic interpretation.

We now premise the following theorem for the proof of which we refer to [9].

Theorem 17 *The function $g(s)$ is holomorphic and not identically zero in a connected field T.*

The derivative of the set of zeros of $g(s)$ belonging to T, is contained in ∂T.

Roughly speaking, the set of zeros of a holomorphic function in a connected field T is at most countably infinite, and any accumulation points belong to the boundary of T.

By the theorem 16 the intersections of the cross sections of $S_\pm$ which give rise to non trivial zeros are realized only for $x = 1/2$, corresponding to the critical line. As previously

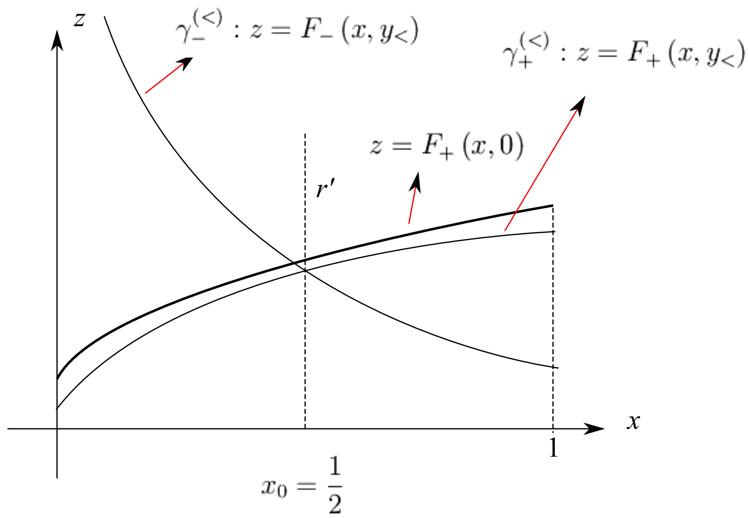

Figure 22: Value $y_<$ of parameter y.

established, there is necessarily a minimum value $y_<$ of the parameter y corresponding to the first intersection (fig. 22).

It follows that the orthogonal projection $\mathcal{C}_0'$ on the xy plane of the place $\mathcal{C}_0$ of intersection of S_- with S_+, is :

$$\mathcal{C}_0' = \left\{ (x, z) \in A \mid x = \frac{1}{2}, \quad |y| > y_> \right\} \tag{57}$$

where the presence of the absolute value derives from the symmetry of the $S_\pm$ with respect to the xz plane. From (57) we see that $\mathcal{C}_0'$ is the critical line without the open segment of extremes $\left(\frac{1}{2}, \pm y_>\right)$ (fig. 23). In fig. 24 we report the qualitative trend of the projection $\mathcal{C}_0''$ of $\mathcal{C}_0$ on the coordinate plane zy.

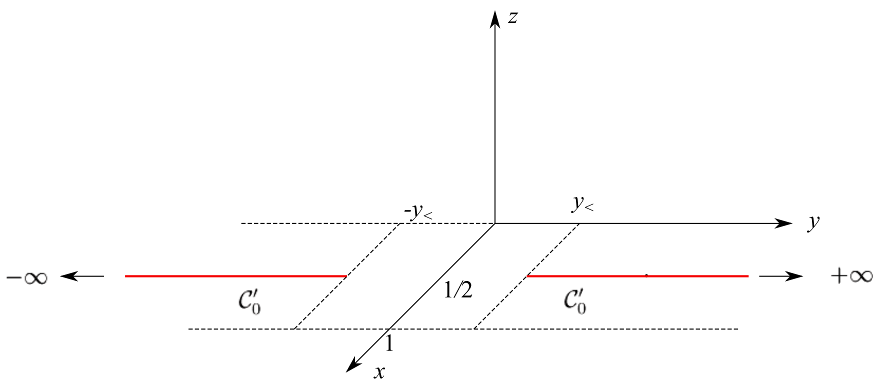

Figure 23: Projection of the point of intersection of $S_-$with S_+ on the xy coordinate plane.

By the Hardy-Littlewood theorem [4] on the critical line there are infinitely many zeros of $\zeta(s)$ and therefore of $\hat{f}(s)$. Denoting this set with H we have $H \subset \mathcal{C}_0'$ with H countably infinite by virtue of the theorem 17.

If $s_0 = \frac{1}{2} + iy_0$ is an element of H i.e. $\hat{f}\left(\frac{1}{2}, y_0\right) = 0$, for the (30)

$$\begin{cases} F_-\left(\frac{1}{2}, y_0\right) = F_+\left(\frac{1}{2}, y_0\right) \\ \varphi_-\left(\frac{1}{2}, y_0\right) = \pi + \varphi_+\left(\frac{1}{2}, y_0\right) \end{cases} \tag{58}$$

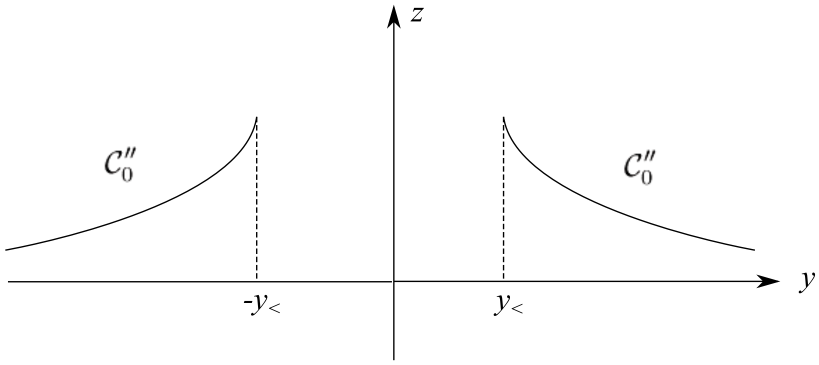

Figure 24: Projection of the point of intersection of $S_-$with S_+ on the zy coordinate plane.

In other words, the condition $F_-(x,y) = F_+(x,y)$ generates the set of points $\mathcal{C}_0$ and therefore the set $\mathcal{C}_0'$. The condition on the phase $\varphi_-\left(\frac{1}{2}, y_0\right) = \pi + \varphi_+\left(\frac{1}{2}, y_0\right)$ whatever $\left(\frac{1}{2}, y_0\right) \in H$, generates the countability of H. We note incidentally that by virtue of the proposition 14 the case $F_\pm\left(\frac{1}{2}, y_0\right) = 0$ never arises, which would reduce the equation $F_-(x,y) = F_+(x,y)$ to the identity $0 = 0$ and to the condition on the phase to an indeterminacy.

Recalling that $F_\pm(x,y) = |I_\pm(x,y)|$ we have the graphical representation of fig. 25 in the complex plane containing the co-domain of the functions

$$I_\pm(x,y) : A \to \mathbb{C}$$

Let $y_< < y_0' \neq y_0$ such that $s_0 = \left(\frac{1}{2} + iy_0'\right) \notin H$ i.e. $\hat{f}\left(\frac{1}{2}, y_0'\right) \neq 0$. We still have $F_-\left(\frac{1}{2}, y_0'\right) = F_+\left(\frac{1}{2}, y_0'\right)$ but

$$\varphi_-\left(\frac{1}{2}, y_0'\right) \neq \pi + \varphi_+\left(\frac{1}{2}, y_0'\right)$$

as illustrated in fig. 26.

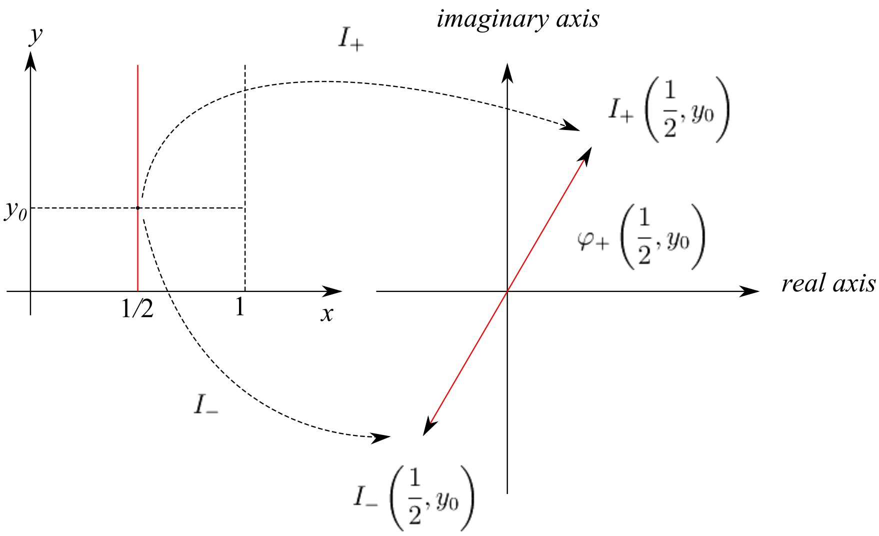

Figure 25: The complex numbers $I_-\left(\frac{1}{2}, y_0\right) = \int_{-\infty}^{0} \frac{e^{t//2}}{e^{e^t}+1} e^{iy_0 t} dt$, $I_+\left(\frac{1}{2}, y_0\right) = \int_0^{+\infty} \frac{e^{t//2}}{e^{e^t}+1} e^{iy_0 t} dt$ have phases that differ by π.

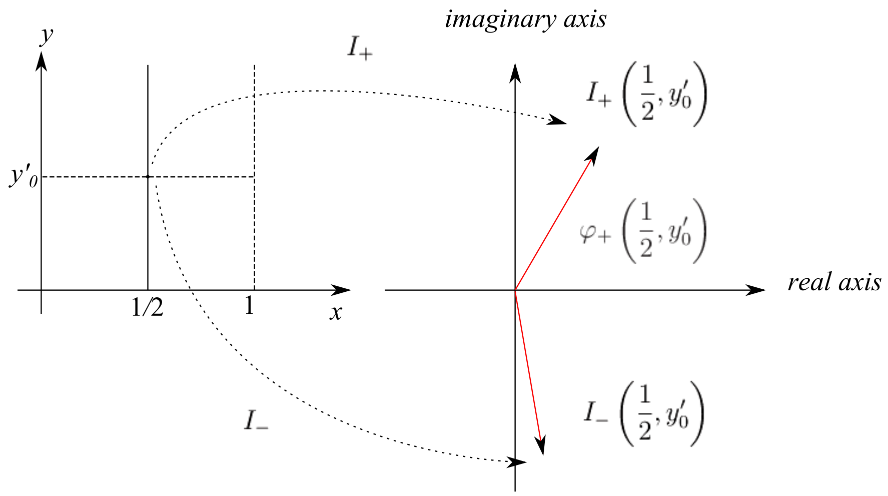

Figure 26: I numeri complessi $I_-\left(\frac{1}{2}, y_0'\right) = \int_{-\infty}^{0} \frac{e^{t//2}}{e^{e^t}+1} e^{iy_0't} dt$, $I_+\left(\frac{1}{2}, y_0'\right) = \int_0^{+\infty} \frac{e^{t//2}}{e^{e^t}+1} e^{iy_0't} dt$ non sono in opposizione di fase.

We define

$$\Delta\varphi(y) = \varphi_-\left(\frac{1}{2}, y\right) - \varphi_+\left(\frac{1}{2}, y\right), \quad \forall y \in Y \tag{59}$$
$$0 \leq \Delta\varphi(y) < 2\pi$$

being

$$Y = (-\infty, y_<] \cup [y_<, +\infty)$$

i.e. the projection of $\mathcal{C}_0'$ onto the xy coordinate plane. It follows

$$H = \left\{\left(\frac{1}{2}, y\right) \in \mathcal{C}_0' \mid \Delta\varphi(y) = \pi\right\} \tag{60}$$

As stated above, the equation $\Delta\varphi(y) = \pi$ admits infinitely many roots y_k (con $k \in \mathbb{Z}$). Follows:

$$z_k = F_-\left(\frac{1}{2}, y_k\right) = F_+\left(\frac{1}{2}, y_k\right), \quad k \in \mathbb{Z}$$

let's say

$$Z = \left\{z_k = F_-\left(\frac{1}{2}, y_k\right)\right\}_{k\in\mathbb{Z}} \subset [0, z_{\max}], \quad z_{\max} = F_+\left(\frac{1}{2}, 0\right)$$

which is the image of the sequence of elements of $\mathbb{R}$ defined by the restriction of $F_-\left(\frac{1}{2}, y\right)$ to the set whose elements are the imaginary part y_k of the zeros. For the above, Z is countably infinite and being limited, by the Bolzano-Weierstrass theorem, admits at least one point of accumulation. The latter is the image, through $F_-\left(\frac{1}{2}, y_k\right)$ of the points at infinity along the critical line.

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