Via Geometric Algebra: A Solution to the Snellius-Pothenot Resection (Surveying) Problem

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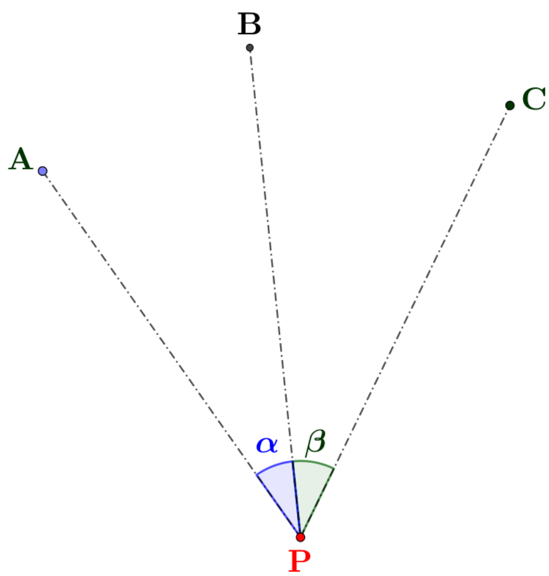
James Smith

Abstract

Using geometric algebra (GA), we derive a solution to the classic Snellius-Pothenot problem. We note two types of cases where that solution does not apply, and present a GA-based solution for one of those cases.



The points A, B, C and the angles α and β are known. Determine the location of the observer point P.

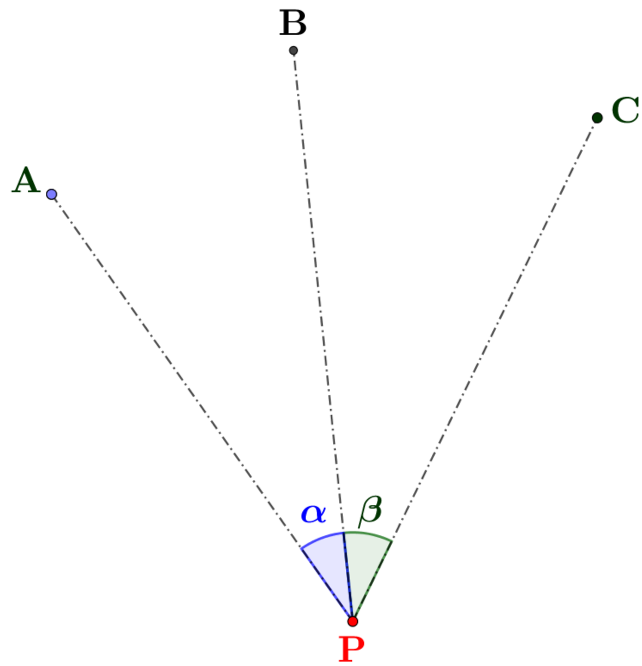


Figure 1: The points A, B, C and the angles α and β are known. Determine the location of the observer point P.

1 Statement of the Problem

As viewed from point P, the angles between the known points A, B , and C. are as shown in Fig. 1. What is the position of point P in terms of the positions of A, B , and C?

2 Some of the Ideas that We Will Find Useful

1. An angle that is inscribed in a circle is half as large as the central angle that subtends the same chord (Fig. 2).

2. As a consequence of the first idea: If a chord of length d is subtended by an inscribed angle whose measure is θ, then the half-chord is subtended by a central angle with that same measure (Fig. 3).

3. The "rejection" of one vector from another (Fig. 4). See also [1].

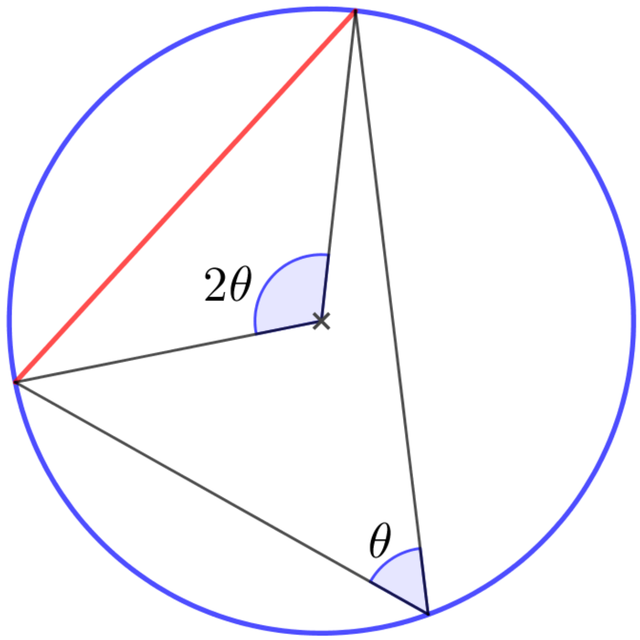


Figure 2: An angle that is inscribed in a circle is half as large as the central angle that subtends the same chord.

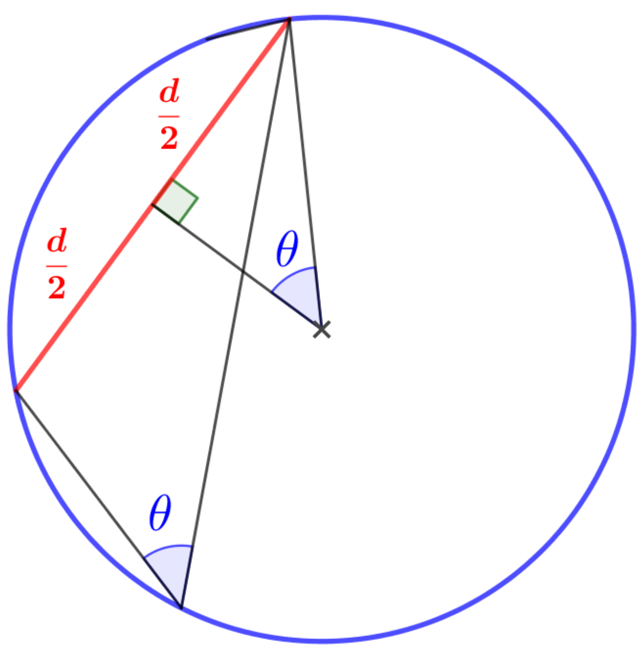


Figure 3: If a chord of length d is subtended by an inscribed angle whose measure is θ, then the half-chord is subtended by a central angle that has that same measure.

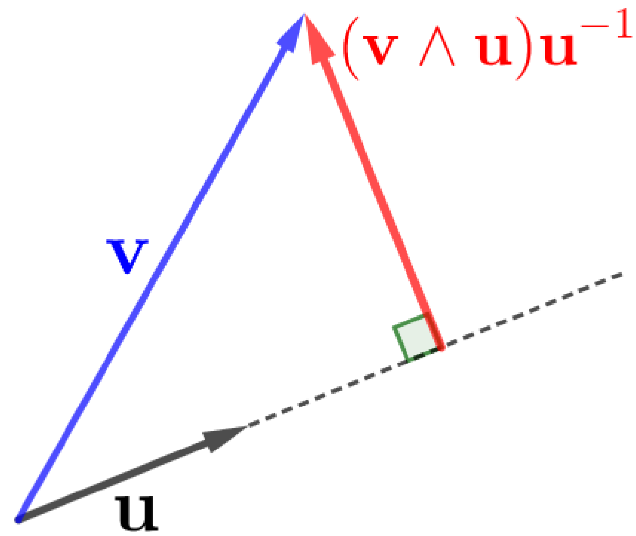


Figure 4: The red vector is the "rejection" of $\mathbf{v}$ with respect to $\mathbf{u}$.

3 Preliminary Analysis, and Formulation in GA Terms

3.1 Preliminary Analysis

P and B are the points of intersection of the two circles shown in Fig. 5 . The points B and P are equidistant from the line that connects the centers of the two circles that are shown, because (1) the perpendicular bisector of any chord of a circle passes through that circle's center, and (2) the chord BP is common to the two circles (Fig. 6).

3.2 Formulation in GA Terms

The problem is formulated via the vectors (with point B as origin) shown in Figs. 7 and 8.

4 The Solution, and Its Limitations

4.1 The Solution

As shown in Fig. 9, the vector from point B to P is twice the rejection of the vector from B to the center of either circle, with respect to the vector between the circles' centers.

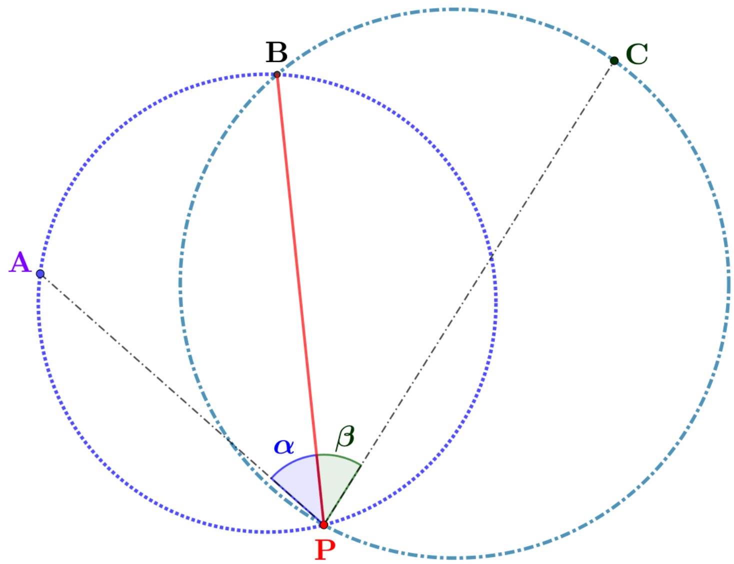


Figure 5: Point P is one of the two points of intersection of the two circles that are shown here. Point B is the other.

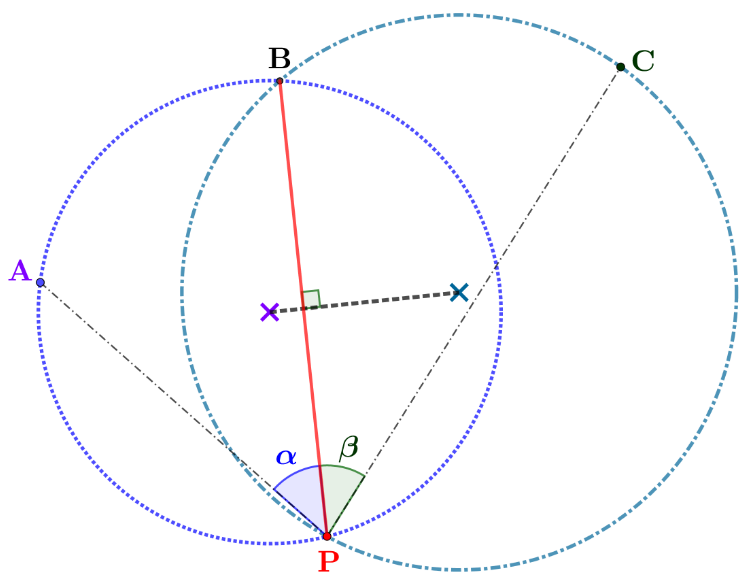


Figure 6: The points B and P are equidistant from the line that connects the centers of the two circles that are shown, because (1) the perpendicular bisector of any chord of a circle passes through that circle's center, and (2) the chord BP is common to the two circles.

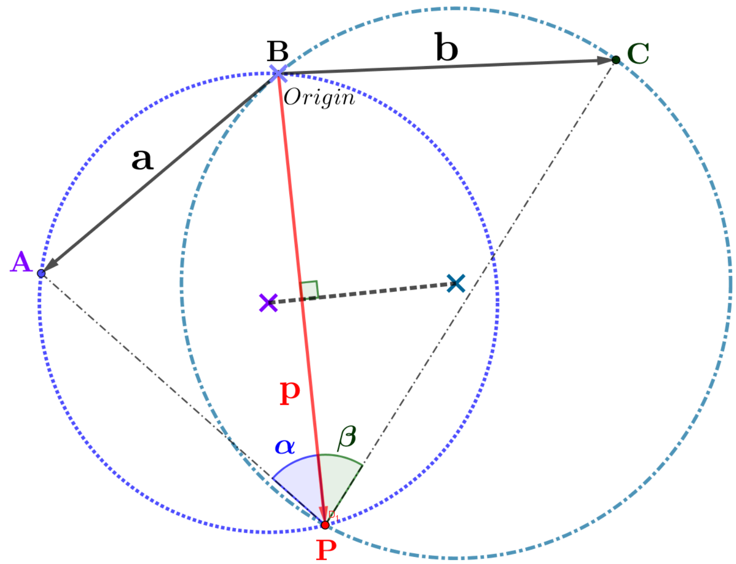


Figure 7: Formulation of the problem in terms that will allow us to use GA.

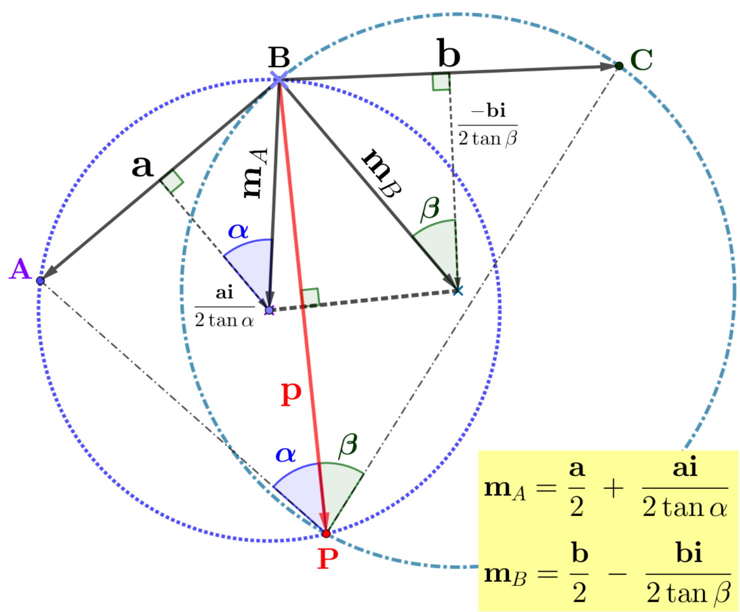
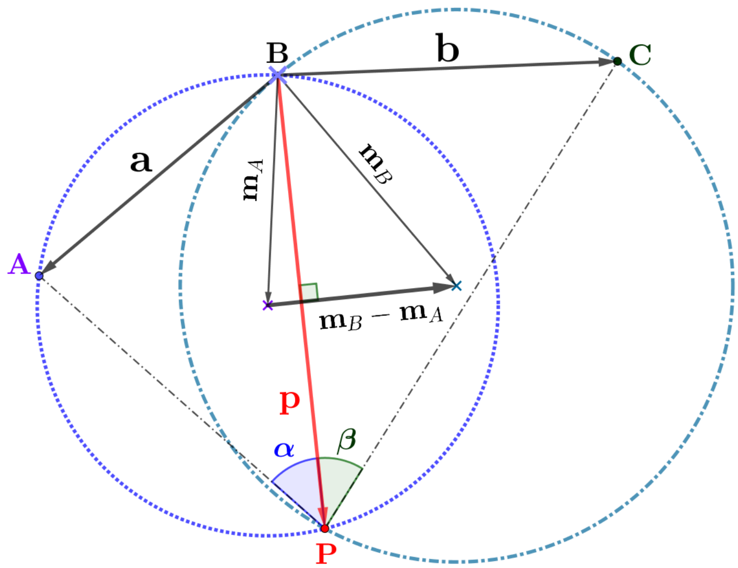


Figure 8: Identifying the vectors from B to the two circles' centers.



$$\mathbf{p} = 2\Big\{[\mathbf{m}_A \wedge (\mathbf{m}_B - \mathbf{m}_A)](\mathbf{m}_B - \mathbf{m}_A)^{-1}\Big\}$$
$$= 2\Big\{[\mathbf{m}_A \wedge \mathbf{m}_B](\mathbf{m}_B - \mathbf{m}_A)^{-1}\Big\}$$

Figure 9: The vector $\mathbf{p}$ is twice the rejection of $\mathbf{m}_A$ (or also $\mathbf{m}_B$)with respect to the vector between the circles' centers.

5 Limitations of this Solution

Professor Francisco G. Montoya, of the *Universidad de Almería,* Spain, has pointed out that the solution presented above does not work when P is aligned with $\overline{AB}$ or $\overline{BC}$, because in those cases the radius of one of the circles becomes infinite. Fig. 10 shows one such case, and its solution.

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).

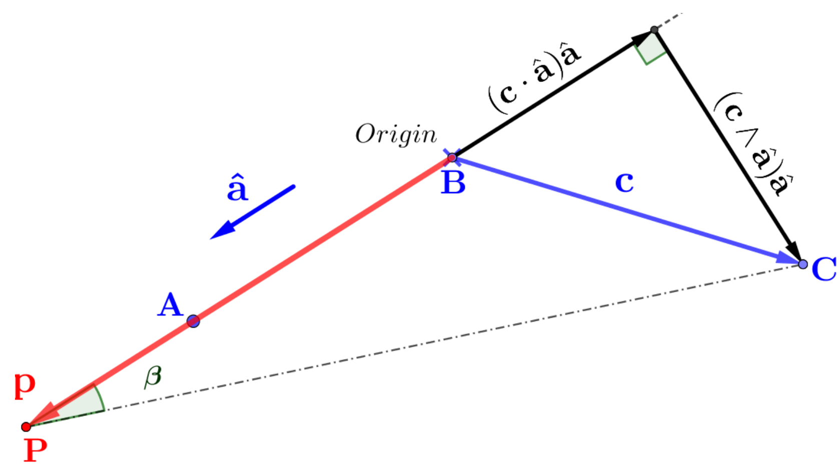


Figure 10: When point P is aligned with A and B as shown here, we can use the relation $\tan\beta = \|\mathbf{c}\wedge\hat{\mathbf{a}}\|/\|\mathbf{p}-(\mathbf{c}\cdot\hat{\mathbf{a}})\,\hat{\mathbf{a}}\|$. Consequently, $\mathbf{p} = \left[\frac{\|\mathbf{c}\wedge\hat{\mathbf{a}}|}{\tan\beta} - \mathbf{c}\cdot\hat{\mathbf{a}}\right]\hat{\mathbf{a}}$.