Syracuse Conjecture Quadrature

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ABSTRACT

After circa 2300 years (*Circle Quadrature*; Archimèdès, Syracuse 287 – 212 BC) the history of mathematics repeats itself in a different problem.

The conjecture of Syracuse, or Collatz conjecture, is approached from a completely dissimilar point of view than many previous attempts. One of its features suggests a process that leads to Theorem 2n+1, whose demonstration subdivided the set of odd numbers in seven subsets which have different behaviors applying algorithm of Collatz. It allows us to replace the Collatz cycles with the cycles of links, transforming their oscillating sequences in monotone decreasing sequences. By Theorem of Independence we can manage cycles of links as we like, also to reach very high horizons and when we decide go back to lower horizons. In this article it's proved that Collatz conjecture is not fully demonstrable. In fact, if we consider the banal link n < 2n, there are eight cycles which connect each other in an endless of possible links. It is a particular type of *Circle Quadrature*, but its statement is confirmed. In other words: BIG CRUNCH (go back to 1) is always possible, but BIG BANG (to move on) has no End.

INTRODUCTION

This paper shows the true nature of Syracuse Conjecture (better known as Collatz Conjecture), in fact it's proved that it is a particular kind of the squaring of the circle or *Circle Quadrature*. In addition to the important result are reported many properties and peculiarities of this conjecture. It hides the magical harmony of odd numbers, and maybe a type of law on the expansion of Cosmos based on the powers of 2, as prophesied by Plato in some of his writings. In this paper is used only arithmetic calculus, elementary number theory and binomial algebraic inequalities. In spite of its simple enunciation, Syracuse conjecture is a difficult topic, therefore this article needs patience in reading for a well-understanding. A variety of applications and examples were needed for better explain the work. "*Syracuse conjecture: a wolf disguised in a lamb*". (Maddmaths, web 2011).

Remark : This article is an exhaustive synthesized makeover of the book (110 pages plus 30 tables of links odd numbers 5 – 2999, A4 format) self-published on Amazon [**1**].

CONCISE PRESENTATION OF CONTENTS

The famous Syracuse Conjecture (SC), or Collatz Conjecture (CC), it's approached by a completely dissimilar point of view than many previous attempts, highlighting some of its features. One of them suggests a process that leads to Theorem 2n+1 whose proof subdivided the set of odd numbers in seven subsets which have different behaviors applying algorithm of Collatz. It allows us to replace the Collatz cycles with the cycles of links, transforming their oscillating sequences in monotone decreasing sequences, which, after a finite number of steps (very low), fall down to 1 always respecting the final cycles {10; 5; 4; 2; 1} or {7; 5; 4; 2; 1}. The General List of binomial inequalities until a number of steps N(s) ≤ 21 cover *circa* 96% of ℕ. By Theorem of Independence we can cover the remaining 4% managing the cycles of links as we like, also to reach very high main horizons $\Theta(m)$ and, when we decide, go back to the lower horizon $\Theta(l) < \Theta(m)$. In this article it is proved that CC is not entirely demonstrable. In fact, if we consider the cycle of even numbers, there are eight cycles which connect each other in an endless of links. The unique feasible procedure for its demonstration is to prove that any sequence obtained by Collatz algorithm contains a value lesser than the starting number (or *generating number*). Given that the even numbers 2n go to n < 2n, we can consider only odd numbers. For every biggest horizon we are able to cover by binomial inequality after N(s) steps, there is an upper horizon that needs a greater number of steps to become lower than itself. So it's proved that SC is a sort of *Circle Quadrature*. In other words: BIG CRUNCH (go back to 1) is always possible, but BIG BANG (to move on) has no End. Therefore we can claim that its initial statement is true.

"*The numbers reign on the Universe*" (Pythagoras)
"*ONE and DIADE are generators of Cosmos*" (Plato)

Remark : At the end three appendices were added to be used for a desirable teaching of CC in High Schools.

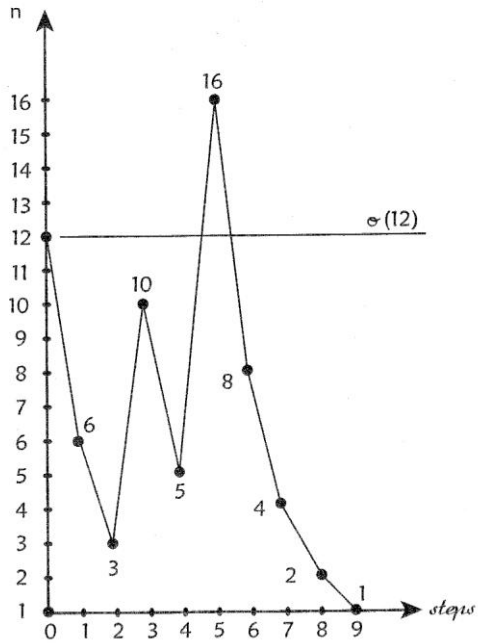
CONTENTS

1. **The conjecture of Syracuse**

The conjecture of Syracuse states: *If at any natural integer n, non-zero, we apply the algorithm 3n+1 if n is odd, n/2 if n is even, the sequence of the values obtained, precipitates to 1 after a finite number of steps, always in compliance with the final cycle {4; 2; 1}.*

$$\text{For every } n \in \mathbb{N} : n \neq 0 \rightarrow \begin{cases} 3n+1 \; if \; n \; is \; odd \\ \frac{n}{2} \; if \; n \; is \; even \end{cases}$$

Chosen n = 12 (even), applying the algorithm of Collatz we obtain the following sequence:
n = 12 → S(12) = {12; 6; 3; 10; 5; 16; 8; 4; 2; 1} the number 12 falls down to 1 in nine steps.
The sequence (or succession) generated by the number 12 is oscillating (neither increasing nor decreasing) and can be represented with a graph on a Cartesian plane having its origin in (0,1). The number 1 can be understood as the minimum level or the sea level. The steps are shown on the horizontal axis and the generating number on the vertical axis. The graph of fluctuation of the sequence S(12) is represented in figure. The horizontal line parallel to the abscissa axis (steps) is the horizon of the number 12 (indicated with the Greek letter θ). The number 12, during the oscillation of the sequence S(12), exceeds its horizon once, exactly in step 5, in correspondence of which reaches a peak 16.



It is to be noted that if we choose as starting number an odd number, the first element of the sequence is always an even number, since the product of 3 for another odd number still gives an odd number, which becomes even with the addition of 1. This happens for all the successors of an odd number contained in the sequence. An even number, instead, may be followed by an odd number or by one or more even numbers: n = 48 → S(48) = {48; 24; 12; 6; 3; 10; 5; 16; 8; 4; 2; 1}.
If you choose an even number power of 2, $n = 2^p$ = 2; 4; 8; 16; 32; 64; 128; 256; 512; 1024; …, it reaches 1 after a cycle of p applications of the algorithm. The same observation also applies to the odd numbers when $3n+1 = 2^p \rightarrow n = (2^p-1)/3 : p \in E \subset \mathbb{N}$; hence $n = (4^p-1)/3$. In this case the corresponding sequence S(n) contains p+2 values and it reaches 1 in p+1 steps.

If we choose the number 25 we will have the following sequence:
25 → 76 → 38 → **19** → 58 → 29 → 88 → 44 → 22 → **11** → 34 → 17 → 52 → 26 → 13 → 40 → 20 → **10** → **5** → 16 → 8 → **4** → **2** → **1**
It consists of 24 values, 23 steps. We observe that the sequence of values does not contain two same numbers, but they are all different from each other. In short, the sequences obtained through algorithm of Collatz precipitate to 1 taking different values between them. This is obvious. In fact, if in a succession two numbers were the same, from one of them onwards we would have the same sequence, giving rise to a vicious circle (*loop*), and this is impossible (clearly with the exclusion of 1 which generates the final cycle).
Again with reference to the number 25, but this reasoning is valid for any other number, another *important observation* is that when the value of the sequence falls below 25, in this case 19, its succession from here on becomes perfectly equal to that of this number. In this way the sequence of number 25 is linked to the succession of number 19. In short, the endless sequences generated by the algorithm give rise to a kind of interaction, in which each sequence is linked to a previous one whose generating number is smaller. For 25 we have the following chain of links:
25 → 19 → 11 → 10 → 5 → 4 → 2 → 1
In this way we can associate a specific procedure to the Collatz sequences.

If we take 49 as generating number, we have:

49 → 148 → 74 → **37** < 49
37 → 112 → 56 → **28** < 37
28 → **14** < 28
14 → **7** < 14
7 → 22 → 11 → 34 → 17 → 52 → 26 → 13 → 40 → 20 → 10 → **5** < 7
5 → 16 → 8 → **4** < 5
4 → **2** → **1** final cycle.

49 → 37 → 28 → 14 → 7 → 5 → 4 → 2 → 1

If we take 204 as generating number, applying the algorithm, we obtain:

204 → **102** < 204
102 → **51** < 102
51 → 154 → 77 → 232 → 116 → 58 → **29** < 51
29 → 88 → 44 → **22** < 29
22 → **11** < 22
11 → 34 → 17 → 52 → 26 → 13 → 40 → 20 → **10** < 11
10 → **5** < 10
5 → 16 → 8 → **4** < 5
4 → **2** → **1** final cycle.

204 → 102 → 51 → 29 → 22 → 11 → 10 → 5 → 4 → 2 → 1

Such a proceeding is applicable for every $n \in \mathbb{N}$: $n \neq 0$ and $n \neq 1$.

In brief, the endless sequences can be imagined as a maze in which the numbered rooms are connected to each other in a descending order in accordance with the rules imposed by the algorithm, up to the final cycle 4 → 2 → 1 finding the exit. SC would be proved if we could demonstrate that any sequence contains a value lesser than the starting number (generating number). Because if this happens it hooks to this, which in turn hooks to another lesser than it, and so on, up to precipitate necessarily to 1 (exit of the maze). Or, as they say, the main horizon of generating number descends to the next lower horizons until it reaches the sea level. This can be stated with certainty for even numbers $n \in E : n = 2^p$ and for odd numbers $n \in O : n = (4^p-1)/3$. For all other numbers there is uncertainty.

1.1. Theorem 2n+1

We have seen that the Collatz sequences are oscillating, excluding those whose generating number is power of 2. These are decreasing monotonic and fall down to 1 after a number of steps previously calculated. For the oscillating sequences we found that they certainly fall down to 1 if a term of the sequence, generated by a generic number n, assumes a smaller value of n, i.e. below the horizon of n that we can mark with $\Theta(n)$. Marking with n_i this value ($n_i < n$; $\Theta(n_i)$ lower of $\Theta(n)$), the sequence generated by n is linked to that generated by n_i, which, in turn, is linked to that of $n_j < n_i$ ($\Theta(n_j)$ lower of $\Theta(n_i)$), which ... and so it precipitates to 1, exiting from the labyrinth, or, how to say, reaches the sea level. We wonder: does this happens in all sequences? In any succession there is always an element whose value is less than the generator? This is certainly true for the even starting numbers. For them, the first element of the sequence is n/2, and n/2 < n. If, instead, the generating number is odd, the first element is 3n+1, which, as we have seen, it's certainly even and therefore divisible by 2. That is: $3n+1 \rightarrow (3n+1)/2$. Unless it is of the type $(3n+1)/2 = 2^p$ or $(3n+1)/2 = (4^p-1)/3$ ($n \in O$), it turns out to be $(3n+1)/2 > n$. But if 3n+1 were doubly even (twice divisible by 2, or divisible by 4), triply even (three times divisible by 2, or divisible by 8), then $(3n+1)/4 < n$ and $(3n+1)/8 < n$ they would definitely be true.

We denote the generic odd number with 2n+1.

Statement: *There are iterative cycles obtained by the algorithm of Collatz applied to odd numbers 2n+1 which contain terms a_n lesser than their generators.*

Demonstration: Bearing in mind that in the sequences generated by the algorithm of Collatz any odd number has as its successor an even number, we will have:
$2n+1 \rightarrow 3(2n+1)+1 = 6n+4 \rightarrow 3n+2$.

3n+2 can be even or odd.

a) If 3n+2 is even, then $3n+2 \rightarrow (3n+2)/2 < 2n+1$ ($3n+2 < 4n+2$, from which $n > 0$, always true).
But 3n+2 is even if $n \in E = \{2; 4; 6; 8; 10; 12; 14; 16; ... ; 2n\}$.
It follows that 2n+1 is in $\mathbf{O_1 = \{5; 9; 13; 17; 21; 25; 29; 33; \ldots ; 4n+1\}}$ [n > 0]

b) If 3n+2 is odd, then: $3n+2 \rightarrow 3(3n+2)+1 = 9n+7$, which being even in virtue of algorithm, then $9n+7 \rightarrow (9n+7)/2$, which can be even or odd.

b1) If (9n+7)/2 is even, then $(9n+7)/2 \rightarrow (9n+7)/4 < 4n+2 = 2(2n+1) \rightarrow (9n+7)/8 < 2n+1$. And this happens if (9n+7)/2 is at least doubly even (divisible by 4). So: $(9n+7)/2 \rightarrow (9n+7)/4 \rightarrow (9n+7)/8 < 2n+1$. But (9n+7)/2 is even for $n \in O_1 = \{1; 5; 9; 13; 17; 21; 25; 29; \ldots; 4n+1\}$.
It follows that 2n+1 is in $\mathbf{O_2 = \{3; 11; 19; 27; 35; 43; 51; 59; \ldots; 8n+3\}}$ [$n \geq 0$]

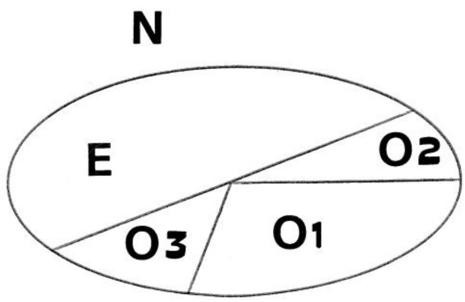
b2) If (9n+7)/2 is odd, then $(9n+7)/2 \rightarrow 3((9n+7)/2)+1 = (27n+21)/2+1 = (27n+23)/2$, which being even in virtue of the algorithm: $(27n+23)/2 \rightarrow (27n+23)/4 < 8n+4 = 4(2n+1) \rightarrow (27n+23)/16 < 2n+1$. And this happens if (27n+23)/2 is at least triply even (divisible by 8). So: $(27n+23)/2 \rightarrow (27n+23)/4 \rightarrow (27n+23)/8 \rightarrow (27n+23)/16 < 2n+1$.
But (27n+23)/2 is even for $n \in O_4 = \{3; 7; 11, 15; 19; 23; 27; 31; \ldots ; 4n-1; 4n+3\}$.

But $(9n+7)/2$ is odd for $n \in O_4 = \{3; 7; 11, 15; 19; 23; 27; 31; \dots ; 4n-1; 4n+3\}$.
It follows that 2n+1 is in **$O_3 = \{7; 15; 23; 31; 39; 47; 55; 63; \dots; 8n-1; 8n+7\}$** [n > 0]

If we denote with S(2n+1) any sequence obtained applying Collatz algorithm starting from a generic odd generating number 2n+1; we have identified three subsets in whose iterative cycles there are terms $a_n \in S(2n+1)$ such that $a_n < 2n+1$. They are:

$O_1 = \{5; 9; 13; 17; 21; 25; 29; 33; \dots ; 4n+1\}$
$O_2 = \{3; 11; 19; 27; 35; 43; 51; 59; \dots; 8n+3\}$
$O_3 = \{7; 15; 23; 31; 39; 47; 55; 63; \dots; 8n-1; 8n+7\}$

$O_1 \cup O_2 \cup O_3 = \{3; 5; 7; 9; 11; 13; 15; 17; 19; 21; 23; 25; 27; 29; 31; \dots; 2n+1; 2n+3\} = O - \{1\}$, with 1 generator of the end cycle.



1.2. Addition to the theorem 2n+1

Note: Bold font is used in this article for highlighting substitutions.

a) $2n+1 \in O_1$

3n+2 is even if $n \in E = \{0; 2; 4; 6; 8; \dots; 2n\}$ $[n \geq 0]$
It is simply even if $n \in E' = \{0; 4; 8; 12; 16; \dots; 4n\}$
n = **4n** : 2n+1 = 2(**4n**)+1 = **8n+1**; 3n+2 = 3(**4n**)+2 = 12n+2 → **6n+1** < **8n+1** O_1
It is doubly even if $n \in E'' = \{6; 14; 22; 30; 38; \dots; 8n+6\}$
n = **8n+6** : 2n+1 = 2(**8n+6**)+1 = **16n+13**; 3n+2 = 3(**8n+6**)+2 = 24n+20 → **12n+10** < **16n+13** O_1
It is at least triply even if $n \in E''' = \{2; 10; 18; 26; 34, \dots; 8n+2\}$
n = **8n+2** : 2n+1 = 2(**8n+2**)+1 = **16n+5**; 3n+2 = 3(**8n+2**)+2 = 24n+8 → **12n+4** < **16n+5** O_1
$E' \cup E'' \cup E''' = E$ → for every $n \in E$: **(3n+2)/2 < 2n+1** $\in O_1$
n = **2n** : 2n+1 = 2(**2n**)+1 = **4n+1**; (3n+2)/2 **=** (3(**2n**)+2)/2 = **3n+1 < 4n+1** $\in O_1$

b1) $2n+1 \in O_2$

(9n+7)/2 is even if $n \in O_1 = \{1; 5; 9; 13; \dots; 4n+1\}$ $[n \geq 0]$
It is doubly even if $n \in O_1' = \{9; 25; 41; 57; 73; \dots; 16n+9\}$
n = **16n+9** → 2n+1 = 2(**16n+9**)+1 = **32n+19**; (9n+7)/2 = [9(**16n+9**)+7]/2 = 72n+44 → 36n+22 → **18n+11** < **32n+19**

It is at least triply even if $n \in O_1'' = \{1; 17; 33; 49; \ldots; 16n+1\}$
$n = \mathbf{16n+1} \rightarrow 2n+1 = 2(\mathbf{16n+1})+1 = \mathbf{32n+3}$; $(9n+7)/2 = [9(\mathbf{16n+1})+7]/2 = 72n+8 \rightarrow 36n+4 \rightarrow \mathbf{18n+2} < \mathbf{32n+3}$
$O_1' \cup O_1'' = O_1''' = \{1; 9; 17; 25; \ldots; 8n+1\} \subset O_1 \rightarrow$ for every $n \in O_1'''$: $\mathbf{(9n+7)/8 < 2n+1} \in O_2$
$n = \mathbf{8n+1}$: $2n+1 = 2(\mathbf{8n+1})+1 = \mathbf{16n+3}$; $(9(\mathbf{8n+1})+7)/8 \ \mathbf{= 9n+2 < 16n+3} \in O_2$

b2) $2n+1 \in O_3$

$\mathbf{(27n+23)/2}$ is even if $n \in O_4 = \{3; 7; 11; 15; \ldots; 4n+3\}$ $[n \geq 0]$
It is at least triply even if $n \in O_4' = \{11; 27; 43; 59; \ldots; 16n+11\}$
$n = \mathbf{16n+11} \rightarrow 2n+1 = 2(\mathbf{16n+11})+1 = \mathbf{32n+23}$; $(27n+23)/2 = [27(\mathbf{16n+11})+23]/2 = 216n+160 \rightarrow 108n+80 \rightarrow 54n+40 \rightarrow \mathbf{27n+20} < \mathbf{32n+23}$
$O_4' \subset O_4 \rightarrow$ for every $n \in O_4'$: $\mathbf{(27n+23)/16 < 2n+1} \in O_3$
$n = \mathbf{16n+11}$: $2n+1 = 2(\mathbf{16n+11})+1 = \mathbf{32n+23}$; $(27(\mathbf{16n+11})+23)/16 \ \mathbf{= 27n+20 < 32n+23} \in O_3$

1.2.1. Summary

a)
$2n+1 \in O_1 \rightarrow 6n+4 \rightarrow 3n+2 \rightarrow (3n+2)/2 < 2n+1 = 3n+1 < 4n+1$ O_1 : an odd number in O_1 assumes a value less than itself after three applications of Collatz algorithm (3 steps). This always happens.

b1)
$2n+1 \in O_2 \rightarrow 6n+4 \rightarrow 3n+2 \rightarrow 9n+7 \rightarrow (9n+7)/2 \rightarrow (9n+7)/4 \rightarrow (9n+7)/8 < 2n+1 = 9n+2 < 16n+3$ O_2 : an odd number in O_2 assumes a value less than itself after six applications of Collatz algorithm (6 steps). If this does not happen it is transformed into an odd number or in O_1 or in O_2 or in O_3 .

b2)
$2n+1 \in O_3 \rightarrow 6n+4 \rightarrow 3n+2 \rightarrow 9n+7 \rightarrow (9n+7)/2 \rightarrow (27n+23)/2 \rightarrow (27n+23)/4 \rightarrow (27n+23)/8 \rightarrow (27n+23)/16 < 2n+1 = 27n+20 < 32n+23$ O_3 : an odd number in O_3 assumes a value less than itself after eight applications of Collatz algorithm (8 steps). If this does not happen it is transformed into an odd number or in O_1 or in O_2 or in O_3 .

1.2.2. Corollaries

Every sequence obtained by application of Collatz algorithm do not contain (never) odd numbers divisible by 3. In fact : $(2n+1) \in O \rightarrow 3(2n+1)+1 = 6n+4 \rightarrow 3n+2$ can be even or odd and it is not divisible by 3 for every $n \in \mathbb{N}$. We can therefore state the following corollaries.

Corollary 1: *In the cycles of links no odd number connects to an odd number divisible by 3.*
Corollary 2: *In the cycles of links no odd number connects to an even number divisible by 3.*
Corollary 3: *If the sequences obtained by applying the algorithm of Collatz start from an odd generating number, then the odd numbers divisible by 3 are present only as generating numbers*.
Corollary 4: *If a sequence obtained by applying the algorithm of Collatz starts from an odd generating number, then in it are not presents even numbers divisible by 3.*

2. Explanatory notes and odd numbers subsets

Syracuse conjecture plan Theorem 2n+1

2n+1 O → 6n+4 E → 3n+2 O ∨ **E → (3n+2)/2 O_1 : 3** steps
↓
9n+7 E → (9n+7)/2 O ∨ **E → (9n+7)/4 O_2 : 5** steps
↓
(27n+23)/2 E → (27n+23)/4 O_3 : 6 steps

Notes

a) Definition: *A cycle of link is the number of steps required for to arrive from the main horizon of a generating number* (Start) *to the lower horizon of its link* (End).

We point: Ɵ(m) : main horizon
Ɵ(l) : lower horizon
N(s) : number of steps

b) Definition: *Two cycles of links,* c1, c2, *are equivalent if contain the same number of steps, i.e. :* c1 ~ c2 ↔ N(s)[c1] = N(s)[c2].

c) **O_1 = {5; 9; 13; 17; 21; 25; 29; 33; …; 4n+1}** [n > 0]
O_2 = {11; 19; 27; 35; 43; 51; 59; 67; …; 8n+3} [n > 0]
O_3 = {7; 15; 23; 31; 39; 47; 55; 63; …; 8n-1} [n > 0]

A number n is in O_1 if n-1 is divisible by 4 :
We pose : u = (n-1)/4 ; x = 2u = (n-1)/2 → f(x) = (3x+2)/2 after N(s) = **3**
A number n is in O_2 if n-3 is divisible by 8 :
We pose : v = (n-3)/8 ; y = 4v+1 = (n-1)/2 → f(y) = (9y+7)/4 after N(s) = **5**
A number n is in O_3 if n+1 is divisible by 8 :
We pose : w = (n+1)/8 : z = 4w-1 = (n-1)/2 → f(z) = (27z+23)/4 after N(s) = **6**

d) Important remark
We use binomials of 4 types:
1) (even number)·n + (even number) → divisible by 2
2) (odd number)·n + (even number) : it can be Odd or Even. Is Odd if n = **2n+1** ; is Even if n = **2n**
3) (odd number)·n + (odd number): it can be Odd or Even. Is Odd if n = **2n** ; is Even if n = **2n+1**
4) (even number)·n + (odd number): it can be in O_1 or in O_2 or in O_3 :
4a) It's in O_1 if n = **2n** or if n = **2n+1**.
4b) It's in O_2 if n = **4n+1** or if n = **4n+3** ; if n = **4n** or if n = **4n+2**
4c) It's in O_3 if n = **4n+3** or if n = **4n+1** ; if n = **4n+2** or if n = **4n**

e) If it is written E; O; O_1; O_2; O_3 it is an obligation; instead [E]; [O]; [O_1]; [O_2]; [O_3] it is a choice

2.1. O_1 ; O_2 ; O_3

O_1

4n+1 O_1 : x = 2n → f(x) = **3n+1 < 4n+1** O_1 : N(s) = **3**

O_2

8n+3 O_2 : y = 4n+1 → f(y) = 9n+4 [E] : n = **2n**

8n+3 O_2 → 8(**2n**)+3 = **16n+3** O_2*
9n+4 → 9(**2n**)+4 = 18n+4 → **9n+2 < 16n+3** O_2* = {19; 35; …} ⊂ O_2 : N(s) = **6**

For generating odd numbers $O_2 – O_2$* = **16n+11** O_{2a} = {11; 27; …} we have:

16n+11 O_{2a} : y = 8n+5 → f(y) = 18n+13 $O_1 \vee O_2 \vee O_3$ after 5 steps

O_3

8n-1 O_3 : z = 4n-1 → f(z) = 27n-1 [E] : n = **2n+1**

8n-1 O_3 → 8(**2n+1**)-1 = **16n+7** O_3
27n-1 → 27(**2n+1**)-1 = 54n+26 → 27n+13 [E] : n = **2n+1**

16n+7 O_3 → 16(**2n+1**)+7 = **32n+23** O_3*
27n+13 → 27(**2n+1**)+13 = 54n+40 → **27n+20 < 32n+23** O_3* = {23; 55; …} ⊂ O_3 : N(s) = **8**

For generating odd numbers $O_3 – O_3^*$ of type:

32n-1 O_{3a} = {31; 63; 95; …} [n > 0]
32n+7 O_{3b} = {7; 39; 71; …} [n ≥ 0]
32n+15 O_{3c} = {15; 47; 79; …} [n ≥ 0]
we have:
32n-1 O_{3a} : z = 16n-1 → f(z) = 108n-1 $O_2 \vee O_3$ after 6 steps
32n+7 O_{3b} : z = 16n+3 → f(z) = 108n+26 → 54n+13 $O_1 \vee O_2 \vee O_3$ after 7 steps
32n+15 O_{3c} : z = 16n+7 → f(z) = 108n+53 O_1 : x = 54n+26 → f(x) = 81n+40 E ∨ O after 9 steps

Keep in mind that: 32n-9 O_3* ~ 32n+23 O_3* ; 32n-1 O_{3a} ~ 32n+31 O_{3a} .

Summarizing :

$O = O_1 \cup O_2 \cup O_3$
$O_2 = O_2^* \cup O_{2a}$
$O_3 = O_3^* \cup O_{3a} \cup O_{3b} \cup O_{3c}$

2.2. Further analysis O_2

16n+11 O_{2a} : y = 8n+5 → f(y) = 18n+13 $O_1 \vee O_2 \vee O_3$: N(s) = **5**

16n+11 O_{2a} : y = 8n+5 → f(y) = 18n+13 [O_1] : n = **2n**
16n+11 O_{2a} → **32n+11** O_{2a}
18n+13 → **36n+13** O_1 : x = 18n+6 → f(x) = **27n+10 < 32n+11** O_{2a}

Check :
32n+11 O_{2a} : y = 16n+5 → f(y) = 36n+13 O_1 : x = 18n+6 → f(x) = **27n+10 < 32n+11** O_{2a} = {11; 43; …} ⊂ O_2 : N(s) = **8**

Chain of connections :
O_{2a} → O_1 → **27n+10 < 32n+11** O_{2a}

$f(y) = 18n+13$ $[O_2]$: $n =$ **4n+3**
16n+11 $O_{2a} \rightarrow$ **64n+59** O_{2a}
$18n+13 \rightarrow 72n+67$ O_2 : $y = 36n+33 \rightarrow f(y) = 81n+76$ [E] : $n =$ **2n**
64n+59 $O_{2a} \rightarrow$ **128n+59** O_{2a}
$81n+76 \rightarrow 162n+76 \rightarrow$ **81n+38 < 128n+59** O_{2a}

Check :
$128n+59$ O_{2a} : $y = 64n+29 \rightarrow f(y) = 144n+67$ O_2* : $y = 72n+33 \rightarrow f(y) = 162n+76 \rightarrow$
81n+38 < 128n+59 $O_{2a} = \{59; 187; \ldots\} \subset O_2$: N(s) = **11**

Chain of connections :
$O_{2a} \rightarrow O_2^* \rightarrow E \rightarrow$ **81n+38 < 128n+59** O_{2a}

$f(y) = 81n+76$ [O] : $n =$ **2n+1**
64n+59 $O_{2a} \rightarrow$ **128n+123** O_{2a}
$81n+76 \rightarrow 162n+157$ $[O_1]$: $n =$ **2n**
128n+123 $O_{2a} \rightarrow$ **256n+123** O_{2a}
$162n+157 \rightarrow 324n+157$ O_1 : $x = 162n+78 \rightarrow f(x) =$ **243n+118 < 256n+123** O_{2a}

Check :
256n+123 O_{2a} : $y = 128n+61 \rightarrow f(y) = 288n+139$ O_{2a} : $y = 144n+69 \rightarrow f(y) = 324n+157$ O_1 :
$x = 162n+78 \rightarrow f(x) =$ **243n+118 < 256n +123** $O_{2a} = \{123; 379; \ldots\} \subset O_2$: N(s) = **13**

Chain of connections :
$O_{2a} \rightarrow O_{2a} \rightarrow O_1 \rightarrow$ **243n+118 < 256n +123** O_{2a}

$162n+157$ $[O_2]$: $n =$ **4n+3**
128n+123 $O_{2a} \rightarrow$ **512n+507** O_{2a}
$162n+157 \rightarrow 648n+643$ O_2 : $y = 324n+321 \rightarrow f(y) = 729n+724$ [E] : $n =$ **2n**
512n+507 $O_{2a} \rightarrow$ **1024n+507** O_{2a}
$729n+724 \rightarrow 1458n+724 \rightarrow$ **729n+362 < 1024n+507** O_{2a}

Check :
1024n+507 O_{2a} : $y = 512n+253 \rightarrow f(y) = 1152n+571$ O_{2a} : $y = 576n+285 \rightarrow$
$f(y) = 1296n+643$ O_2* : $y = 648n+321 \rightarrow f(y) = 1458n+728 \rightarrow$
729n+362 < 1024n+507 $= \{507; 1531; \ldots\} \subset O_2$: N(s) = **16**

Chain of connections :
$O_{2a} \rightarrow O_{2a} \rightarrow O_2^* \rightarrow E \rightarrow$ **729n+362 < 1024n+507**

$f(y) = 729n+724$ [O] : $n =$ **2n+1**
512n+507 $O_{2a} \rightarrow 512($**2n+1**$)+507 =$ **1024n+1019** O_{2a}
$729n+724 \rightarrow 1458n+1453$ $O_1 \vee O_2 \vee O_3$ … …

$162n+157$ $[O_3]$: $n =$ **4n+1**
128n+123 $O_{2a} \rightarrow$ **512n+251** O_{2a}
$162n+157 \rightarrow 648n+319$ O_3 : $z = 324n+159 \rightarrow f(z) = 2187n+1079$ [E] : $n =$ **2n+1**
512n+251 $O_{2a} \rightarrow$ **1024n+763** O_{2a}
$2187n+1079 \rightarrow 4374n+3266 \rightarrow 2187n+1633$ [E] : $n =$ **2n+1**
1024n+763 $O_{2a} \rightarrow$ **2048n+1787** O_{2a}
$2187n+1633 \rightarrow 4374n+3820 \rightarrow 2187n+1910$ [E] : $n =$ **2n**

2048n+1787 O_{2a} → **4096n+1787** O_{2a}
2187n+1910 → 4374n+1910 → **2187n+955 < 4096n+1787** O_{2a}

Check :
4096n+1787 O_{2a} : y = 2048n+893 → f(y) = 4608n+2011 O_{2a} : y = 2304+1005 →
f(y) = 5184n+2263 O_3* : z = 2592n+1131 → f(z) = 17496n+7640 → 8748n+3820 →
4374n+1910→ **2187n+955 < 4096n+1787** O_{2a} = {1787; …} ⊂ O_2 : N(s) = **19**

Chain of connections :
O_{2a} → O_{2a} → O_3* → E → E → E → **2187n+955 < 4096n+1787** O_{2a}

f(z) = 2187n+1079 [O] : n = **2n**
512n+251 O_{2a} → **1024n+251** O_{2a}
2187n+1079 → 4374n+1079 O_1 ∨ O_2 ∨ O_3 … …

18n+13 [O_3] : n = **4n+1**
16n+11 O_{2a} → **64n+27** O_{2a}
18n+13 → 72n+31 O_3 : z = 36n+15 → f(z) = 243n+107 [E] : n = **2n+1** … …

f(z) = 243n+107 [O] : n = **2n**
64n+27 O_{2a} → **128n+27** O_{2a}
243n+107 → 486n+107 [O_3] : n = **4n+2**
128n+27 O_{2a} → **512n+283** O_{2a}
486n+107 → 1944n+1079 O_3 : z = 972n+539 → f(z) = 6561n+3644 [E] : n = **2n**
512n+283 O_{2a} → **1024n+283** O_{2a}
6561n+3644 → 13122n+3644 → 6561n+1822 [E] : n = **2n**
1024n+283 O_{2a} → **2048n+283** O_{2a}
6561n+1822 → 13122n+1822 → 6561n+911 [E] : n = **2n+1**
2048n+283 O_{2a} → **4096n+2331** O_{2a}
6561n+911 → 13122n+7472 → 6561n+3736 [E] : n = **2n**
4096n+2331 O_{2a} → **8192n+2331** O_{2a}
6561n+3736 → 13122n+3736 → **6561n+1868** < **8192n+2331** O_{2a}

Check :
8192n+2331 O_{2a} : y = 4096n+1165 → f(y) = 9216n+2623 O_{3a} : z = 4608n+1311 →
f(z) = 31104n+8855 O_3* : z = 15552n+4427 → f(z) = 104976n+29888 →
52488n+14944 → 26244n+7472 → 13122n+3736 →
6561n+1868 < 8192n+2331 O_{2a} = {2331; …} ⊂ O_2 : N(s) = **21**

Chain of connections :
O_{2a} → O_{3a} → O_3* → E → E → E → E → **6561n+1868 < 8192n+2331** O_{2a} =
{2331; …} ⊂ O_2 : N(s) = **21**
… …
… …
… …

2.3. Further analysis O_3

O_{3a}

32n-1 O_{3a} : z = 16n-1 → f(z) = 108n-1 O_2 ∨ O_3 → 324n-2 → 162n-1 O_1 ∨ O_2 ∨ O_3 : N(s) = **8**

162n-1 [O_1] : n = **2n+1**
32n-1 O_{3a} → **64n+31** O_{3a}
162n-1 → 324n+161 O_1 : x = 162n+80 → f(x) = 243n+121 [E] : n = **2n+1**
64n+31 O_{3a} → **128n+95** O_{3a}
243n+121 → 486n+364 → 243n+182 [E] : n = **2n**
128n+95 O_{3a} → **256n+95** O_{3a}
243n+182 → 486n+182 → **243n+91 < 256n+95** O_{3a}

Check :
256n+95 O_{3a} : z = 128n+47 → f(z) = 864n+323 O_2* : y = 432n+161 → f(y) = 972n+364 → 486n+182 → **243n+91 < 256n+95** O_{3a} = {95; 351; …} ⊂ O_3 : N(s) = **13**

Chain of connections :
O_{3a} → O_2* → E → E → **243n+91 < 256n+95** O_{3a}

f(x) = 243n+121 [O] : n = **2n**
64n+31 O_{3a} → 1**28n+31** O_{3a}
243n+121 → 486n+121 [O_1] : n = **2n**
128n+31 O_{3a} → **256n+31** O_{3a}
486n+121 → 972n+121 O_1 : x = 486n+60 → f(x) = 729n+91 [E] : n = **2n+1**
256n+31 O_{3a} → **512n+287** O_{3a}
729n+91 → 1458n+820 → 729n+410 [E] : n = **2n**
512n+287 O_{3a} → **1024n+287** O_{3a}
729n+410 → 1458n+410 → **729n+205 < 1024n+287** O_{3a}

Check :
1024n+287 O_{3a} : z = 512n+143 → f(z) = 3456n+971 O_{2a} : y = 1728n+485 →
f(y) = 3888n+1093 O_1 : x = 1944n+546 → f(x) = 2916n+820 → 1458n+410 →
729n+205 < 1024n+287 O_{3a} = {287; 1311; …} ⊂ O_3 : N(s) = **16**

Chain of connections :
O_{3a} → O_{2a} → O_1 → E → E → **729n+205 < 1024n+287** O_{3a}

… …

f(z) = 162n-1 [O_2] : n = **4n+2**
32n-1 O_{3a} → **128n+63** O_{3a}
162n-1 → 648n+323 O_2 : y = 324n+161 → f(y) = 729n+364 [O] : n = **2n+1** … …

f(z) = 162n-1 [O_3] : n = **4n**
32n-1 O_{3a} → **128n-1** O_{3a}
162n-1 → 648n-1 O_3 : z = 324n-1 → f(z) = 2187n-1 [E] : n = **2n+1**
128n-1 O_{3a} → **256n+127** O_{3a}
2187n-1 → 4374n+2186 → 2187n+1093 [E] : n = **2n+1**
256n+127 O_{3a} → **512n+383** O_{3a}
2187n+1093 → 4374n+3280 → 2187n+1640 [E] : n = **2n**
512n+383 O_{3a} → 512(**2n**)+383 = **1024n+383** O_{3a}
2187n+1640 → 4374n+1640 → 2187n+820 [E] : n = **2n**
1024n+383 O_{3a} → **2048n+383** O_{3a}
2187n+820 → 4374n+820 → 2187n+410 [E] : n = **2n**

2048n+383 O_{3a} → **4096n+383** O_{3a}
2187n+410 → 4374n+410 → **2187n+205 < 4096n+383** O_{3a}

Check :
4096n+383 O_{3a} : z = 2048n+191 → f(z) = 13824n+1295 O_{3c} : z = 6912n+647 →
f(z) = 46656n+4373 O_1 : x = 23328n+2186 → f(x) = 34992n+3280 → 17496n+1640 →
8748n+820 → 4374n+410 → **2187n+205 < 4096n+383** O_{3a} = {383; …} ⊂ O_3 : N(s) = **19**

Chain of connections :
O_{3a} → O_{3c} → O_1 → E → E → E → E → **2187n+205 < 4096n+383** O_{3a}

f(z) = 2187n-1 [O] : n = **2n**
128n-1 O_{3a} → **256n-1** O_{3a}
2187n-1 → 4374n-1 [O_1] : n = **2n+1**
256n-1 O_{3a} → **512n+255** O_{3a}
4374n-1 → 8748n+4373 O_1 : x = 4374n+2186 → f(x) = 6561n+3280 [E] : n = **2n**
512n+255 O_{3a} → **1024n+255** O_{3a}
6561n+3280 → 13122n+3280 → 6561n+1640 [E] : n = **2n**
1024n+255 O_{3a} → 1024(**2n**)+255 = **2048n+255** O_{3a}
6561n+1640 → 13122n+1640 → 6561n+820 [E] : n = **2n**
2048n+255 O_{3a} → **4096n+255** O_{3a}
6561n+820 → 13122n+820 → 6561n+410 [E] n = **2n**
4096n+255 O_{3a} → **8192n+255** O_{3a}
6561n+410 → 13122n+410 → **6561n+205 < 8192n+255** O_{3a}

Check :
8192n+255 O_{3a} : z = 4096n+127 → f(z) = 27648n+863 O_{3a} : z = 13824n+431 →
f(z) = 93312n+2915 O_2* : y = 46656n+1457 → f(y) = 104976n+3280 →
52488n+1640 → 26244n+820 → 13122n+410 →
6561n+205 < 8192n+255 O_{3a} = {255; …} ⊂ O_{3a} : N(s) = **21**

Chain of connections :
O_{3a} → O_{3a} → O_2* → E → E → E → E → **6561n+205 < 8192n+255** O_{3a}

… …
… …

O_{3b}

32n+7 O_{3b} : z = 16n+3 → f(z) = 108n+26 → 54n+13 $O_1 \vee O_2 \vee O_3$: N(s) = **7**

54n+13 [O_1] : n = **2n**
32n+7 O_{3b} → **64n+7** O_{3b}
54n+13 → 108n+13 O_1 : x = 54n+6 → f(x) = 81n+10 [E] : n = **2n**
64n+7 O_{3b} → 64(**2n**)+7 = **128n+7** O_{3b}
81n+10 → 81(**2n**)+10 = 162n+10 → **81n+5 < 128n+7** O_{3b}

Check :
128n+7 O_{3b} : z = 64n+3 → f(z) = 432n+26 → 216n+13 O_1 : x = 108n+6 → f(x) = 162n+10 →
81n+5 < 128n+7 O_{3b} = {7; 135; …} ⊂ O_3 : N(s) = **11**

Chain of connections :
O_{3b} → E → O_1 → E → **81n+5 < 128n+7** O_{3b}

54n+13 [O_2] : n = **4n+1**
32n+7 O_{3b} → **128n+39** O_{3b}
54n+13 → 216n+67 O_2 : y = 108n+33 → f(y) = 243n+76 [E] : n = **2n**
128n+39 O_{3b} → **256n+39** O_{3b}
243n+76 → 486n+76 → **243n+38 < 256n+39** O_{3b}

Check :
256n+39 O_{3b} : z = 128n+19 → f(z) = 864n+134 → 432n+67 O_2* : y = 216n+33 →
f(y) = 486n+76 → **243n+38 < 256n+39** O_{3b} = {39; 295; …} ⊂ O_3 : N(s) = **13**

Chain of connections :
O_{3b} → E → O_2* → E → **243n+38 < 256n+39** O_{3b}

f(x) = 81n+10 [O] : n = **2n+1**
64n+7 O_{3b} → **128n+71** O_{3b}
81n+10 → 162n+91 O_1 ∨ O_2 ∨ O_3 … …

243n+76 [O] : n = **2n+1**
128n+39 O_{3b} → **256n+167** O_{3b}
243n+76 → 486n+319 [O_1] : n = **2n+1**
256n+167 O_{3b} → **512n+423** O_{3b}
486n+319 → 972n+805 O_1 : x = 486n+402 → f(x) = 729n+604 [E] : n = **2n**
512n+423 O_{3b} → **1024n+423** O_{3b}
729n+604 → 1458n+604 = **729n+302 < 1024n+423** O_{3b}

Check :
1024n+423 O_{3b} : z = 512n+211 → f(z) = 3456n+1430 → 1728n+715 O_{2a} : y = 864n+357 →
f(y) = 1944n+805 O_1 : x = 972n+402 → f(x) = 1458n+604 → **729n+302 < 1024n+423** O_{3b} =
{423; 1447; …} ⊂ O_3 : N(s) = **16**

Chain of connections :
O_{3b} → E → O_{2a} → O_1 → E → **729n+302 < 1024n+423** O_{3b}

… …

54n+13 [O_3] : n = **4n+3**
32n+7 O_{3b} → **128n+103** O_{3b}
54n+13 → 216n+175 O_3 : z = 108n+87 → f(z) = 729n+593 [O] : n = **2n+1**
128n+103 O_{3b} → **256n+103** O_{3b}
729n+593 → 1458n+593 [O_1] : n = **2n**
256n+103 O_{3b} → **512n+103** O_{3b}
1458n+593 → 2916n+593 O_1 : x = 1458n+296 → f(x) = 2187n+445 [E] : n = **2n+1**
512n+103 O_{3b} → **1024n+615** O_{3b}
2187n+445 → 4374n+2632 → 2187n+1316 [E] : n = **2n**
1024n+615 O_{3b} → **2048n+615** O_{3b}
2187n+1316 → 4374n+2632 → 2187n+658 [E] : n = **2n**
2048n+615 O_{3b} → **4096n+615** O_{3b}
2187n+658 → 4374n+658 → **2187n+329** < **4096n+615** O_{3b}

Check :
4096n+615 O_{3b} : z = 2048n+307 → f(z) = 13824n+2078 → 6912n+1039 O_{3c} : z = 3456n+519 → f(z) = 23328n+3509 O_1 : x = 11664n+1754 → f(x) = 17496n+2632 → 8748n+1316 → 4374n+658 → **2187n+329** < **4096n+615** O_{3b} = {615; …} ⊂ O_3 : N(s) = **19**

Chain of connections :
O_{3b} → E → O_{3c} → O_1 → E → E → E → **2187n+329** < **4096n+615** O_{3b}

486n+319 [O_3] : n = **4n**
256n+167 O_{3b} → **1024n+167** O_{3b}
486n+319 → 1944n+319 O_3 : z = 972n+159 → f(z) = 6561n+1079 [E] : n = **2n+1**
1024n+167 O_{3b} → **2048n+1191** O_{3b}
6561n+1079 → 13122n+7640 → 6561n+3820 [E] : n = **2n**
2048n+1191 O_{3b} → **4096+1191** O_{3b}
6561n+3820 → 13122n+3820 → 6561n+1910 [E] : n = **2n**
4096+1191 O_{3b} → **8192n+1191** O_{3b}
6561n+1910 → 13122n+1910 → **6561n+955** < **8192n+1191** O_{3b}

Check :
8192n+1191 O_{3b} : z = 4096n+595 → f(z) = 27648n+4022 → 13824n+2011 O_{2a} : y = 6912n+1005 → f(y) = 15552n+2263 O_3* : z = 7776n+1131 → f(z) = 52488n+7640 → 26244n+3820 → 13122n+1910 → **6561n+955 < 8192n+1191** O_{3b} = {1191; …} ⊂ O_3 : N(s) = **21**

Chain of connections :
O_{3b} → E → O_{2a} → O_3* → E → E → E → **6561n+955 < 8192n+1191** O_{3b}

… …
… …

O_{3c}

32n+15 O_{3c} : z = 16n+7 → f(z) = 108n+53 O_1 : x = 54n+26 → f(x) = 81n+40 E ∨ O : N(s) = **9**

81n+40 [E] : n = **2n**
32n+15 O_{3c} → **64n+15** O_{3c}
81n+40 → 162n+40 → 81n+20 [E] : n = **2n**
64n+15 O_{3c} → **128n+15** O_{3c}
81n+20 → 162n+20 → **81n+10 < 128n+15** O_{3c}

Check :
128n+15 O_{3c} : z = 64n+7 → f(z) = 432n+53 O_1 : x = 216n+26 → f(x) = 324n+40 → 162n+20 → **81n+10** < **128n+15** O_{3c} = {15; 143; …} ⊂ O_3 : N(s) = **11**

Chain of connections :
O_{3c} → O_1 → E → E → **81n+10** < **128n+15** O_{3c} = {15; 143; …} ⊂ O_3 : N(s) = **11**

81n+20 [O] : n = **2n+1**
64n+15 O_{3c} → **128n+79** O_{3c}
81n+20 → 162n+101 [O_1] : n = **2n**

128n+79 O_{3c} → **256n+79** O_{3c}
162n+101 → 324n+101 O_1 : x = 162n+50 → f(x) = **243n+76 < 256n+79** O_{3c}

Check :
256n+79 O_{3c} : z = 128n+39 → f(z) = 864n+269 O_1 : x = 432n+134 →
f(x) = 648n+202 → 324n+101 O_1 : x = 162n+50 → f(x) = **243n+76** < **256n+79** O_{3c} =
{79; 335; …} ⊂ O_3 : N(s) = **13**

Chain of connections :
O_{3c} → O_1 → E → O_1 → **243n+76** < **256n+79** O_{3c}

… …
… …

81n+40 [O]: n = **2n+1**
32n+15 O_{3c} → **64n+47** O_{3c}
81n+40 → 162n+121 [O_1] : n = **2n**
64n+47 O_{3c} → 64(**2n**)+47 = **128n+47** O_{3c}
162n+121 → 162(**2n**)+121 = 324n+121 O_1 : x = 162n+60 → f(x) = 243n+91 E ∨ O … …

162n+121 [O_2] : n = **4n+1**
64n+47 O_{3c} → **256n+111** O_{3c}
162n+121 → 648n+283 O_2 : y = 324n+141 → f(y) = 729n+319 [E] : n = **2n+1**
256n+111 O_{3c} → **512n+367** O_{3c}
729n+319 → 729(**2n+1**)+319 = 1458n+1048 → 729n+524 [E] : n = **2n**
512n+367 O_{3c} → **1024n+367** O_{3c}
729n+524 → 1458n+524 → **729n+262** < **1024n+367** O_{3c}

Check :
1024n+367 O_{3c} : z = 512n+183 → f(z) = 3456n+1241 O_1 : x = 1728n+620 →
f(x) = 2592n+931 O_2* : y = 1296n+465 → f(y) = 2916n+1048 → 1458n+524 →
729n+262 < **1024n+367** O_{3c} = {367; …} ⊂ O_3 : N(s) = **16**

Chain of connections :
O_{3c} → O_1 → O_2* → E → E → **729n+262** < **1024n+367** O_{3c}

162n+101 [O_2] : n = **4n+3**
128n+79 O_{3c} → **512n+463** O_{3c}
162n+101 → 648n+587 O_2 : y = 324n+293 → f(y) = 729n+661 [O] : n = **2n**
512n+463 O_{3c} → 512(**2n**)+463 = **1024n+463** O_{3c}
729n+661 → 729(**2n**)+661 = 1458n+661 [O_1] : n = **2n**
1024n+463 O_{3c} → **2048n+463** O_{3c}
1458n+661 → 2916n+661 O_1 : x = 1458n+330 → f(x) = 2187n+496 [E] : n = **2n**
2048n+463 O_{3c} → **4096n+463** O_{3c}
2187n+496 → 4374n+496 → **2187n+248 < 4096n+463** O_{3c}

Check :
4096n+463 O_{3c} : z = 2048n+231 → f(z) = 13824n+1565 O_1 : x = 6912n+782 →
f(x) = 10368n+1174 → 5184n+587 O_{2a} : y = 2592n+293 →
f(y) = 5832n+661 O_1 : x = 2916n+330 → f(x) = 4374n+496 →
2187n+248 < 4096n+463 O_{3c} = {463; …} ⊂ O_3 : N(s) = **19**

Chain of connections :
$O_{3c} \rightarrow O_1 \rightarrow E \rightarrow O_{2a} \rightarrow O_1 \rightarrow E \rightarrow$ **2187n+248 < 4096n+463** O_{3c}

… …
… …

Let's consider f(x) = 243n+91 [O] : n = **2n**
128n+47 $O_{3c} \rightarrow$ **256n+47** O_{3c}
243n+91 → 486n+91 [O_3] : n = **4n+2**
256n+47 $O_{3c} \rightarrow$ **1024n+559** O_{3c}
486n+91 → 1944n+1063 O_3 : z = 972n+531 → f(z) = 6561n+3590 [E] : n = **2n**
1024n+559 $O_{3c} \rightarrow$ **2048n+559** O_{3c}
6561n+3590 → 13122n+3590 → 6561n+1795 [E] : n = **2n+1**
2048n+559 $O_{3c} \rightarrow$ **4096n+2607** O_{3c}
6561n+1795 → 13122n+8356 → 6561n+4178 [E] : n = **2n**
4096n+2607 $O_{3c} \rightarrow$ **8192n+2607** O_{3c}
6561n+4178 → 13122n+4178 → **6561n+2089 < 8192n+2607** O_{3c}

Check:
8192n+2607 O_{3c} : z = 4096n+1303 → f(z) = 27648n+8801 O_1 : x = 13824n+ 4400 → f(x) = 20736n+6601 O_1 : x = 10368n+3300 → f(x) = 15552n+4951 O_3* : z = 7776n+2475 → f(z) = 52488n+16712 → 26244n+8356 → 13122n+4178 → **6561n+2089** < **8192n+2607** O_{3c} = {2607; …} ⊂ O_3 : N(s) = **21**

Chain of connections :
$O_{3c} \rightarrow O_1 \rightarrow O_1 \rightarrow O_3^* \rightarrow E \rightarrow E \rightarrow E \rightarrow$ **6561n+2089** < **8192n+2607** O_{3c}

… …
… …
… …

By the procedure previously illustrated, appropriately choosing the connections between the eight cycles, applying the formulas obtained from Theorem 2n+1, using Theorem of Independence explained later (§ 3.2.); we arrive to the following list binomial inequalities N(s) ≤ 21.

2.4. General List binomial inequalities N(s) ≤ 21

N(s) = **3**

3n+1 < 4n+1 O_1 = **3n+1 <** $\mathbf{2^2 \cdot n+1}$ O_1

N(s) = **6**

9n+2 < 16n+3 O_2* = $\mathbf{3^2 \cdot n+2 < 2^4 \cdot n+3}$ O_2*

N(s) = **8**

27n+10 < 32n+11 O_{2a} = $\mathbf{3^3 \cdot n+10 < 2^5 \cdot n+11}$ O_{2a}
27n+20 < 32n+23 O_3* = $\mathbf{3^3 \cdot n+20 < 2^5 \cdot n+23}$ O_3*

N(s) = **11**

$81n+5 < 128n+7\ O_{3b}$ = $\mathbf{3^{4}\cdot n+5 < 2^{7}\cdot n+7}$ O_{3b}
$81n+10 < 128n+15\ O_{3c}$ = $\mathbf{3^{4}\cdot n+10 < 2^{7}\cdot n+15}$ O_{3c}
$81n+38 < 128n+59\ O_{2a}$ = $\mathbf{3^{4}\cdot n+38 < 2^{7}\cdot n+59}$ O_{2a}

N(s) = **13**

$243n+38 < 256n+39\ O_{3b}$ = $\mathbf{3^{5}\cdot n+38 < 2^{8}\cdot n+39}$ O_{3b}
$243n+76 < 256n+79\ O_{3c}$ = $\mathbf{3^{5}\cdot n+76 < 2^{8}\cdot n+79}$ O_{3c}
$243n+91 < 256n+95\ O_{3a}$ = $\mathbf{3^{5}\cdot n+91 < 2^{8}\cdot n+95}$ O_{3a}
$243n+118 < 256n+123\ O_{2a}$ = $\mathbf{3^{5}\cdot n+118 < 2^{8}\cdot n+123}$ O_{2a}
$243n+167 < 256n+175\ O_{3c}$ = $\mathbf{3^{5}\cdot n+167 < 2^{8}\cdot n+175}$ O_{3c}
$243n+190 < 256n+199\ O_{3b}$ = $\mathbf{3^{5}\cdot n+190 < 2^{8}\cdot n+199}$ O_{3b}
$243n+209 < 256n+219\ O_{2a}$ = $\mathbf{3^{5}\cdot n+209 < 2^{8}\cdot n+219}$ O_{2a}

N(s) = **16**

$729n+205 < 1024n+287\ O_{3a}$ = $\mathbf{3^{6}\cdot n+205 < 2^{10}\cdot n+287}$ O_{3a}
$729n+248 < 1024n+347\ O_{2a}$ = $\mathbf{3^{6}\cdot n+248 < 2^{10}\cdot n+347}$ O_{2a}
$729n+262 < 1024n+367\ O_{3c}$ = $\mathbf{3^{6}\cdot n+262 < 2^{10}\cdot n+367}$ O_{3c}
$729n+302 < 1024n+423\ O_{3b}$ = $\mathbf{3^{6}\cdot n+302 < 2^{10}\cdot n+423}$ O_{3b}
$729n+362 < 1024n+507\ O_{2a}$ = $\mathbf{3^{6}\cdot n+362 < 2^{10}\cdot n+507}$ O_{2a}
$729n+410 < 1024n+575\ O_{3a}$ = $\mathbf{3^{6}\cdot n+410 < 2^{10}\cdot n+575}$ O_{3a}
$729n+416 < 1024n+583\ O_{3b}$ = $\mathbf{3^{6}\cdot n+416 < 2^{10}\cdot n+583}$ O_{3b}
$729n+524 < 1024n+735\ O_{3a}$ = $\mathbf{3^{6}\cdot n+524 < 2^{10}\cdot n+735}$ O_{3a}
$729n+581 < 1024n+815\ O_{3c}$ = $\mathbf{3^{6}\cdot n+581 < 2^{10}\cdot n+815}$ O_{3c}
$729n+658 < 1024n+923\ O_{2a}$ = $\mathbf{3^{6}\cdot n+658 < 2^{10}\cdot n+923}$ O_{2a}
$729n+695 < 1024n+975\ O_{3c}$ = $\mathbf{3^{6}\cdot n+695 < 2^{10}\cdot n+975}$ O_{3c}
$729n+712 < 1024n+999\ O_{3b}$ = $\mathbf{3^{6}\cdot n+712 < 2^{10}\cdot n+999}$ O_{3b}

N(s) = **19**

$2187n+124 < 4096n+231\ O_{3b}$ = $\mathbf{3^{7}\cdot n+124 < 2^{12}\cdot n+231}$ O_{3b}
$2187n+205 < 4096n+383\ O_{3a}$ = $\mathbf{3^{7}\cdot n+205 < 2^{12}\cdot n+383}$ O_{3a}
$2187n+248 < 4096n+463\ O_{3c}$ = $\mathbf{3^{7}\cdot n+248 < 2^{12}\cdot n+463}$ O_{3c}
$2187n+329 < 4096n+615\ O_{3b}$ = $\mathbf{3^{7}\cdot n+329 < 2^{12}\cdot n+615}$ O_{3b}
$2187n+470 < 4096n+879\ O_{3c}$ = $\mathbf{3^{7}\cdot n+470 < 2^{12}\cdot n+879}$ O_{3c}
$2187n+500 < 4096n+935\ O_{3b}$ = $\mathbf{3^{7}\cdot n+500 < 2^{12}\cdot n+935}$ O_{3b}
$2187n+545 < 4096n+1019\ O_{2a}$ = $\mathbf{3^{7}\cdot n+545 < 2^{12}\cdot n+1019}$ O_{2a}
$2187n+581 < 4096n+1087\ O_{3a}$ = $\mathbf{3^{7}\cdot n+581 < 2^{12}\cdot n+1087}$ O_{3a}
$2187n+974 < 4096n+1215\ O_{3a}$ = $\mathbf{3^{7}\cdot n+974 < 2^{12}\cdot n+1215}$ O_{3a}
$2187n+658 < 4096n+1231\ O_{3c}$ = $\mathbf{3^{7}\cdot n+658 < 2^{12}\cdot n+1231}$ O_{3c}
$2187n+767 < 4096n+1435\ O_{2a}$ = $\mathbf{3^{7}\cdot n+767 < 2^{12}\cdot n+1435}$ O_{2a}
$2187n+880 < 4096n+1647\ O_{3c}$ = $\mathbf{3^{7}\cdot n+880 < 2^{12}\cdot n+1647}$ O_{3c}
$2187n+910 < 4096n+1703\ O_{3b}$ = $\mathbf{3^{7}\cdot n+910 < 2^{12}\cdot n+1703}$ O_{3b}
$2187n+955 < 4096n+1787\ O_{2a}$ = $\mathbf{3^{7}\cdot n+955 < 2^{12}\cdot n+1787}$ O_{2a}
$2187n+974 < 4096n+1823\ O_{3a}$ = $\mathbf{3^{7}\cdot n+974 < 2^{12}\cdot n+1823}$ O_{3a}
$2187n+991 < 4096n+1855\ O_{3a}$ = $\mathbf{3^{7}\cdot n+991 < 2^{12}\cdot n+1855}$ O_{3a}
$2187n+1085 < 4096n+2031\ O_{3c}$ = $\mathbf{3^{7}\cdot n+1085 < 2^{12}\cdot n+2031}$ O_{3c}
$2187n+1177 < 4096n+2203\ O_{2a}$ = $\mathbf{3^{7}\cdot n+1177 < 2^{12}\cdot n+2203}$ O_{2a}

2187n+1196 < 4096n+2239 O_{3a} = $\mathbf{3^7 \cdot n+1196 < 2^{12} \cdot n+2239}$ O_{3a}
2187n+1256 < 4096n+2351 O_{3c} = $\mathbf{3^7 \cdot n+1256 < 2^{12} \cdot n+2351}$ O_{3c}
2187n+1382 < 4096n+2587 O_{2a} = $\mathbf{3^7 \cdot n+1382 < 2^{12} \cdot n+2587}$ O_{2a}
2187n+1384 < 4096n+2591 O_{3a} = $\mathbf{3^7 \cdot n+1384 < 2^{12} \cdot n+2591}$ O_{3a}
2187n+1553 < 4096n+2907 O_{2a} = $\mathbf{3^7 \cdot n+1553 < 2^{12} \cdot n+2907}$ O_{2a}
2187n+1589 < 4096n+2975 O_{3a} = $\mathbf{3^7 \cdot n+1589 < 2^{12} \cdot n+2975}$ O_{3a}
2187n+1666 < 4096n+3119 O_{3c} = $\mathbf{3^7 \cdot n+1666 < 2^{12} \cdot n+3119}$ O_{3c}
2187n+1679 < 4096n+3143 O_{3b} = $\mathbf{3^7 \cdot n+1679 < 2^{12} \cdot n+3143}$ O_{3b}
2187n+1760 < 4096n+3295 O_{3a} = $\mathbf{3^7 \cdot n+1760 < 2^{12} \cdot n+3295}$ O_{3a}
2187n+1901 < 4096n+3559 O_{3b} = $\mathbf{3^7 \cdot n+1901 < 2^{12} \cdot n+3559}$ O_{3b}
2187n+1963 < 4096n+3675 O_{2a} = $\mathbf{3^7 \cdot n+1963 < 2^{12} \cdot n+3675}$ O_{2a}
2187n+2089 < 4096n+3911 O_{3b} = $\mathbf{3^7 \cdot n+2089 < 2^{12} \cdot n+3911}$ O_{3b}
2187n+2170 < 4096n+4063 O_{3a} = $\mathbf{3^7 \cdot n+2170 < 2^{12} \cdot n+4063}$ O_{3a}

N(s) = **21**

6561n+154 < 8192n+191 O_{3a} = $\mathbf{3^8 \cdot n+154 < 2^{13} \cdot n+191}$ O_{3a}
6561n+167 < 8192n+207 O_{3c} = $\mathbf{3^8 \cdot n+167 < 2^{13} \cdot n+207}$ O_{3c}
6561n+205 < 8192n+255 O_{3a} = $\mathbf{3^8 \cdot n+205 < 2^{13} \cdot n+255}$ O_{3a}
6561n+244 < 8192n+303 O_{3c} = $\mathbf{3^8 \cdot n+244 < 2^{13} \cdot n+303}$ O_{3c}
6561n+433 < 8192n+539 O_{2a} = $\mathbf{3^8 \cdot n+433 < 2^{13} \cdot n+539}$ O_{2a}
6561n+436 < 8192n+543 O_{3a} = $\mathbf{3^8 \cdot n+436 < 2^{13} \cdot n+543}$ O_{3a}
6561n+500 < 8192n+623 O_{3c} = $\mathbf{3^8 \cdot n+500 < 2^{13} \cdot n+623}$ O_{3a}
6561n+545 < 8192n+679 O_{3a} = $\mathbf{3^8 \cdot n+545 < 2^{13} \cdot n+679}$ O_{3a}
6561n+577 < 8192n+719 O_{3c} = $\mathbf{3^8 \cdot n+577 < 2^{13} \cdot n+719}$ O_{3c}
6561n+641 < 8192n+799 O_{3a} = $\mathbf{3^8 \cdot n+641 < 2^{13} \cdot n+799}$ O_{3a}
6561n+859 < 8192n+1071 O_{3c} = $\mathbf{3^8 \cdot n+859 < 2^{13} \cdot n+1071}$ O_{3c}
6561n+910 < 8192n+1135 O_{3c} = $\mathbf{3^8 \cdot n+910 < 2^{13} \cdot n+1135}$ O_{3c}
6561n+955 < 8192n+1191 O_{3b} = $\mathbf{3^8 \cdot n+955 < 2^{13} \cdot n+1191}$ O_{3b}
6561n+974 < 8192n+1215 O_{3a} = $\mathbf{3^8 \cdot n+974 < 2^{13} \cdot n+1215}$ O_{3a}
6561n+1000 < 8192n+1247 O_{3a} = $\mathbf{3^8 \cdot n+1000 < 2^{13} \cdot n+1247}$ O_{3a}
6561n+1064 < 8192n+1327 O_{3c} = $\mathbf{3^8 \cdot n+1064 < 2^{13} \cdot n+1327}$ O_{3c}
6561n+1253 < 8192n+1563 O_{2a} = $\mathbf{3^8 \cdot n+1253 < 2^{13} \cdot n+1563}$ O_{2a}
6561n+1256 < 8192n+1567 O_{3a} = $\mathbf{3^8 \cdot n+1256 < 2^{13} \cdot n+1567}$ O_{3a}
6561n+1384 < 8192n+1727 O_{3a} = $\mathbf{3^8 \cdot n+1384 < 2^{13} \cdot n+1727}$ O_{3a}
6561n+1589 < 8192n+1983 O_{3a} = $\mathbf{3^8 \cdot n+1589 < 2^{13} \cdot n+1983}$ O_{3a}
6561n+1615 < 8192n+2015 O_{3a} = $\mathbf{3^8 \cdot n+1615 < 2^{13} \cdot n+2015}$ O_{3a}
6561n+1663 < 8192n+2075 O_{2a} = $\mathbf{3^8 \cdot n+1663 < 2^{13} \cdot n+2075}$ O_{2a}
6561n+1666 < 8192n+2079 O_{3a} = $\mathbf{3^8 \cdot n+1666 < 2^{13} \cdot n+2079}$ O_{3a}
6561n+1679 < 8192n+2095 O_{3c} = $\mathbf{3^8 \cdot n+1679 < 2^{13} \cdot n+2095}$ O_{3c}
6561n+1820 < 8192n+2271 O_{2a} = $\mathbf{3^8 \cdot n+1820 < 2^{13} \cdot n+2271}$ O_{3a}
6561n+1868 < 8192n+2331 O_{2a} = $\mathbf{3^8 \cdot n+1868 < 2^{13} \cdot n+2331}$ O_{2a}
6561n+1948 < 8192n+2431 O_{3a} = $\mathbf{3^8 \cdot n+1948 < 2^{13} \cdot n+2431}$ O_{3a}
6561n+2089 < 8192n+2607 O_{3c} = $\mathbf{3^8 \cdot n+2089 < 2^{13} \cdot n+2607}$ O_{3c}
6561n+2134 < 8192n+2663 O_{3b} = $\mathbf{3^8 \cdot n+2134 < 2^{13} \cdot n+2663}$ O_{3b}
6561n+2435 < 8192n+3039 O_{3a} = $\mathbf{3^8 \cdot n+2435 < 2^{13} \cdot n+3039}$ O_{3a}
6561n+2458 < 8192n+3067 O_{2a} = $\mathbf{3^8 \cdot n+2458 < 2^{13} \cdot n+3067}$ O_{2a}
6561n+2512 < 8192n+3135 O_{3a} = $\mathbf{3^8 \cdot n+2512 < 2^{13} \cdot n+3135}$ O_{3a}
6561n+2768 < 8192n+3455 O_{3a} = $\mathbf{3^8 \cdot n+2768 < 2^{13} \cdot n+3455}$ O_{3a}
6561n+2791 < 8192n+3483 O_{2a} = $\mathbf{3^8 \cdot n+2791 < 2^{13} \cdot n+3483}$ O_{2a}
6561n+2845 < 8192n+3551 O_{3a} = $\mathbf{3^8 \cdot n+2845 < 2^{13} \cdot n+3551}$ O_{3a}

$6561n+2954 < 8192n+3687$ O_{3b} = $\mathbf{3^8 \cdot n+2954 < 2^{13} \cdot n+3687}$ O_{3b}
$6561n+3073 < 8192n+3835$ O_{2a} = $\mathbf{3^8 \cdot n+3073 < 2^{13} \cdot n+3835}$ O_{2a}
$6561n+3127 < 8192n+3903$ O_{3a} = $\mathbf{3^8 \cdot n+3127 < 2^{13} \cdot n+3903}$ O_{3a}
$6561n+3178 < 8192n+3967$ O_{3a} = $\mathbf{3^8 \cdot n+3178 < 2^{13} \cdot n+3967}$ O_{3a}
$6561n+3268 < 8192n+4079$ O_{3c} = $\mathbf{3^8 \cdot n+3268 < 2^{13} \cdot n+4079}$ O_{3c}
$6561n+3278 < 8192n+4091$ O_{2a} = $\mathbf{3^8 \cdot n+3278 < 2^{13} \cdot n+4091}$ O_{2a}
$6561n+3332 < 8192n+4159$ O_{3a} = $\mathbf{3^8 \cdot n+3332 < 2^{13} \cdot n+4159}$ O_{3a}
$6561n+3364 < 8192n+4199$ O_{3b} = $\mathbf{3^8 \cdot n+3664 < 2^{13} \cdot n+4199}$ O_{3b}
$6561n+3383 < 8192n+4223$ O_{3a} = $\mathbf{3^8 \cdot n+3383 < 2^{13} \cdot n+4223}$ O_{3a}
$6561n+3406 < 8192n+4251$ O_{2a} = $\mathbf{3^8 \cdot n+3406 < 2^{13} \cdot n+4251}$ O_{2a}
$6561n+3569 < 8192n+4455$ O_{3b} = $\mathbf{3^8 \cdot n+3569 < 2^{13} \cdot n+4455}$ O_{3b}
$6561n+3661 < 8192n+4507$ O_{2a} = $\mathbf{3^8 \cdot n+3661 < 2^{13} \cdot n+4507}$ O_{2a}
$6561n+3893 < 8192n+4859$ O_{2a} = $\mathbf{3^8 \cdot n+3893 < 2^{13} \cdot n+4859}$ O_{2a}
$6561n+3947 < 8192n+4927$ O_{3a} = $\mathbf{3^8 \cdot n+3947 < 2^{13} \cdot n+4927}$ O_{3a}
$6561n+3970 < 8192n+4955$ O_{2a} = $\mathbf{3^8 \cdot n+3970 < 2^{13} \cdot n+4955}$ O_{2a}
$6561n+4024 < 8192n+5023$ O_{3a} = $\mathbf{3^8 \cdot n+4024 < 2^{13} \cdot n+5023}$ O_{3a}
$6561n+4088 < 8192n+5103$ O_{3c} = $\mathbf{3^8 \cdot n+4088 < 2^{13} \cdot n+5103}$ O_{3c}
$6561n+4159 < 8192n+5191$ O_{3b} = $\mathbf{3^8 \cdot n+4159 < 2^{13} \cdot n+5191}$ O_{3b}
$6561n+4226 < 8192n+5275$ O_{2a} = $\mathbf{3^8 \cdot n+4226 < 2^{13} \cdot n+5275}$ O_{2a}
$6561n+4303 < 8192n+5371$ O_{2a} = $\mathbf{3^8 \cdot n+4303 < 2^{13} \cdot n+5371}$ O_{2a}
$6561n+4357 < 8192n+5439$ O_{3a} = $\mathbf{3^8 \cdot n+4357 < 2^{13} \cdot n+5439}$ O_{3a}
$6561n+4492 < 8192n+5607$ O_{3b} = $\mathbf{3^8 \cdot n+4492 < 2^{13} \cdot n+5607}$ O_{3b}
$6561n+4498 < 8192n+5615$ O_{3c} = $\mathbf{3^8 \cdot n+4498 < 2^{13} \cdot n+5615}$ O_{3c}
$6561n+4585 < 8192n+5723$ O_{2a} = $\mathbf{3^8 \cdot n+4585 < 2^{13} \cdot n+5723}$ O_{2a}
$6561n+4636 < 8192n+5787$ O_{2a} = $\mathbf{3^8 \cdot n+4636 < 2^{13} \cdot n+5787}$ O_{2a}
$6561n+4703 < 8192n+5871$ O_{3c} = $\mathbf{3^8 \cdot n+4703 < 2^{13} \cdot n+5871}$ O_{3c}
$6561n+4774 < 8192n+5959$ O_{3b} = $\mathbf{3^8 \cdot n+4774 < 2^{13} \cdot n+5959}$ O_{3b}
$6561n+4790 < 8192n+5979$ O_{2a} = $\mathbf{3^8 \cdot n+4790 < 2^{13} \cdot n+5979}$ O_{2a}
$6561n+4844 < 8192n+6047$ O_{3a} = $\mathbf{3^8 \cdot n+4844 < 2^{13} \cdot n+6047}$ O_{3a}
$6561n+4979 < 8192n+6215$ O_{3b} = $\mathbf{3^8 \cdot n+4979 < 2^{13} \cdot n+6215}$ O_{3b}
$6561n+5107 < 8192n+6375$ O_{3b} = $\mathbf{3^8 \cdot n+5107 < 2^{13} \cdot n+6375}$ O_{3b}
$6561n+5254 < 8192n+6559$ O_{3a} = $\mathbf{3^8 \cdot n+5254 < 2^{13} \cdot n+6559}$ O_{3a}
$6561n+5293 < 8192n+6607$ O_{3a} = $\mathbf{3^8 \cdot n+5293 < 2^{13} \cdot n+6607}$ O_{3a}
$6561n+5312 < 8192n+6631$ O_{3b} = $\mathbf{3^8 \cdot n+5312 < 2^{13} \cdot n+6631}$ O_{3b}
$6561n+5405 < 8192n+6747$ O_{2a} = $\mathbf{3^8 \cdot n+5405 < 2^{13} \cdot n+6747}$ O_{2a}
$6561n+5671 < 8192n+6783$ O_{3a} = $\mathbf{3^8 \cdot n+5671 < 2^{13} \cdot n+6783}$ O_{3a}
$6561n+5459 < 8192n+6815$ O_{3a} = $\mathbf{3^8 \cdot n+5459 < 2^{13} \cdot n+6815}$ O_{3a}
$6561n+5594 < 8192n+6983$ O_{3b} = $\mathbf{3^8 \cdot n+5594 < 2^{13} \cdot n+6983}$ O_{3b}
$6561n+5626 < 8192n+7023$ O_{3c} = $\mathbf{3^8 \cdot n+5626 < 2^{13} \cdot n+7023}$ O_{3c}
$6561n+5671 < 8192n+7079$ O_{3b} = $\mathbf{3^8 \cdot n+5671 < 2^{13} \cdot n+7079}$ O_{3b}
$6561n+5815 < 8192n+7259$ O_{2a} = $\mathbf{3^8 \cdot n+5815 < 2^{13} \cdot n+7259}$ O_{2a}
$6561n+5908 < 8192n+7375$ O_{3c} = $\mathbf{3^8 \cdot n+5908 < 2^{13} \cdot n+7375}$ O_{3c}
$6561n+5927 < 8192n+7399$ O_{3b} = $\mathbf{3^8 \cdot n+5927 < 2^{13} \cdot n+7399}$ O_{3b}
$6561n+6004 < 8192n+7495$ O_{3b} = $\mathbf{3^8 \cdot n+6004 < 2^{13} \cdot n+7495}$ O_{3b}
$6561n+6113 < 8192n+7631$ O_{3c} = $\mathbf{3^8 \cdot n+6113 < 2^{13} \cdot n+7631}$ O_{3c}
$6561n+6241 < 8192n+7791$ O_{3c} = $\mathbf{3^8 \cdot n+6241 < 2^{13} \cdot n+7791}$ O_{3c}
$6561n+6286 < 8192n+7847$ O_{3b} = $\mathbf{3^8 \cdot n+6286 < 2^{13} \cdot n+7847}$ O_{3b}
$6561n+6337 < 8192n+7911$ O_{3b} = $\mathbf{3^8 \cdot n+6337 < 2^{13} \cdot n+7911}$ O_{3b}
$6561n+6382 < 8192n+7967$ O_{3a} = $\mathbf{3^8 \cdot n+6382 < 2^{13} \cdot n+7967}$ O_{3a}
$6561n+6446 < 8192n+8047$ O_{3c} = $\mathbf{3^8 \cdot n+6446 < 2^{13} \cdot n+8047}$ O_{3c}
$6561n+6491 < 8192n+8103$ O_{3b} = $\mathbf{3^8 \cdot n+6491 < 2^{13} \cdot n+8103}$ O_{3b}

The binomial inequalities are of type:

$$3^h \cdot n + q < \ 2^k \cdot n + p$$

- p is the well-know term of the binomial main horizons; q is the corresponding well-know term of the binomial lower horizons : $q < p$.
- h + k = number of steps N(s) : N(s) increment = 2 or 3
- Increase of k 1 or 2 : 1 if N(s) increment = 2 ; 2 if N(s) increment = 3
- Increase of h always 1

We note that Collatz sequences arrive to the links by unpredictable ways and without generalized rules. For N(s) = 21 there are 86 chains of connections and related links. The number of links increase considerably with the increase of N(s).

Remark

To compile the above list, in some cases of the same N(s), the following method was used.
If Θ(m1) < Θ(m2) ($p_1 < p_2$) are two consecutive main horizons and p is a generic well-know term, such that $p_1 < \ p \ < p_2$, then: p = {$p_2$mod4 - $p \in O_2^* \cup p \in O_3^* \cup p \sim$ N(s) previously calculated} (§ 3.3.3.).

2.4.1. Percentages

From the list above we have the following percentages:

N(s) = 1 E : 2n → n … … … … … … … … … … … … … … … … … 50
N(s) = 3 O_1 : 4n+1 → 3n + 1 … … … … … … … … … … … … … … 25
N(s) = 6 O_2^* : 16n+3 → 9n+2 … … … … … … … … … … … … … … 6.25
N(s) = 8 O_{2a} ; O_3^* … … … … … … … … … … … … … … … … … … 6.25
N(s) = 11 : 0.78125 x 3 … … … … … … … … … … … … … … … … … … 2.344
N(s) = 13 : 0.39062… x 7 … … … … … … … … … … … … … … … … … 2.734
N(s) = 16 : 0.09765… x 12 … … … … … … … … … … … … … … … … 1.172
N(s) = 19 : 0.02441… x 31 … … … … … … … … … … … … … … … … 0.757
N(s) = 21 : 0.01221… x 86 … … … … … … … … … … … … … … … … 1.050

≈ 96 %

The remaining 4% is covered by the binomial inequalities of the infinite cycles of links. By apposite calculation tools it is possible to reach close by 100% of ℕ, but it's not possible to arrive 100% coverage of ℕ. As it will be proved, for every biggest horizon we are able to cover by a binomial inequality after N(s) steps, there is an upper horizon that needs a greater number of steps N(s) to become lower than itself. So we can affirm that SC is a sort of *Circle Quadrature*.

3. SC Quadrature

3.1. Theorem of duplication

Statement

If a binomial inequality is valid for the horizon $2^{\alpha}\cdot n+p$ then it also applies to the horizon $2\cdot 2^{\alpha}\cdot n+p$, i.e.: $2^{\alpha+1}\cdot n+p$.

Demonstration

O_1 : $\mathbf{2^{\alpha+1}\cdot n+1}$ O_1 : x = $2^{\alpha}\cdot n$ → f(x) = $\mathbf{3\cdot 2^{\alpha-1}\cdot n+1}$ < $\mathbf{2^{\alpha+1}\cdot n+1}$ O_1 : [α ≥ 1]
α = 1 → **3n+1** < **4n+1** O_1 principal inequality

In O_2 we take into consideration just O_{2a} (O_2* is perfectly similar)

O_{2a} : $\mathbf{2^{\alpha+1}\cdot n+11}$ O_{2a} : y = $2^{\alpha}\cdot n+5$ → f(y) = $3^2\cdot 2^{\alpha-2}\cdot n+13$ O_1 : x = $3^2\cdot 2^{\alpha-3}\cdot n+6$ →
f(x) = $\mathbf{3^3\cdot 2^{\alpha-4}\cdot n+10}$ < $\mathbf{2^{\alpha+1}\cdot n+11}$ O_{2a} : [α ≥ 4]
α = 4 → **27n+10 < 32n+11** O_{2a} principal inequality

In O_3 we take into consideration just O_{3c} (O_3*, O_{3a} and O_{3b} are perfectly similar).

O_{3c} : $\mathbf{2^{\alpha+1}\cdot n+15}$ O_{3c} : z = $2^{\alpha}\cdot n+7$ → f(z) = $3^3\cdot 2^{\alpha-2}\cdot n+53$ O_1 : x = $3^3\cdot 2^{\alpha-3}\cdot n+26$ → f(x) = $3^4\cdot 2^{\alpha-4}\cdot n+40$ → $3^4\cdot 2^{\alpha-5}\cdot n+20$ → $\mathbf{3^4\cdot 2^{\alpha-6}\cdot n+10}$ < $\mathbf{2^{\alpha+1}\cdot n+15}$ O_{3c} : [α ≥ 6]

α = 6 → **81n+10 < 128n+15** O_{3c} principal inequality
… …

3.1.1. Corollary

If a binomial inequality is valid for the horizon $2^{\alpha}\cdot n+p$ then it's valid for the horizon $2^{\alpha+i}\cdot n+p$ for every i ∈ ℕ

We use the principle of induction :

(1) i = 1 → $2^{\alpha+1}\cdot n+p$ is true for Theorem of duplication
(2) Supposed true $2^{\alpha+i}\cdot n+p$
(3) $2^{\alpha+i+1}\cdot n+p = 2\cdot(2^{\alpha+i})\cdot n+p \sim 2^{\alpha+i}\cdot n+p$ (1) → $2^{\alpha+i}\cdot n+p$ is true (2) → QED

3.2. Theorem of Independence

Statement

In every horizon $2^{\alpha}\cdot n+p$ the well-know term of the principal binomial inequality is independent of the term parametric from an appropriate exponent α onwards.

Demonstration

$\mathbf{O_1}$: 4k+1 O_1 : k = {1; 2; 3; …}
$\mathbf{2^{\alpha}\cdot n+4k+1}$ O_1 : x = $2^{\alpha-1}\cdot n+2k$ → f(x) = $\mathbf{3\cdot 2^{\alpha-2}\cdot n+3k+1}$ < $\mathbf{2^{\alpha}\cdot n+4k+1}$ O_1 : [α ≥ 2]

α = 2 → 3n+3k+1 < 4n+4k+1 → 3(n + k)+1 < 4(n + k)+1 O_1
n = 0 → **3k+1 < 4k+1** O_1 principal inequality : N(s) = **3**

$\mathbf{O_2}$* : 16k+3 O_2* : k = {1; 2; 3; …}

$\mathbf{2^{\alpha}\cdot n+16k+3}$ O_2* : y = $2^{\alpha-1}\cdot n+8k+1 \rightarrow f(y) = 3^2\cdot 2^{\alpha-3}\cdot n+18k+4 \rightarrow$
$\mathbf{3^2\cdot 2^{\alpha-4}\cdot n+9k+2} < \mathbf{2^{\alpha}\cdot n+16k+3}$ O_2* : $[\alpha \geq 4]$

$\alpha = 4 \rightarrow 9n+9k+2 < 16n+16k+3 \rightarrow 9(n + k)+2 < 16(n + k)+3$ O_2*
n = 0 → **9k+2 < 16k+3** O_2* principal inequality : N(s) = **6**

O_3* : 32k-9 ~ 32k+23 O_3* : k = {0; 1; 2; …}
$\mathbf{2^{\alpha}\cdot n+32k+23}$ O_3* : z = $2^{\alpha-1}\cdot n+16k+11 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+80 \rightarrow 3^3\cdot 2^{\alpha-4}\cdot n+54k+40 \rightarrow$
$\mathbf{3^3\cdot 2^{\alpha-5}\cdot n+27k+20} < \mathbf{2^{\alpha}\cdot n+32k+23}$ O_3* : $[\alpha \geq 5]$

$\alpha = 5 \rightarrow 27n+27k+20 < 32n+32k+23 \rightarrow 27(n + k)+20 < 32(n + k)+23$ O_3*
n = 0 → **27k+20 < 32k+23** O_3* principal inequality : N(s) = **8**

If 2k+1 ∈ O_{2a} ∨ O_{3a} ∨ O_{3b} ∨ O_{3c} now we prove the theorem of independence for the most favorable cycles with the fewest number of steps. As it will be proved it's applicable to all other cycles.

O_{2a} : 16k+11 O_{2a} : k = {0; 1; 2; …}

$\mathbf{2^{\alpha}\cdot n+16k+11}$ O_{2a} : y = $2^{\alpha-1}\cdot n+8k+5 \rightarrow f(y) = 3^2\cdot 2^{\alpha-3}\cdot n+18k +13$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+16k+11} \rightarrow 2^{\alpha}\cdot n+16(\mathbf{2k})+11 = \mathbf{2^{\alpha}\cdot n+32k+11}$ O_{2a}
$3^2\cdot 2^{\alpha-3}\cdot n+18k+13 \rightarrow 3^2\cdot 2^{\alpha-3}\cdot n+18(\mathbf{2k})+13 = 3^2\cdot 2^{\alpha-3}\cdot n+36k+13$ O_1 : x = $3^2\cdot 2^{\alpha-4}\cdot n+18k+6 \rightarrow$
f(x) = $\mathbf{3^3\cdot 2^{\alpha-5}\cdot n+27k+10 < 2^{\alpha}\cdot n+32k+11}$ O_{2a} : $[\alpha \geq 5]$

$\alpha = 5 \rightarrow 27n+27k+10 < 32n+32k+11 \rightarrow 27(n + k)+10 < 32(n + k)+11$ O_{2a}
n = 0 → **27k+10 < 32k+11** O_{2a} principal inequality : N(s) = **8**
n = k = 0 → **10 < 11** O_{2a} nude number

O_{3a} : 32k-1 ~ 32k+31 O_{3a} : k = {0; 1; 2; …}

$\mathbf{2^{\alpha}\cdot n+32k+31}$ O_{3a} : z = $2^{\alpha-1}\cdot n+16k+15 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+107$ [O_2] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+32k+31} \rightarrow 2^{\alpha}\cdot n+32(\mathbf{2k})+31 = \mathbf{2^{\alpha}\cdot n+64k+31}$ O_{3a}
$3^3\cdot 2^{\alpha-3}\cdot n+108k+107 \rightarrow 3^3\cdot 2^{\alpha-3}\cdot n+108(\mathbf{2k})+107 = 3^3\cdot 2^{\alpha-3}\cdot n+216k+107$ O_2 :
y = $3^3\cdot 2^{\alpha-4}\cdot n+108k+53 \rightarrow f(y) = 3^5\cdot 2^{\alpha-6}\cdot n+243k+121$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+64k+31} \rightarrow 2^{\alpha}\cdot n+64(\mathbf{2k+1})+31 \rightarrow \mathbf{2^{\alpha}\cdot n+128k+95}$ O_{3a}
$3^5\cdot 2^{\alpha-6}\cdot n+243k+121 \rightarrow 3^5\cdot 2^{\alpha-6}\cdot n+243(\mathbf{2k+1})+121 \rightarrow 3^5\cdot 2^{\alpha-6}\cdot n+486k+364 \rightarrow$
$3^5\cdot 2^{\alpha-7}\cdot n+243k+182$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+128k+95} \rightarrow 2^{\alpha}\cdot n+128(\mathbf{2k})+5 = \mathbf{2^{\alpha}\cdot n+256k+95}$ O_{3a}
$3^5\cdot 2^{\alpha-7}\cdot n+243k+182 \rightarrow 3^5\cdot 2^{\alpha-7}\cdot n+243(\mathbf{2k})+182 \rightarrow 3^5\cdot 2^{\alpha-7}\cdot n+486k+182 \rightarrow$
$\mathbf{3^5\cdot 2^{\alpha-8}\cdot n+243k+91} < \mathbf{2^{\alpha}\cdot n+256k+95}$ O_{3a} : $[\alpha \geq 8]$

$\alpha = 8 \rightarrow 243n+243k+91 < 256n+256k+95 \rightarrow 243(n + k)+91 < 256(n + k)+95$ O_{3a}
n = 0 → **243k+91 < 256k+95** O_{3a} principal inequality : N(s) = **13**
n = k = 0 → **91** O_{2a} **< 95** O_{3a} nude number

O_{3b} : 32k+7 O_{3b} : k = {0; 1; 2; …}

$\mathbf{2^{\alpha}\cdot n+32k+7}$ O_{3b} : z = $2^{\alpha-1}\cdot n+16k+3 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+26 \rightarrow$

$3^3 \cdot 2^{\alpha-4} \cdot n+54k+13$ [O_1] : k = **2k**

$\mathbf{2^{\alpha} \cdot n+32k+7}$ → $2^{\alpha} \cdot n+32(\mathbf{2k})+7$ = $\mathbf{2^{\alpha} \cdot n+64k+7}$ O_{3b}
$3^3 \cdot 2^{\alpha-4} \cdot n+54k+13$ → $3^3 \cdot 2^{\alpha-4} \cdot n+54(\mathbf{2k})+13 = 3^3 \cdot 2^{\alpha-4} \cdot n+108k+13$ O_1 :
x = $3^3 \cdot 2^{\alpha-5} \cdot n+54k+6$ → f(x) = $3^4 \cdot 2^{\alpha-6} \cdot n+81k+10$ [E] : k = **2k**

$\mathbf{2^{\alpha} \cdot n+64k+7}$ → $2^{\alpha} \cdot n+64(\mathbf{2k})+7$ = $\mathbf{2^{\alpha} \cdot n+128k+7}$ O_{3b}
$3^4 \cdot 2^{\alpha-6} \cdot n+81k+10$ → $3^4 \cdot 2^{\alpha-6} \cdot n+81(\mathbf{2k})+10 = 3^4 \cdot 2^{\alpha-6} \cdot n+162k+10$ →
$\mathbf{3^4 \cdot 2^{\alpha-7} \cdot n+81k+5}$ < $\mathbf{2^{\alpha} \cdot n+128k+7}$ O_{3b} : [$\alpha \geq 7$]

$\alpha = 7$ → 81n+81k+5 < 128n+128k+7 → 81(n + k)+5 < 128(n + k)+7 O_{3b}
n = 0 → **81k+5 < 128k+7** O_{3b} principal inequality : N(s) = **11**
n = k = 0 → **5** O_1 **< 7** O_{3b} nude number

$\mathbf{O_{3c}}$: 32k+15 O_{3c} : k = {0; 1; 2; …}

$\mathbf{2^{\alpha} \cdot n+32k+15}$ O_{3c} : z = $2^{\alpha-1} \cdot n+16k+7$ → f(z) = $3^3 \cdot 2^{\alpha-3} \cdot n+108k+53$ O_1 :
x = $3^3 \cdot 2^{\alpha-4} \cdot n+54k+26$ → f(x) = $3^4 \cdot 2^{\alpha-5} \cdot n+81k+40$ [E] : k = **2k**

$\mathbf{2^{\alpha} \cdot n+32k+15}$ → $2^{\alpha} \cdot n+32(\mathbf{2k})+15$ = $\mathbf{2^{\alpha} \cdot n+64k+15}$ O_{3c}
$3^4 \cdot 2^{\alpha-5} \cdot n+81k+40$ → $3^4 \cdot 2^{\alpha-5} \cdot n+81(\mathbf{2k})+40 = 3^4 \cdot 2^{\alpha-5} \cdot n+162k+40$ →
$3^4 \cdot 2^{\alpha-6} \cdot n+81k+20$ [E] : k = **2k**

$\mathbf{2^{\alpha} \cdot n+64k+15}$ → $2^{\alpha} \cdot n+64(\mathbf{2k})+15$ = $\mathbf{2^{\alpha} \cdot n+128k+15}$ O_{3c}
$3^4 \cdot 2^{\alpha-6} \cdot n+81k+20$ → $3^4 \cdot 2^{\alpha-6} \cdot n+81(\mathbf{2k})+20 = 3^4 \cdot 2^{\alpha-6} \cdot n+162k+20$ →
$\mathbf{3^4 \cdot 2^{\alpha-7} \cdot n+81k+10}$ < $\mathbf{2^{\alpha} \cdot n+128k+15}$ O_{3c} : [$\alpha \geq 7$]

$\alpha = 7$ → 81n+81k+10 < 128n+128k+15 → 81(n + k)+10 < 128(n + k)+15 O_{3c}
n = 0 → **81k+10 < 128k+15** O_{3c} principal inequality : N(s) = **11**
n = k = 0 → **10 < 15** O_{3c} nude number

Now we take into consideration 16k+11 O_{2a} and we apply a cycle managed by us:

$\mathbf{2^{\alpha} \cdot n+16k+11}$ O_{2a} : y = $2^{\alpha-1} \cdot n+8k+5$ → f(y) = $3^2 \cdot 2^{\alpha-3} \cdot n+18k+13$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha} \cdot n+16k+11}$ O_{2a} → $2^{\alpha} \cdot n+16(\mathbf{4k+1})+11$ = $\mathbf{2^{\alpha} \cdot n+64k+27}$ O_{2a}
$3^2 \cdot 2^{\alpha-3} \cdot n+18k+13$ → $3^2 \cdot 2^{\alpha-3} \cdot n+18(\mathbf{4k+1})+13 = 3^2 \cdot 2^{\alpha-3} \cdot n+72k+31$ O_3 :
z = $3^2 \cdot 2^{\alpha-4} \cdot n+36k+15$ → f(z) = $3^5 \cdot 2^{\alpha-6} \cdot n+243k+107$ [E] : k = **2k+1**

$\mathbf{2^{\alpha} \cdot n+64k+27}$ O_{2a} → $2^{\alpha} \cdot n+64(\mathbf{2k+1})+27$ = $\mathbf{2^{\alpha} \cdot n+128k+91}$ O_{2a}
$3^5 \cdot 2^{\alpha-6} \cdot n+243k+107$ → $3^5 \cdot 2^{\alpha-6} \cdot n+243(\mathbf{2k+1})+107 = 3^5 \cdot 2^{\alpha-6} \cdot n+486k+350$ →
$3^5 \cdot 2^{\alpha-7} \cdot n+243k+175$ [O] : k = **2k**

$\mathbf{2^{\alpha} \cdot n+128k+91}$ O_{2a} → $2^{\alpha} \cdot n+128(\mathbf{2k})+91$ = $\mathbf{2^{\alpha} \cdot n+256k+91}$ O_{2a}
$3^5 \cdot 2^{\alpha-7} \cdot n+243k+175$ → $3^5 \cdot 2^{\alpha-7} \cdot n+243(\mathbf{2k})+175 = 3^5 \cdot 2^{\alpha-7} \cdot n+486k+175$ [O_3] : k = **4k**

$\mathbf{2^{\alpha} \cdot n+256k+91}$ O_{2a} → $2^{\alpha} \cdot n+256(\mathbf{4k})+91$ = $\mathbf{2^{\alpha} \cdot n+1024k+91}$ O_{2a}
$3^5 \cdot 2^{\alpha-7} \cdot n+486k+175$ → $3^5 \cdot 2^{\alpha-7} \cdot n+486(\mathbf{4k})+175 = 3^5 \cdot 2^{\alpha-7} \cdot n+1944k+175$ O_3 :
z = $3^5 \cdot 2^{\alpha-8} \cdot n+972k+87$ → f(z) = $3^8 \cdot 2^{\alpha-10} \cdot n+6561k+593$ [E] : k = **2k+1**

$\mathbf{2^{\alpha} \cdot n+1024k+91}$ O_{2a} → $2^{\alpha} \cdot n+1024(\mathbf{2k+1})+91$ = $\mathbf{2^{\alpha} \cdot n+2048k+1115}$ O_{2a}

$3^8\cdot 2^{\alpha-10}\cdot n+6561k+593 \rightarrow 3^8\cdot 2^{\alpha-10}\cdot n+6561(\mathbf{2k+1})+593 = 3^8\cdot 2^{\alpha-10}\cdot n+13122k+7154 \rightarrow$
$3^8\cdot 2^{\alpha-11}\cdot n+6561k+3577$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+2048k+1115}$ $O_{2a} \rightarrow 2^{\alpha}\cdot n+2048(\mathbf{2k})+1115 = \mathbf{2^{\alpha}\cdot n+4096k+1115}$ O_{2a}
$3^8\cdot 2^{\alpha-11}\cdot n+6561k+3577 \rightarrow 3^8\cdot 2^{\alpha-11}\cdot n+6561(\mathbf{2k})+3577 = 3^8\cdot 2^{\alpha-11}\cdot n+13122k+3577$ $[O_1]$: k = **2k**

$\mathbf{2^{\alpha}\cdot n+4096k+1115}$ $O_{2a} \rightarrow 2^{\alpha}\cdot n+40968(\mathbf{2k})+1115 = \mathbf{2^{\alpha}\cdot n+8192k+1115}$ O_{2a}
$3^8\cdot 2^{\alpha-11}\cdot n+13122k+3577 \rightarrow 3^8\cdot 2^{\alpha-11}\cdot n+13122(\mathbf{2k})+3577 = 3^8\cdot 2^{\alpha-11}\cdot n+26244k+3577$ O_1 :
$x = 3^8\cdot 2^{\alpha-12}\cdot n+13122k+1788 \rightarrow f(x) = 3^9\cdot 2^{\alpha-13}\cdot n+19683k+2683$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+8192k+1115}$ $O_{2a} \rightarrow 2^{\alpha}\cdot n+8192(\mathbf{2k+1})+1115 = \mathbf{2^{\alpha}\cdot n+16384k+9307}$ O_{2a}
$3^9\cdot 2^{\alpha-13}\cdot n+19683k+2683 \rightarrow 3^9\cdot 2^{\alpha-13}\cdot n+19683(\mathbf{2k+1})+2683 = 3^9\cdot 2^{\alpha-13}\cdot n+39366k+22366 \rightarrow$
$3^9\cdot 2^{\alpha-14}\cdot n+19683k+11183$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+16384k+9307}$ $O_{2a} \rightarrow 2^{\alpha}\cdot n+16384(\mathbf{2k+1})+9307 = \mathbf{2^{\alpha}\cdot n+32768k+25691}$ O_{2a}
$3^9\cdot 2^{\alpha-14}\cdot n+19683k+11183 \rightarrow 3^9\cdot 2^{\alpha-14}\cdot n+19683(\mathbf{2k+1})+11183 = 3^9\cdot 2^{\alpha-14}\cdot n+39366k+30866 \rightarrow$
$3^9\cdot 2^{\alpha-15}\cdot n+19683k+15433 < 2^{\alpha}\cdot n+32768k+25691$ O_{2a} $[\alpha \geq 15]$

$\alpha = 15 \rightarrow 3^9\cdot n+3^9\cdot k+15433 < 2^{15}\cdot n+2^{15}\cdot k+25691$ O_{2a} $= 3^9\cdot(n + k)+15433 < 2^{15}\cdot(n + k)+25691$ O_{2a}
$n = 0 \rightarrow \mathbf{3^9\cdot k+15433 < 2^{15}\cdot k+25691}$ $\mathbf{O_{2a}}$ principal inequality : N(s) = **24**
$n = k = 0 \rightarrow$ **15433** O_1 < **25691** O_{2a} nude number

Check link nude number :
25691 O_{2a} : y = 12845 → f(y) = 28903 O_{3b} : z = 14451 → f(z) = 97550 → 48775 O_{3b} :
z = 24387 → f(z) = 164618 → 82309 O_1 : x = 41154 → f(x) = 61732 → 30866 →
15433 O_1 < **25691** O_{2a} : N(s) = **24**

Chain of connections :
$O_{2a} \rightarrow O_{3b} \rightarrow E \rightarrow O_{3b} \rightarrow E \rightarrow O_1 \rightarrow E \rightarrow E \rightarrow$ **15433** O_1 < **25691** O_{2a}

Another cycle of link with immediate substitutions managed so:

$O_{3a} \rightarrow O_2 \rightarrow O_3 \rightarrow O_2 \rightarrow O_3 \rightarrow O_2 \rightarrow E \rightarrow E \rightarrow \ldots \ldots <$

$\mathbf{2^{\alpha}\cdot n+32k+31}$ O_{3a} : $z = 2^{\alpha-1}\cdot n+16k+15 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+107 \rightarrow 3^4\cdot 2^{\alpha-3}\cdot n+324k+322 \rightarrow$
$3^4\cdot 2^{\alpha-4}\cdot n+162k+161$ $[O_2]$: k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+32k+31}$ $O_{3a} \rightarrow \mathbf{2^{\alpha}\cdot n+128k+63}$ O_{3a}
$3^4\cdot 2^{\alpha-4}\cdot n+162k+161 \rightarrow 3^4\cdot 2^{\alpha-4}\cdot n+648k+323$ O_2 : $y = 3^4\cdot 2^{\alpha-5}\cdot n+324k+161 \rightarrow$
$f(y) = 3^6\cdot 2^{\alpha-7}\cdot n+729k+364$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+128k+63}$ $O_{3a} \rightarrow \mathbf{2^{\alpha}\cdot n+256k+191}$ O_{3a}
$3^6\cdot 2^{\alpha-7}\cdot n+729k+364 \rightarrow 3^6\cdot 2^{\alpha-7}\cdot n+1458k+1093$ $[O_3]$: k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+256k+191}$ $O_{3a} \rightarrow \mathbf{2^{\alpha}\cdot n+1024k+447}$ O_{3a}
$3^6\cdot 2^{\alpha-7}\cdot n+1458k+1093 \rightarrow 3^6\cdot 2^{\alpha-7}\cdot n+5832k+2551$ O_3 : $z = 3^6\cdot 2^{\alpha-8}\cdot n+2916k+1275 \rightarrow$
$f(z) = 3^9\cdot 2^{\alpha-10}\cdot n+19683k+8612$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+1024k+447}$ $O_{3a} \rightarrow \mathbf{2^{\alpha}\cdot n+2048k+1471}$ O_{3a}
$3^9\cdot 2^{\alpha-10}\cdot n+19683k+8612 \rightarrow 3^9\cdot 2^{\alpha-10}\cdot n+39366k+28295$ $[O_2]$: k = **4k+2**

$\mathbf{2^{\alpha}\cdot n+2048k+1471}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+8192k+5567}$ O_{3a}
$3^9\cdot 2^{\alpha-10}\cdot n+39366k+28295$ → $3^9\cdot 2^{\alpha-10}\cdot n+157464k+107027$ O_2 : $y = 3^9\cdot 2^{\alpha-11}\cdot n+78732k+53513$ →
$f(y) = 3^{11}\cdot 2^{\alpha-13}\cdot n+177147k+120406$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+8192k+5567}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+16384k+13759}$ O_{3a}
$3^{11}\cdot 2^{\alpha-13}\cdot n+177147k+120406$ → $3^{11}\cdot 2^{\alpha-13}\cdot n+354294k+297553$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+16384k+13759}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+65536k+30143}$ O_{3a}
$3^{11}\cdot 2^{\alpha-13}\cdot n+354294k+297553$ → $3^{11}\cdot 2^{\alpha-13}\cdot n+1417176k+651847$ O_3 :
$z = 3^{11}\cdot 2^{\alpha-14}\cdot n+708588k+325923$ → $f(z) = 3^{14}\cdot 2^{\alpha-16}\cdot n+4782969k+2199986$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+65536k+30143}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+131072k+95679}$ O_{3a}
$3^{14}\cdot 2^{\alpha-16}\cdot n+4782969k+2199986$ → $3^{14}\cdot 2^{\alpha-16}\cdot n+9565938k+6982955$ [O_2] : k = **4k**

$\mathbf{2^{\alpha}\cdot n+131072k+95679}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+524288k+95679}$ O_{3a}
$3^{14}\cdot 2^{\alpha-16}\cdot n+9565938k+6982955$ → $3^{14}\cdot 2^{\alpha-16}\cdot n+38263752k+6982955$ O_2 :
$y = 3^{14}\cdot 2^{\alpha-17}\cdot n+19131876k+3491477$ → $3^{16}\cdot 2^{\alpha-19}\cdot n+43046721k+7855825$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+524288k+95679}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+1048576k+619967}$ O_{3a}
$3^{16}\cdot 2^{\alpha-19}\cdot n+43046721k+7855825$ → $3^{16}\cdot 2^{\alpha-19}\cdot n+86093442k+50902546$ →
$3^{16}\cdot 2^{\alpha-20}\cdot n+43046721k+25451273$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+1048576k+619967}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+2097152k+1668543}$ O_{3a}
$3^{16}\cdot 2^{\alpha-20}\cdot n+43046721k+25451273$ → $3^{16}\cdot 2^{\alpha-20}\cdot n+86093442k+68497994$ →
$3^{16}\cdot 2^{\alpha-21}\cdot n+43046721k+34248997$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+2097152k+1668543}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+4194304k+3765695}$ O_{3a}
$3^{16}\cdot 2^{\alpha-21}\cdot n+43046721k+34248997$ → $3^{16}\cdot 2^{\alpha-21}\cdot n+86093442k+77295718$ →
$3^{16}\cdot 2^{\alpha-22}\cdot n+43046721k+38647859$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+4194304k+3765695}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+8388608k+7959999}$ O_{3a}
$3^{16}\cdot 2^{\alpha-22}\cdot n+43046721k+38647859$ → $3^{16}\cdot 2^{\alpha-22}\cdot n+86093442k+81694580$ →
$3^{16}\cdot 2^{\alpha-23}\cdot n+43046721k+40847290$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+8388608k+7959999}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+16777216k+7959999}$ O_{3a}
$3^{16}\cdot 2^{\alpha-23}\cdot n+43046721k+40847290$ → $3^{16}\cdot 2^{\alpha-23}\cdot n+86093442k+40847290$ →
$3^{16}\cdot 2^{\alpha-24}\cdot n+43046721k+20423645$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+16777216k+7959999}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+33554432k+24737215}$ O_{3a}
$3^{16}\cdot 2^{\alpha-24}\cdot n+43046721k+20423645$ → $3^{16}\cdot 2^{\alpha-24}\cdot n+86093442k+63470366$ →
$3^{16}\cdot 2^{\alpha-25}\cdot n+43046721k+31735183$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+33554432k+24737215}$ O_{3a} → $\mathbf{2^{\alpha}\cdot n+67108864k+58291647}$ O_{3a}
$3^{16}\cdot 2^{\alpha-25}\cdot n+43046721k+31735183$ → $3^{16}\cdot 2^{\alpha-25}\cdot n+86093442k+74781904$ →
$3^{16}\cdot 2^{\alpha-26}\cdot n+43046721k+37390952 < 2^{\alpha}\cdot n+67108864k+58291647$ O_{3a} [$\alpha \geq 26$]

$\alpha = 26$ → $3^{16}\cdot n+3^{16}\cdot k+37390952 < 2^{26}\cdot n+2^{26}\cdot k+58291647$ O_{3a} =
$3^9\cdot(n + k)+15433 < 2^{15}\cdot(n + k)+25691$ O_{2a}

n = 0 → $\mathbf{3^{16}\cdot k+37390952 < 2^{26}\cdot k+58291647}$ O_{3a} principal inequality : N(s) = **42**
n = k = 0 → **37390952** < **58291647** O_{3a} nude number

Check link nude number :
58291647 O_{3a} : z = 29145823 → f(z) = 196734311 O_{3b} : z = 98367155 →
f(z) = 663978302 → 331989151 O_{3a} : z = 165994575 → f(z) = 1120463387 O_{2a} :
y = 560231693 → f(y) = 1260521311 O_{3a} : z = 630260655 → f(z) = 4254259427 O_2^* :
y = 2127129713 → f(y) = 4786041856 → 2393020928 → 1196510464 → 598255232 →
299127616 → 149563808 → 74781904 → **37390952** < **58291647** O_{3a} : N(s) = **42**

Chain of connections :
O_{3a} → O_{3b} → E → O_{3a} → O_{2a} → O_{3a} → O_2^* → E → E → E → E → E → E → E →
37390952 < **58291647** O_{3a}

At beginning of the chain we observe a small difference from our initial choices because we have applied the formulas obtained from Theorem 2n+1 to do the calculations.

… …

Whatever cycle of link we decide to choose Theorem of Independence is always valid, also in the worst conditions for to reach highest horizons.

3.2.1. A cycle of link in worst conditions

$\mathbf{2^{\alpha}\cdot n+16k+11}$ O_{2a} : y = $2^{\alpha-1}\cdot n+8k+5$ → f(y) = $3^2\cdot 2^{\alpha-3}\cdot n+18k+13$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+16k+11}$ O_{2a} → $2^{\alpha}\cdot n+16(\mathbf{4k+1})+11$ = $\mathbf{2^{\alpha}\cdot n+64k+27}$ O_{2a}
$3^2\cdot 2^{\alpha-3}\cdot n+18k+13$ → $3^2\cdot 2^{\alpha-3}\cdot n+18(\mathbf{4k+1})+13$ = $3^2\cdot 2^{\alpha-3}\cdot n+72k+31$ O_3 : z = $3^2\cdot 2^{\alpha-4}\cdot n+36k+15$ →
f(z) = $3^5\cdot 2^{\alpha-6}\cdot n+243k+107$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+64k+27}$ O_{2a} → $2^{\alpha}\cdot n+64(\mathbf{2k})+27$ = $\mathbf{2^{\alpha}\cdot n+128k+27}$ O_{2a}
$3^5\cdot 2^{\alpha-6}\cdot n+243k+107$ → $3^5\cdot 2^{\alpha-6}\cdot n+243(\mathbf{2k})+107$ = $3^5\cdot 2^{\alpha-6}\cdot n+486k+107$ [O_3] : k = **4k+2**

$\mathbf{2^{\alpha}\cdot n+128k+27}$ O_{2a} → $2^{\alpha}\cdot n+128(\mathbf{4k+2})+27$ = $\mathbf{2^{\alpha}\cdot n+512k+283}$ O_{2a}
$3^5\cdot 2^{\alpha-6}\cdot n+486k+107$ → $3^5\cdot 2^{\alpha-6}\cdot n+486(\mathbf{4k+2})+107$ = $3^5\cdot 2^{\alpha-6}\cdot n+1944k+1079$ O_3 :
z = $3^5\cdot 2^{\alpha-7}\cdot n+972k+539$ → f(z) = $3^8\cdot 2^{\alpha-9}\cdot n+6561k+3644$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+512k+283}$ O_{2a} → $2^{\alpha}\cdot n+512(\mathbf{2k+1})+283$ = $\mathbf{2^{\alpha}\cdot n+1024k+795}$ O_{2a}
$3^8\cdot 2^{\alpha-9}\cdot n+6561k+3644$ → $3^8\cdot 2^{\alpha-9}\cdot n+6561(\mathbf{2k+1})+3644$ = $3^8\cdot 2^{\alpha-9}\cdot n+13122k+10205$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+1024k+795}$ O_{2a} → $2^{\alpha}\cdot n+1024(\mathbf{4k+1})+795$ = $\mathbf{2^{\alpha}\cdot n+4096k+1819}$ O_{2a}
$3^8\cdot 2^{\alpha-9}\cdot n+13122k+10205$ → $3^8\cdot 2^{\alpha-9}\cdot n+13122(\mathbf{4k+1})+10205$ = $3^8\cdot 2^{\alpha-9}\cdot n+52488k+23327$ O_3 :
z = $3^8\cdot 2^{\alpha-10}\cdot n+26244k+11663$ → f(z) = $3^{11}\cdot 2^{\alpha-12}\cdot n+177147k+78731$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+4096k+1819}$ O_{2a} → $2^{\alpha}\cdot n+4096(\mathbf{2k})+1819$ = $\mathbf{2^{\alpha}\cdot n+8192k+1819}$ O_{2a}
$3^{11}\cdot 2^{\alpha-12}\cdot n+177147k+78731$ → $3^{11}\cdot 2^{\alpha-12}\cdot n+177147(\mathbf{2k})+78731$ = $3^{11}\cdot 2^{\alpha-12}\cdot n+354294k+78731$ [O_3] :
k = **4k+2**

$\mathbf{2^{\alpha}\cdot n+8192k+1819}$ O_{2a} → $2^{\alpha}\cdot n+8192(\mathbf{4k+2})+1819$ = $\mathbf{2^{\alpha}\cdot n+32768k+18203}$ O_{2a}
$3^{11}\cdot 2^{\alpha-12}\cdot n+354294k+78731$ → $3^{11}\cdot 2^{\alpha-12}\cdot n+354294(\mathbf{4k+2})+78731$ =
$3^{11}\cdot 2^{\alpha-12}\cdot n+1417176k+787319$ O_3 : z = $3^{11}\cdot 2^{\alpha-13}\cdot n+708588k+393659$ →
f(z) = $3^{14}\cdot 2^{\alpha-15}\cdot n+4782969k+2657204$ [O] : k = **2k+1** … … we decide to come back, then:

$f(z) = 3^{14}\cdot 2^{\alpha-15}\cdot n+4782969k+2657204$ [E] : k = **2k**

$2^{\alpha}\cdot n+32768k+18203$ O_{2a} → $2^{\alpha}\cdot n+32768(\mathbf{2k})+18203$ = **$2^{\alpha}\cdot n+65536k+18203$** O_{2a}
$3^{14}\cdot 2^{\alpha-15}\cdot n+4782969k+2657204 \rightarrow 3^{14}\cdot 2^{\alpha-15}\cdot n+4782969(\mathbf{2k})+2657204 =$
$3^{14}\cdot 2^{\alpha-15}\cdot n+9565938k+2657204 \rightarrow 3^{14}\cdot 2^{\alpha-16}\cdot n+4782969k+1328602$ [E] : k = **2k**

$2^{\alpha}\cdot n+65536k+18203$ O_{2a} → $2^{\alpha}\cdot n+65536(\mathbf{2k})+18203$ = **$2^{\alpha}\cdot n+131072k+18203$** O_{2a}
$3^{14}\cdot 2^{\alpha-16}\cdot n+4782969k+1328602 \rightarrow 3^{14}\cdot 2^{\alpha-16}\cdot n+4782969(\mathbf{2k})+1328602 =$
$3^{14}\cdot 2^{\alpha-16}\cdot n+9565938k+1328602 \rightarrow 3^{14}\cdot 2^{\alpha-17}\cdot n+4782969k+664301$ [E] : k = **2k+1**

$2^{\alpha}\cdot n+131072k+18203$ O_{2a} → $2^{\alpha}\cdot n+131072(\mathbf{2k+1})+18203$ = **$2^{\alpha}\cdot n+262144k+149275$** O_{2a}
$3^{14}\cdot 2^{\alpha-17}\cdot n+4782969k+664301 \rightarrow 3^{14}\cdot 2^{\alpha-17}\cdot n+4782969(\mathbf{2k+1})+664301 =$
$3^{14}\cdot 2^{\alpha-17}\cdot n+9565938k+5447270 \rightarrow 3^{14}\cdot 2^{\alpha-18}\cdot n+4782969k+2723635$ [E] : k = **2k+1**

$2^{\alpha}\cdot n+262144k+149275$ O_{2a} → $2^{\alpha}\cdot n+262144(\mathbf{2k+1})+149275$ = **$2^{\alpha}\cdot n+524288k+411419$** O_{2a}
$3^{14}\cdot 2^{\alpha-18}\cdot n+4782969k+2723635 \rightarrow 3^{14}\cdot 2^{\alpha-18}\cdot n+4782969(\mathbf{2k+1})+2723635 =$
$3^{14}\cdot 2^{\alpha-18}\cdot n+9565938k+7506604 \rightarrow 3^{14}\cdot 2^{\alpha-19}\cdot n+4782969k+3753302$ [E] : k = **2k**

$2^{\alpha}\cdot n+524288k+411419$ O_{2a} → $2^{\alpha}\cdot n+524288(\mathbf{2k})+411419$ = **$2^{\alpha}\cdot n+1048576k+411419$** O_{2a}
$3^{14}\cdot 2^{\alpha-19}\cdot n+4782969k+3753302 \rightarrow 3^{14}\cdot 2^{\alpha-19}\cdot n+4782969(\mathbf{2k})+3753302 =$
$3^{14}\cdot 2^{\alpha-19}\cdot n+9565938k+3753302 \rightarrow 3^{14}\cdot 2^{\alpha-20}\cdot n+4782969k+1876651$ [E] : k = **2k+1**

$2^{\alpha}\cdot n+1048576k+411419$ O_{2a} → $2^{\alpha}\cdot n+1048576(\mathbf{2k+1})+411419$ = **$2^{\alpha}\cdot n+2097152k+1459995$** O_{2a}
$3^{14}\cdot 2^{\alpha-20}\cdot n+4782969k+1876651 \rightarrow 3^{14}\cdot 2^{\alpha-20}\cdot n+4782969(\mathbf{2k+1})+1876651 =$
$3^{14}\cdot 2^{\alpha-20}\cdot n+9565938k+6659620 \rightarrow 3^{14}\cdot 2^{\alpha-21}\cdot n+4782969k+3329810$ [E] : k = **2k**

$2^{\alpha}\cdot n+2097152k+1459995$ O_{2a} → $2^{\alpha}\cdot n+2097152(\mathbf{2k})+1459995$ = **$2^{\alpha}\cdot n+4194304k+1459995$** O_{2a}
$3^{14}\cdot 2^{\alpha-21}\cdot n+4782969k+3329810 \rightarrow 3^{14}\cdot 2^{\alpha-21}\cdot n+4782969(\mathbf{2k})+3329810 =$
$3^{14}\cdot 2^{\alpha-21}\cdot n+9565938k+3329810 \rightarrow 3^{14}\cdot 2^{\alpha-22}\cdot n+4782969k+1664905$ [E] : k = **2k+1**

$2^{\alpha}\cdot n+4194304k+1459995$ O_{2a} → $2^{\alpha}\cdot n+4194304(\mathbf{2k+1})+1459995$ = **$2^{\alpha}\cdot n+8388608k+5654299$** O_{2a}
$3^{14}\cdot 2^{\alpha-22}\cdot n+4782969k+1664905 \rightarrow 3^{14}\cdot 2^{\alpha-22}\cdot n+4782969(\mathbf{2k+1})+1664905 =$
$3^{14}\cdot 2^{\alpha-22}\cdot n+9565938k+6447874 \rightarrow$
$3^{14}\cdot 2^{\alpha-23}\cdot n+4782969k+3223937 < 2^{\alpha}\cdot n+8388608k+5654299$ O_{2a} $[\alpha \geq 23]$

$\alpha = 23 \rightarrow 3^{14}\cdot n+4782969k+3223937 < 2^{23}\cdot n+8388608k+5654299$ O_{2a} =
$3^{14}\cdot n+3^{14}\cdot k+3223937 < 2^{23}\cdot n+2^{23}\cdot k+5654299$ O_{2a} =
$3^{14}\cdot (n + k)+3223937 < 2^{23}\cdot (n + k)+5654299$ O_{2a}
n = 0 → **$3^{14}\cdot k+3223937$** < **$2^{23}\cdot n+5654299$** O_{2a} principal inequality : N(s) = **37**
n = k = 0 → **3223937** O_1 < **5654299** O_{2a} nude number.

Check link nude number :
5654299 O_{2a} : y = 2827149 → f(y) = 6361087 O_{3a} : z = 3180543 → f(z) = 21468671 O_{3a} :
z = 10734335 → f(z) = 72456767 O_{3a} : z = 36228383 → f(z) = 244541591 O_3* :
z = 122270795 → f(z) = 825327872 → 412663936 → 206331968 → 103165984 →
51582992 → 25791496 → 12895748 → 6447874 → **3223937** O_1 < **5654299** O_{2a} : N(s) = **37**

Chain of connections :
O_{2a} → O_{3a} → O_{3a} → O_{3a} → O_3* → E → E → E → E → E → E → E → E →
3223937 O_1 < **5654299** O_{2a}

Another cycle of link in worst conditions with immediate substitutions

$\mathbf{2^{\alpha}\cdot n+32k+7}$ O_{3b} : $z = 2^{\alpha-1}\cdot n+16k+3 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+26 \rightarrow$
$3^3\cdot 2^{\alpha-4}\cdot n+54k+13$ $[O_3]$: k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+32k+7}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+128k+103}$ O_{3b}
$3^3\cdot 2^{\alpha-4}\cdot n+54k+13 \rightarrow 3^3\cdot 2^{\alpha-4}\cdot n+216k+175$ O_3 : $z = 3^3\cdot 2^{\alpha-5}\cdot n+108k+87 \rightarrow$
$f(z) = 3^6\cdot 2^{\alpha-7}\cdot n+729k+593$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+128k+103}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+256k+103}$ O_{3b}
$3^6\cdot 2^{\alpha-7}\cdot n+729k+593 \rightarrow 3^6\cdot 2^{\alpha-7}\cdot n+1458k+593$ $[O_3]$: k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+256k+103}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+1024k+871}$ O_{3b}
$3^6\cdot 2^{\alpha-7}\cdot n+1458k+593 \rightarrow 3^6\cdot 2^{\alpha-7}\cdot n+5832k+4967$ O_3 : $z = 3^6\cdot 2^{\alpha-8}\cdot n+2916k+2483 \rightarrow$
$f(z) = 3^9\cdot 2^{\alpha-10}\cdot n+19683k+16766$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+1024k+871}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+2048k+1895}$ O_{3b}
$3^9\cdot 2^{\alpha-10}\cdot n+19683k+16766 \rightarrow 3^9\cdot 2^{\alpha-10}\cdot n+39366k+36449$ $[O_3]$: k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+2048k+1895}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+8192k+3943}$ O_{3b}
$3^9\cdot 2^{\alpha-10}\cdot n+39366k+36449 \rightarrow 3^9\cdot 2^{\alpha-10}\cdot n+157464k+75815$ O_3 : $z = 3^9\cdot 2^{\alpha-11}\cdot n+78732k+37907 \rightarrow$
$f(z) = 3^{12}\cdot 2^{\alpha-13}\cdot n+531441k+255878$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+8192k+3943}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+16384k+12135}$ O_{3b}
$3^{12}\cdot 2^{\alpha-13}\cdot n+531441k+255878 \rightarrow 3^{12}\cdot 2^{\alpha-13}\cdot n+1062882k+787319$ … … We come back, then:

$3^{12}\cdot 2^{\alpha-13}\cdot n+1062882k+787319$ $[O_1]$: k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+16384k+12135}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+32768k+28519}$ O_{3b}
$3^{12}\cdot 2^{\alpha-13}\cdot n+1062882k+787319 \rightarrow 3^{12}\cdot 2^{\alpha-13}\cdot n+2125764k+1850201$ O_1 :
$x = 3^{12}\cdot 2^{\alpha-14}\cdot n+1062882k+925100 \rightarrow f(x) = 3^{13}\cdot 2^{\alpha-15}\cdot n+1594323k+1387651$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+32768k+28519}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+65536k+61287}$ O_{3b}
$3^{13}\cdot 2^{\alpha-15}\cdot n+1594323k+1387651 \rightarrow 3^{13}\cdot 2^{\alpha-15}\cdot n+3188646k+2981974 \rightarrow$
$3^{13}\cdot 2^{\alpha-16}\cdot n+1594323k+1490987$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+65536k+61287}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+131072k+126823}$ O_{3b}
$3^{13}\cdot 2^{\alpha-16}\cdot n+1594323k+1490987 \rightarrow 3^{13}\cdot 2^{\alpha-16}\cdot n+3188646k+3085310 \rightarrow$
$3^{13}\cdot 2^{\alpha-17}\cdot n+1594323k+1542655$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+131072k+126823}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+262144k+257895}$ O_{3b}
$3^{13}\cdot 2^{\alpha-17}\cdot n+1594323k+1542655 \rightarrow 3^{13}\cdot 2^{\alpha-17}\cdot n+3188646k+3136978 \rightarrow$
$3^{13}\cdot 2^{\alpha-18}\cdot n+1594323k+1568489$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+262144k+257895}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+524288k+520039}$ O_{3b}
$3^{13}\cdot 2^{\alpha-18}\cdot n+1594323k+1568489 \rightarrow 3^{13}\cdot 2^{\alpha-18}\cdot n+3188646k+3162812 \rightarrow$
$3^{13}\cdot 2^{\alpha-19}\cdot n+1594323k+1581406$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+524288k+520039}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+1048576k+520039}$ O_{3b}
$3^{13}\cdot 2^{\alpha-19}\cdot n+1594323k+1581406 \rightarrow 3^{13}\cdot 2^{\alpha-19}\cdot n+3188646k+1581406 \rightarrow$
$3^{13}\cdot 2^{\alpha-20}\cdot n+1594323k+790703$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+1048576k+520039}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+2097152k+1568615}$ O_{3b}
$3^{13}\cdot 2^{\alpha-20}\cdot n+1594323k+790703$ → $3^{13}\cdot 2^{\alpha-20}\cdot n+3188646k+2385026$ →
$3^{13}\cdot 2^{\alpha-21}\cdot n+1594323k+1192513 < 2^{\alpha}\cdot n+2097152k+1568615$ O_{3b} $[\alpha \geq 21]$

$\alpha = 21$ → $3^{13}\cdot n+1594323k+1192513 < 2^{21}\cdot n+2097152k+1568615$ O_{3b} =
$3^{13}\cdot n+3^{13}\cdot k+1192513 < 2^{21}\cdot n+2^{21}\cdot k+1568615$ O_{3b} =
$3^{13}\cdot(n + k)+1192513 < 2^{21}\cdot(n + k)+1568615$ O_{3b}
$n = 0$ → $\mathbf{3^{13}\cdot k+1192513 < 2^{21}\cdot k+1568615}$ O_{3b} principal inequality : N(s) = **34**
$n = k = 0$ → **1192513** O_1 < **1568615** O_{3b} nude number.

Check link nude number :
1568615 O_{3b} : z = 784307 → f(z) = 5294078 → 2647039 O_{3a} : z = 1323519 → f(z) = 8933759 O_{3a} :
z = 4466879 → f(z) = 30151439 O_{3c} : z = 16075719 → f(z) = 101761109 O_1 : x = 50880554 →
f(x) = 76320832 → 38160416 → 19080208 → 9540104 → 4770052 → 2385026 →
1192513 O_1 < **1568615** O_{3b} : N(s) = **34** steps.

Chain of connections :
O_{3b} → E → O_{3a} → O_{3a} → O_{3c} → O_1 → E → E → E → E → E → E →
1192513 O_1 < **1568615** O_{3b}

3.2.2. An example of a cycle of link with an infinite number of steps

16k+11 O_{2a}

$\mathbf{2^{\alpha}\cdot n+16k+11}$ O_{2a} : $y = 2^{\alpha-1}\cdot n+8k+5$ → $f(y) = 3^2\cdot 2^{\alpha-3}\cdot n+18k+13$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+16k+11}$ O_{2a} → $2^{\alpha}\cdot n+16(\mathbf{4k+1})+27$ O_{2a} = $\mathbf{2^{\alpha}\cdot n+64k+27}$ O_{2a}
$3^2\cdot 2^{\alpha-3}\cdot n+18k+13$ → $3^2\cdot 2^{\alpha-3}\cdot n+18(\mathbf{4k+1})+13 = 3^2\cdot 2^{\alpha-3}\cdot n+72k+31$ O_3 : $z = 3^2\cdot 2^{\alpha-4}\cdot n+36k+15$ →
$f(z) = 3^5\cdot 2^{\alpha-6}\cdot n+243k+107$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+64k+27}$ O_{2a} → $2^{\alpha}\cdot n+64(\mathbf{2k})+27$ O_{2a} = $\mathbf{2^{\alpha}\cdot n+128k+27}$ O_{2a}
$3^5\cdot 2^{\alpha-6}\cdot n+243k+107$ → $3^5\cdot 2^{\alpha-6}\cdot n+243(\mathbf{2k})+107 = 3^5\cdot 2^{\alpha-6}\cdot n+486k+107$ [O_1] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n +128k+27}$ O_{2a} → $2^{\alpha}\cdot n +128(\mathbf{2k+1})+27$ = $\mathbf{2^{\alpha}\cdot n +256k+155}$ O_{2a}
$3^5\cdot 2^{\alpha-6}\cdot n+486k+107$ → $3^5\cdot 2^{\alpha-6}\cdot n+486(\mathbf{2k+1})+107 = 3^5\cdot 2^{\alpha-6}\cdot n+972k+593$ O_1 :
$x = 3^5\cdot 2^{\alpha-7}\cdot n+486k+296$ → $f(x) = 3^6\cdot 2^{\alpha-8}\cdot n+729k+445$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+256k+155}$ O_{2a} → $2^{\alpha}\cdot n+256(\mathbf{2k})+155$ = $\mathbf{2^{\alpha}\cdot n+512k+155}$ O_{2a}
$3^6\cdot 2^{\alpha-8}\cdot n+729k+445$ → $3^6\cdot 2^{\alpha-8}\cdot n+729(\mathbf{2k})+445 = 3^6\cdot 2^{\alpha-8}\cdot n+1458k+445$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+512k+155}$ O_{2a} → $2^{\alpha}\cdot n+512(\mathbf{2k})+155$ = $\mathbf{2^{\alpha}\cdot n+1024k+155}$ O_{2a}
$3^6\cdot 2^{\alpha-8}\cdot n+1458k+445$ → $3^6\cdot 2^{\alpha-8}\cdot n+1458(\mathbf{2k})+445 = 3^6\cdot 2^{\alpha-8}\cdot n+2916k+445$ O_1 :
$x = 3^6\cdot 2^{\alpha-9}\cdot n+1458k+222$ → $f(x) = 3^7\cdot 2^{\alpha-10}\cdot n+2187k+334$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+1024k+155}$ O_{2a} → $2^{\alpha}\cdot n+1024(\mathbf{2k+1})+155$ = $\mathbf{2^{\alpha}\cdot n+2048k+1179}$ O_{2a}
$3^7\cdot 2^{\alpha-10}\cdot n+2187k+334$ → $3^7\cdot 2^{\alpha-10}\cdot n+2187(\mathbf{2k+1})+334 = 3^7\cdot 2^{\alpha-10}\cdot n+4374k+2521$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+2048k+1179}$ O_{2a} → $2^{\alpha}\cdot n+2048(\mathbf{2k})+1179$ = $\mathbf{2^{\alpha}\cdot n+4096k+1179}$ O_{2a}
$3^7\cdot 2^{\alpha-10}\cdot n+4374k+2521$ → $3^7\cdot 2^{\alpha-10}\cdot n+4374(\mathbf{2k})+2521 = 3^7\cdot 2^{\alpha-10}\cdot n+8748k+2521$ O_1 :
$x = 3^7\cdot 2^{\alpha-11}\cdot n+4374k+1260$ → $f(x) = 3^8\cdot 2^{\alpha-12}\cdot n+6561k+1891$ [O] : k = **2k** … … here we stop.

We could continue to go on for an infinity of steps, where O_1 is always presents, but its value doesn't becomes (never) less than that of O_{2a} . Only if we decide to go back this is possible; and this is possible if we choose [E] and not [O]. In fact:

$f(x) = 3^8 \cdot 2^{\alpha-12} \cdot n+6561k+1891$ [E] : k = **2k+1**

$\mathbf{2^\alpha \cdot n+4096k+1179}$ O_{2a} → $2^\alpha \cdot n+4096(\mathbf{2k+1})+1179$ = $\mathbf{2^\alpha \cdot n+8192k+5275}$ O_{2a}
$3^8 \cdot 2^{\alpha-12} \cdot n+6561k+1891 \rightarrow 3^8 \cdot 2^{\alpha-12} \cdot n+6561(\mathbf{2k+1})+1891 = 3^8 \cdot 2^{\alpha-12} \cdot n+13122k+8452 \rightarrow$
$3^8 \cdot 2^{\alpha-13} \cdot n+6561k+4226 < 2^\alpha \cdot n+8192k+5275$ O_{2a} $[\alpha \geq 13]$

$\alpha = 13 \rightarrow 3^8 \cdot n+3^8 \cdot k+4226 < 2^{13} \cdot n+2^{13} \cdot k+5275$ O_{2a} $\rightarrow 3^8 \cdot (n + k)+4226 < 2^{13} \cdot (n + k)+5275$ O_{2a}
$n = 0 \rightarrow$ $\mathbf{3^8 \cdot k+4226 < 2^{13} \cdot k+5275}$ O_{2a} principal inequality : N(s) = **21**
$n = k = 0 \rightarrow$ **4226 < 5275** O_{2a} nude number.

Another cycle of link with an infinite number of steps managed in this way:

$O_{3b} \rightarrow E \rightarrow O_3 \rightarrow O_1 \rightarrow O_3 \rightarrow O_1 \rightarrow \ldots \ldots \rightarrow E \rightarrow \ldots \ \ldots$

$\mathbf{2^\alpha \cdot n+32k+7}$ O_{3b} : $z = 2^{\alpha-1} \cdot n+16k+3 \rightarrow f(z) = 3^3 \cdot 2^{\alpha-3} \cdot n+108k+26 \rightarrow 3^3 \cdot 2^{\alpha-4} \cdot n+54k+13$ [O_3] :
k = **4k+3**

$\mathbf{2^\alpha \cdot n+32k+7}$ O_{3b} → $2^\alpha \cdot n+32(\mathbf{4k+3})+7$ = $\mathbf{2^\alpha \cdot n+128k+103}$ O_{3b}
$3^3 \cdot 2^{\alpha-4} \cdot n+54k+13 \rightarrow 3^3 \cdot 2^{\alpha-4} \cdot n+54(\mathbf{4k+3})+13 = 3^3 \cdot 2^{\alpha-4} \cdot n+216k+175$ O_3 : $z = 3^3 \cdot 2^{\alpha-5} \cdot n+108k+87 \rightarrow$
$f(z) = 3^6 \cdot 2^{\alpha-7} \cdot n+729k+593$ [O] : k = **2k**

$\mathbf{2^\alpha \cdot n+128k+103}$ O_{3b} → $2^\alpha \cdot n+128(\mathbf{2k})+103$ = $\mathbf{2^\alpha \cdot n+256k+103}$ O_{3b}
$3^6 \cdot 2^{\alpha-7} \cdot n+729k+593 \rightarrow 3^6 \cdot 2^{\alpha-7} \cdot n+729(\mathbf{2k})+593 = 3^6 \cdot 2^{\alpha-7} \cdot n+1458k+593$ [O_1] : k = **2k**

$\mathbf{2^\alpha \cdot n+256k+103}$ O_{3b} → $2^\alpha \cdot n+256(\mathbf{2k})+103$ = $\mathbf{2^\alpha \cdot n+512k+103}$ O_{3b}
$3^6 \cdot 2^{\alpha-7} \cdot n+1458k+593 \rightarrow 3^6 \cdot 2^{\alpha-7} \cdot n+1458(\mathbf{2k})+593 = 3^6 \cdot 2^{\alpha-7} \cdot n+2916k+593$ O_1 :
$x = 3^6 \cdot 2^{\alpha-8} \cdot n+1458k+296 \rightarrow f(x) = 3^7 \cdot 2^{\alpha-9} \cdot n+2187k+445$ [O] : k = **2k**

$\mathbf{2^\alpha \cdot n+512k+103}$ O_{3b} → $2^\alpha \cdot n+512(\mathbf{2k})+103$ = $\mathbf{2^\alpha \cdot n+1024k+103}$ O_{3b}
$3^7 \cdot 2^{\alpha-9} \cdot n+2187k+445 \rightarrow 3^7 \cdot 2^{\alpha-9} \cdot n+2187(\mathbf{2k})+445 = 3^7 \cdot 2^{\alpha-9} \cdot n+4374k+445$ [O_3] : k = **4k+3**

$\mathbf{2^\alpha \cdot n+1024k+103}$ O_{3b} → $2^\alpha \cdot n+1024(\mathbf{4k+3})+103$ = $\mathbf{2^\alpha \cdot n+4096k+3175}$ O_{3b}
$3^7 \cdot 2^{\alpha-9} \cdot n+4374k+445 \rightarrow 3^7 \cdot 2^{\alpha-9} \cdot n+4374(\mathbf{4k+3})+445 = 3^7 \cdot 2^{\alpha-9} \cdot n+17496k+13567$ O_3 :
$z = 3^7 \cdot 2^{\alpha-10} \cdot n+8748k+6783 \rightarrow f(z) = 3^{10} \cdot 2^{\alpha-12} \cdot n+59049k+45791$ [O] : k = **2k**

$\mathbf{2^\alpha \cdot n+4096k+3175}$ O_{3b} → $2^\alpha \cdot n+4196(\mathbf{2k})+3175$ = $\mathbf{2^\alpha \cdot n+8192k+3175}$ O_{3b}
$3^{10} \cdot 2^{\alpha-12} \cdot n+59049k+45791 \rightarrow 3^{10} \cdot 2^{\alpha-12} \cdot n+59049(\mathbf{2k})+45791 = 3^{10} \cdot 2^{\alpha-12} \cdot n+118098k+45791$ [O_1] :
k = **2k+1**

$\mathbf{2^\alpha \cdot n+8192k+3175}$ O_{3b} → $2^\alpha \cdot n+8192(\mathbf{2k+1})+3175$ = $\mathbf{2^\alpha \cdot n+16384k+11367}$ O_{3b}
$3^{10} \cdot 2^{\alpha-12} \cdot n+118098k+45791 \rightarrow 3^{10} \cdot 2^{\alpha-12} \cdot n+118098(\mathbf{2k+1})+45791 =$
$3^{10} \cdot 2^{\alpha-12} \cdot n+236196k+163889$ O_1 : $x = 3^{10} \cdot 2^{\alpha-13} \cdot n+118098k+81944 \rightarrow$
$f(x) = 3^{11} \cdot 2^{\alpha-14} \cdot n+177147k+122917$ [O] : k = **2k** … … we could go on for an infinite number of steps, but we decide to go back, so:

$f(x) = 3^{11} \cdot 2^{\alpha-14} \cdot n+177147k+122917$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+16384k+11367}$ O_{3b} → $2^{\alpha}\cdot n+16384(\mathbf{2k+1})+11367$ = $\mathbf{2^{\alpha}\cdot n+32768k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-14}\cdot n+177147k+122917 \rightarrow 3^{11}\cdot 2^{\alpha-14}\cdot n+177147(\mathbf{2k+1})+122917$ =
$3^{11}\cdot 2^{\alpha-14}\cdot n+354294k+300064 \rightarrow 3^{11}\cdot 2^{\alpha-15}\cdot n+177147k+150032$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+32768k+27751}$ O_{3b} → $2^{\alpha}\cdot n+32768(\mathbf{2k})+27751$ = $\mathbf{2^{\alpha}\cdot n+65536k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-15}\cdot n+177147k+150032 \rightarrow 3^{11}\cdot 2^{\alpha-15}\cdot n+177147(\mathbf{2k})+150032 = 3^{11}\cdot 2^{\alpha-15}\cdot n+354294k+150032 \rightarrow$
$3^{11}\cdot 2^{\alpha-16}\cdot n+177147k+75016$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+65536k+27751}$ O_{3b} → $2^{\alpha}\cdot n+65536(\mathbf{2k})+27751$ = $\mathbf{2^{\alpha}\cdot n+131072k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-16}\cdot n+177147k+75016 \rightarrow 3^{11}\cdot 2^{\alpha-16}\cdot n+177147(\mathbf{2k})+75016 = 3^{11}\cdot 2^{\alpha-16}\cdot n+354294k+75016 \rightarrow$
$3^{11}\cdot 2^{\alpha-17}\cdot n+177147k+37508$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+131072k+27751}$ O_{3b} → $2^{\alpha}\cdot n+131072(\mathbf{2k})+27751$ = $\mathbf{2^{\alpha}\cdot n+262144k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-17}\cdot n+177147k+37508 \rightarrow 3^{11}\cdot 2^{\alpha-17}\cdot n+177147(\mathbf{2k})+37508 = 3^{11}\cdot 2^{\alpha-17}\cdot n+354294k+37508 \rightarrow$
$3^{11}\cdot 2^{\alpha-18}\cdot n+177147k+18754 < 2^{\alpha}\cdot n+262144k+27751$ O_{3b} [$\alpha \geq 18$]

$\alpha = 18 \rightarrow 3^{11}\cdot n+3^{11}\cdot k+18754 < 2^{18}\cdot n+2^{18}\cdot k+27751$ O_{3b} =
$3^{11}\cdot(n + k)+18754 < 2^{18}\cdot(n + k)+27751$ O_{3b}
n = 0 → $\mathbf{3^{11}\cdot k+18754 < 2^{18}\cdot k+27751}$ O_{3b} principal inequality : N(s) = **29**
n = k = 0 → **18754 < 27751** O_{3b} nude number

Check link nude number :
27751 O_{3b} : z = 13875 → f(z) = 93662 → 46831 O_{3c} : z = 23415 → f(z) = 158057 O_1 :
x = 79028 → f(x) = 118543 O_{3c} : z = 59271 → f(z) = 400085 O_1 : x = 200042 →
f(x) = 300064 → 150032 → 75016 → 37508 → **18754** < **27751** O_{3b} : N(s) = **29**

We observe that the verification satisfies the chain of connections chosen initially

Two chains of connections with an infinite number of steps

$O_{3a} \rightarrow O_3 \rightarrow O_3 \rightarrow E \rightarrow O_3 \rightarrow O_3 \rightarrow O_1 \rightarrow O_3 \rightarrow O_3 \rightarrow O_2 \rightarrow E \rightarrow E \rightarrow E \rightarrow O_3 \rightarrow \ldots \rightarrow O_3 \rightarrow$
$E \rightarrow O_3 \rightarrow E \rightarrow O_3 \rightarrow O_1 \rightarrow O_2 \rightarrow \ldots \rightarrow O_2 \rightarrow O_3 \rightarrow O_1 \rightarrow O_1 \rightarrow E \rightarrow O_3 \rightarrow \ldots$ ... N(s) → ∞

$O_{3a} \rightarrow O_2 \rightarrow O_2 \rightarrow O_3 \rightarrow O_1 \rightarrow E \rightarrow O_2 \rightarrow O_3 \rightarrow E \rightarrow O_2 \rightarrow O_3 \rightarrow O_3 \rightarrow \ldots \rightarrow O_2 \rightarrow O_3 \rightarrow O_3 \rightarrow$
$O_2 \rightarrow E \rightarrow E \rightarrow O_2 \rightarrow O_3 \rightarrow O_1 \rightarrow O_1 \rightarrow O_3 \rightarrow O_2 \rightarrow E \rightarrow E \rightarrow O_3 \rightarrow O_1 \rightarrow \ldots$ … N(s) → ∞

It is clear and evident that there are an infinity of possible combinations of connections in the cycles of links which could contain an infinite number of steps. So, by the procedure applied above, it's proved that Collatz Conjecture is a particular type of *Circle Quadrature*, and Independence Theorem allows us to manage the cycles of links as we like. So, chosen an odd number $p \in O$ we can associate the horizon $2^k\cdot n+p$ to it. By formulas plan Theorem 2n+1 we can calculate $3^h\cdot n+q < 2^k\cdot n+p$ and the number of steps needed to move from the main horizon $2^k\cdot n+p$ to its lower horizon $3^h\cdot n+q$: N(s) = h + k .

Test of some binomial inequalities in General List by Theorem of Independence

N(s) = **8** : **27n+20 < 32n+23** O_3*
$\mathbf{2^k\cdot n+23}$ O_3* : z = $2^{k-1}\cdot n+11 \rightarrow f(z) = 3^3\cdot 2^{k-3}\cdot n+80 \rightarrow 3^3\cdot 2^{k-4}\cdot n+40 \rightarrow$
$\mathbf{3^3\cdot 2^{k-5}\cdot n+20}$ < $\mathbf{2^k\cdot n+23}$ O_3* [k ≥ 5] : k = 5 → **27n+20 < 32n+23** O_3*

N(s) = **11** : **$81n+38 < 128n+59$** O_{2a}
$2^k \cdot n+59$ O_{2a} : $y = 2^{k-1} \cdot n+29 \rightarrow f(y) = 3^2 \cdot 2^{k-3} \cdot n+67$ O_2^* : $y = 3^2 \cdot 2^{k-4} \cdot n+33 \rightarrow$
$f(y) = 3^4 \cdot 2^{k-6} \cdot n+76 \rightarrow$ **$3^4 \cdot 2^{k-7} \cdot n+38$** $<$ **$2^k \cdot n+59$** O_{2a} $[k \geq 7]$: $k = 7 \rightarrow$ **$81n+38 < 128n+59$** O_{2a}

N(s) = **13** : **$243n+91 < 256n+95$** O_{3a}
$2^k \cdot n+95$ O_{3a} : $z = 2^{k-1} \cdot n+47 \rightarrow f(z) = 3^3 \cdot 2^{k-3} \cdot n+323$ O_2^* : $y = 3^3 \cdot 2^{k-4} \cdot n+161 \rightarrow$
$f(y) = 3^5 \cdot 2^{k-6} \cdot n+364 \rightarrow 3^5 \cdot 2^{k-7} \cdot n+182 \rightarrow$ **$3^5 \cdot 2^{k-8} \cdot n+91$** $<$ **$2^k \cdot n+95$** O_{3a} $[k \geq 8]$: $k = 8 \rightarrow$
$243n+91 < 256n+95$ O_{3a}

N(s) = **16** : **$729n+302 < 1024n+423$** O_{3b}
$2^k \cdot n+423$ O_{3b} : $z = 2^{k-1} \cdot n+211 \rightarrow f(z) = 3^3 \cdot 2^{k-3} \cdot n+1430 \rightarrow 3^3 \cdot 2^{k-4} \cdot n+715$ O_{2a} :
$y = 3^3 \cdot 2^{k-5} \cdot n+357 \rightarrow f(y) = 3^5 \cdot 2^{k-7} \cdot n+805$ O_1 : $x = 3^5 \cdot 2^{k-8} \cdot n+402 \rightarrow f(x) = 3^6 \cdot 2^{k-9} \cdot n+604 \rightarrow$
$3^6 \cdot 2^{k-10} \cdot n+302$ $<$ **$2^k \cdot n+423$** O_{3b} $[k \geq 10]$: $k = 10 \rightarrow$ **$729n+302 < 1024n+423$** O_{3b}

N(s) = **19** : **$2187n + 205 < 4096n+383$** O_{3a}
$2^k \cdot n+383$ O_{3a} : $z = 2^{k-1} \cdot n + 191 \rightarrow f(z) = 3^3 \cdot 2^{k-3} \cdot n + 1295$ O_{3c} : $z = 3^3 \cdot 2^{k-4} \cdot n + 647 \rightarrow$
$f(z) = 3^6 \cdot 2^{k-6} \cdot n + 4373$ O_1 : $x = 3^6 \cdot 2^{k-7} \cdot n + 2186 \rightarrow f(x) = 3^7 \cdot 2^{k-8} \cdot n + 3280 \rightarrow$
$3^7 \cdot 2^{k-9} \cdot n + 1640 \rightarrow 3^7 \cdot 2^{k-10} \cdot n + 820 \rightarrow 3^7 \cdot 2^{k-11} \cdot n + 410 \rightarrow$
$3^7 \cdot 2^{k-12} \cdot n + 205 < 2^k \cdot n+383$ O_{3a} $[k \geq 12]$: $k = 12 \rightarrow$ **$2187n + 205 < 4096n+383$** O_{3a}

N(s) **= 21** : **$6561n+205 < 8192n+255$** O_{3a}
$2^k \cdot n+255$ O_{3a} : $z = 2^{k-1} \cdot n+127 \rightarrow f(z) = 3^3 \cdot 2^{k-3} \cdot n+863$ O_{3a} : $z = 3^3 \cdot 2^{k-4} \cdot n+431 \rightarrow$
$f(z) = 3^6 \cdot 2^{k-6} \cdot n+2915$ O_2^* : $y = 3^6 \cdot 2^{k-7} \cdot n+1457 \rightarrow f(y) = 3^8 \cdot 2^{k-9} \cdot n+3280 \rightarrow$
$3^8 \cdot 2^{k-10} \cdot n+1640 \rightarrow 3^8 \cdot 2^{k-11} \cdot n+820 \rightarrow 3^8 \cdot 2^{k-12} \cdot n+410 \rightarrow$
$3^8 \cdot 2^{k-13} \cdot n+205 < 2^k \cdot n+255$ O_{3a} $[k \geq 13]$: $k = 13 \rightarrow$ **$6561n+205 < 8192n+255$** O_{3a}

3.3. Long cycles of links

In this paragraph we analyze the odd generating numbers that need several steps to become less than themselves, i.e. : N(s) > 21 .

3.3.1. The number 27 and its internal binomial inequalities

$2^k \cdot n+27$ $O_{2a} \rightarrow 3 \cdot 2^k \cdot n+82 \rightarrow 3 \cdot 2^{k-1} \cdot n+41$ $O_1 \rightarrow 3^2 \cdot 2^{k-1} \cdot n+124 \rightarrow 3^2 \cdot 2^{k-2} \cdot n+62 \rightarrow$
$3^2 \cdot 2^{k-3} \cdot n+31$ $O_{3a} \rightarrow 3^3 \cdot 2^{k-3} \cdot n+94 \rightarrow 3^3 \cdot 2^{k-4} \cdot n+47$ $O_{3c} \rightarrow 3^4 \cdot 2^{k-4} \cdot n+142 \rightarrow 3^4 \cdot 2^{k-5} \cdot n+71$ $O_{3b} \rightarrow$
$3^5 \cdot 2^{k-5} \cdot n+214 \rightarrow 3^5 \cdot 2^{k-6} \cdot n+107$ $O_{2a} \rightarrow 3^6 \cdot 2^{k-6} \cdot n+322 \rightarrow 3^6 \cdot 2^{k-7} \cdot n+161$ $O_1 \rightarrow 3^7 \cdot 2^{k-7} \cdot n+484 \rightarrow$
$3^7 \cdot 2^{k-8} \cdot n+242 \rightarrow 3^7 \cdot 2^{k-9} \cdot n+121$ $O_1 \rightarrow 3^8 \cdot 2^{k-9} \cdot n+364 \rightarrow 3^8 \cdot 2^{k-10} \cdot n+182 \rightarrow 3^8 \cdot 2^{k-11} \cdot n+91$ $O_{2a} \rightarrow$
$3^9 \cdot 2^{k-11} \cdot n+274 \rightarrow 3^9 \cdot 2^{k-12} \cdot n+137$ $O_1 \rightarrow 3^{10} \cdot 2^{k-12} \cdot n+412 \rightarrow 3^{10} \cdot 2^{k-13} \cdot n+206 \rightarrow$
$3^{10} \cdot 2^{k-14} \cdot n+103$ $O_{3b} \rightarrow 3^{11} \cdot 2^{k-14} \cdot n+310 \rightarrow 3^{11} \cdot 2^{k-15} \cdot n+155$ $O_{2a} \rightarrow 3^{12} \cdot 2^{k-15} \cdot n+466 \rightarrow$
$3^{12} \cdot 2^{k-16} \cdot n+233$ $O_1 \rightarrow 3^{13} \cdot 2^{k-16} \cdot n+700 \rightarrow 3^{13} \cdot 2^{k-17} \cdot n+350 \rightarrow 3^{13} \cdot 2^{k-18} \cdot n+175$ $O_{3c} \rightarrow$
$3^{14} \cdot 2^{k-18} \cdot n+526 \rightarrow 3^{14} \cdot 2^{k-19} \cdot n+263$ $O_{3b} \rightarrow 3^{15} \cdot 2^{k-19} \cdot n+790 \rightarrow 3^{15} \cdot 2^{k-20} \cdot n+395$ $O_{2a} \rightarrow$
$3^{16} \cdot 2^{k-20} \cdot n+1186 \rightarrow 3^{16} \cdot 2^{k-21} \cdot n+593$ $O_1 \rightarrow 3^{17} \cdot 2^{k-21} \cdot n+1780 \rightarrow 3^{17} \cdot 2^{k-22} \cdot n+890 \rightarrow$
$3^{17} \cdot 2^{k-23} \cdot n+445$ $O_1 \rightarrow 3^{18} \cdot 2^{k-23} \cdot n+1336 \rightarrow 3^{18} \cdot 2^{k-24} \cdot n+668 \rightarrow 3^{18} \cdot 2^{k-25} \cdot n+334 \rightarrow$
$3^{18} \cdot 2^{k-26} \cdot n+167$ $O_{3b} \rightarrow 3^{19} \cdot 2^{k-26} \cdot n+502 \rightarrow 3^{19} \cdot 2^{k-27} \cdot n+251$ $O_{2a} \rightarrow 3^{20} \cdot 2^{k-27} \cdot n+754 \rightarrow$
$3^{20} \cdot 2^{k-28} \cdot n+377$ $O_1 \rightarrow 3^{21} \cdot 2^{k-28} \cdot n+1132 \rightarrow 3^{21} \cdot 2^{k-29} \cdot n+566 \rightarrow 3^{21} \cdot 2^{k-30} \cdot n+283$ $O_{2a} \rightarrow$
$3^{22} \cdot 2^{k-30} \cdot n+850 \rightarrow 3^{22} \cdot 2^{k-31} \cdot n+425$ $O_1 \rightarrow 3^{23} \cdot 2^{k-31} \cdot n+1276 \rightarrow 3^{23} \cdot 2^{k-32} \cdot n+638 \rightarrow$
$3^{23} \cdot 2^{k-33} \cdot n+319$ $O_{3a} \rightarrow 3^{24} \cdot 2^{k-33} \cdot n+958 \rightarrow 3^{24} \cdot 2^{k-34} \cdot n+479$ $O_{3a} \rightarrow 3^{25} \cdot 2^{k-34} \cdot n+1438 \rightarrow$
$3^{25} \cdot 2^{k-35} \cdot n+719$ $O_{3c} \rightarrow 3^{26} \cdot 2^{k-35} \cdot n+2158 \rightarrow 3^{26} \cdot 2^{k-36} \cdot n+1079$ $O_3^* \rightarrow 3^{27} \cdot 2^{k-36} \cdot n+3238 \rightarrow$
$3^{27} \cdot 2^{k-37} \cdot n+1619$ $O_2^* \rightarrow 3^{28} \cdot 2^{k-37} \cdot n+4858 \rightarrow 3^{28} \cdot 2^{k-38} \cdot n+2429$ $O_1 \rightarrow 3^{29} \cdot 2^{k-38} \cdot n+7288 \rightarrow$
$3^{29} \cdot 2^{k-39} \cdot n+3644 \rightarrow 3^{29} \cdot 2^{k-40} \cdot n+1822 \rightarrow 3^{29} \cdot 2^{k-41} \cdot n+911$ $O_{3c} \rightarrow 3^{30} \cdot 2^{k-41} \cdot n+2734 \rightarrow$
$3^{30} \cdot 2^{k-42} \cdot n+1367$ $O_3^* \rightarrow 3^{31} \cdot 2^{k-42} \cdot n+4102 \rightarrow 3^{31} \cdot 2^{k-43} \cdot n+2051$ $O_2^* \rightarrow 3^{32} \cdot 2^{k-43} \cdot n+6154 \rightarrow$

$3^{32}\cdot2^{k-44}\cdot n+3077$ O_1 → $3^{33}\cdot2^{k-44}\cdot n+9232$ → $3^{33}\cdot2^{k-45}\cdot n+4616$ → $3^{33}\cdot2^{k-46}\cdot n+2308$ →
$3^{33}\cdot2^{k-47}\cdot n+1154$ → $3^{33}\cdot2^{k-48}\cdot n+577$ O_1 → $3^{34}\cdot2^{k-48}\cdot n+1732$ → $3^{34}\cdot2^{k-49}\cdot n+866$ →
$3^{34}\cdot2^{k-50}\cdot n+433$ O_1 → $3^{35}\cdot2^{k-50}\cdot n+1300$ → $3^{35}\cdot2^{k-51}\cdot n+650$ → $3^{35}\cdot2^{k-52}\cdot n+325$ O_1 →
$3^{36}\cdot2^{k-52}\cdot n+976$ → $3^{36}\cdot2^{k-53}\cdot n+488$ → $3^{36}\cdot2^{k-54}\cdot n+244$ → $3^{36}\cdot2^{k-55}\cdot n+122$ →
$3^{36}\cdot2^{k-56}\cdot n+61$ O_1 → $3^{37}\cdot2^{k-56}\cdot n+184$ → $3^{37}\cdot2^{k-57}\cdot n+92$ → $3^{37}\cdot2^{k-58}\cdot n+46$ →
$\mathbf{3^{37}\cdot2^{k-59}\cdot n+23} < \mathbf{2^{k}\cdot n+27}$ O_{2a}

k = 59 → $\mathbf{3^{37}\cdot n+23} < \mathbf{2^{59}\cdot n+27}$ O_{2a} : N(s) = **96**

Chain of transformations:
O_{2a} → O_1 → O_{3a} → O_{3c}→ O_{3b} → O_{2a} → O_1 → O_1 → O_{2a} → O_1 → O_{3b} → O_{2a} → O_1 → O_{3c} → O_{3b} → O_{2a} → O_1 → O_1 → O_{3b} → O_{2a} → O_1 → O_{2a} → O_1 → O_{3a} → O_{3a} → O_{3c} → O_3* → O_2* → O_1 → O_{3c} → O_3* → O_2* → O_1 → O_1 → O_1 → O_1 → O_1 → E

In the chain of transformations we note that O_1 appears many more times than the other subsets in perfect agreement with Theorem 2n+1 and its addition.

Inside the cycle of link of 27 O_{2a} are present the following binomial inequalities

$3^{37}\cdot2^{k-59}\cdot n+23$ O_3* < $3^{2}\cdot2^{k-3}\cdot n+31$ O_{3a} → $3^{35}\cdot2^{k-59}\cdot n+23 < 2^{k-3}\cdot n+31$: k = 59 →
$\mathbf{3^{35}\cdot n+23 < 2^{56}\cdot n+31}$ O_{3a} : N(s) = **91**

$3^{37}\cdot2^{k-58}\cdot n+46 < 3^{3}\cdot2^{k-4}\cdot n+47$ O_{3c} → $3^{34}\cdot2^{k-58}\cdot n+46 < 2^{k-4}\cdot n+47$: k = 58 →
$\mathbf{3^{34}\cdot n+46 < 2^{54}\cdot n+47}$ O_{3c} : N(s) = **88**

$3^{36}\cdot2^{k-56}\cdot n+61$ O_1 < $3^{4}\cdot2^{k-5}\cdot n+71$ O_{3b} → $3^{32}\cdot2^{k-56}\cdot n+61 < 2^{k-5}\cdot n+71$: k = 56 →
$\mathbf{3^{32}\cdot n+61 < 2^{51}\cdot n+71}$ O_{3b} : N(s) = **83**

$3^{36}\cdot2^{k-56}\cdot n+61$ O_1 < $3^{8}\cdot2^{k-11}\cdot n+91$ O_{2a} → $3^{28}\cdot2^{k-56}\cdot n+61 < 2^{k-11}\cdot n+91$: k = 56 →
$\mathbf{3^{28}\cdot n+61 < 2^{45}\cdot n+91}$ O_{2a} : N(s) = **73**

$3^{36}\cdot2^{k-56}\cdot n+61$ O_1 < $3^{10}\cdot2^{k-14}\cdot n+103$ O_{3b} → $3^{26}\cdot2^{k-56}\cdot n+61 < 2^{k-14}\cdot n+103$: k = 56 →
$\mathbf{3^{26}\cdot n+61 < 2^{42}\cdot n+103}$ O_{3b} : N(s) = **68**

$3^{36}\cdot2^{k-55}\cdot n+122 < 3^{11}\cdot2^{k-15}\cdot n+155$ O_{2a} → $3^{25}\cdot2^{k-55}\cdot n+122 < 2^{k-15}\cdot n+155$: k = 55 →
$\mathbf{3^{25}\cdot n+122 < 2^{40}\cdot n+155}$ O_{2a} : N(s) = **65**

$3^{36}\cdot2^{k-55}\cdot n+122 < 3^{18}\cdot2^{k-26}\cdot n+167$ O_{3b} → $3^{18}\cdot2^{k-55}\cdot n+122 < 2^{k-26}\cdot n+167$: k = 55 →
$\mathbf{3^{18}\cdot n+122 < 2^{29}\cdot n+167}$ O_{3b} : N(s) = **47**

$3^{36}\cdot2^{k-54}\cdot n+244 < 3^{19}\cdot2^{k-27}\cdot n+251$ O_{2a} → $3^{17}\cdot2^{k-54}\cdot n+244 < 2^{k-27}\cdot n+251$: k = 54 →
$\mathbf{3^{17}\cdot n+244 < 2^{27}\cdot n+251}$ O_{2a} : N(s) = **44**

$3^{36}\cdot2^{k-54}\cdot n+244 < 3^{21}\cdot2^{k-30}\cdot n+283$ O_{2a} → $3^{15}\cdot2^{k-54}\cdot n+244 < 2^{k-30}\cdot n+283$: k = 54 →
$\mathbf{3^{15}\cdot n+244 < 2^{24}\cdot n+283}$ O_{2a} : N(s) = **39**

$3^{36}\cdot2^{k-54}\cdot n+244 < 3^{23}\cdot2^{k-33}\cdot n+319$ O_{3a} → $3^{13}\cdot2^{k-54}\cdot n+244 < 2^{k-33}\cdot n+319$: k = 54 →
$\mathbf{3^{13}\cdot n+244 < 2^{21}\cdot n+319}$ O_{3a} : N(s) = **34**

$3^{34}\cdot2^{k-50}\cdot n+433$ O_1 < $3^{24}\cdot2^{k-34}\cdot n+479$ O_{3a} → $3^{10}\cdot2^{k-50}\cdot n+433 < 2^{k-34}\cdot n+479$: k = 50 →
$\mathbf{3^{10}\cdot n+433 < 2^{16}\cdot n+479}$ O_{3a} : N(s) = **26**

$3^{33}\cdot 2^{k-48}\cdot n+577$ O_1 < $3^{25}\cdot 2^{k-35}\cdot n+719$ O_{3c} → $3^{8}\cdot 2^{k-48}\cdot n+577 < 2^{k-35}\cdot n+719$: k = 48 →
$\mathbf{3^{8}\cdot n+577 < 2^{13}\cdot n+719}$ O_{3c} : N(s) = **21**

Of course in the cycle of link of 27 we could apply Independence Theorem and formulas plan Theorem 2n+1, but in this case we would have lost some of the internal binomial inequalities.

3.3.2. Applications Independence Theorem

We apply Theorem of Independence to odd numbers in O_{2a} ; O_{3a} ; O_{3b} ; O_{3c} . We don't take into consideration O_1 ; O_2* ; O_3* because they are completely covered by their own respective binomial inequalities.

Remembering : $\begin{cases} x = \frac{n-1}{2} \rightarrow f(x) = \frac{3x+2}{2} : N(s) = 3 \\ y = \frac{n-1}{2} \rightarrow f(y) = \frac{9y+7}{4} : N(s) = 5 \\ z = \frac{n-1}{2} \rightarrow f(z) = \frac{27z+23}{4} : N(s) = 6 \end{cases}$

n = **2407**

16n+11 = 38523 O_{2a}
$\mathbf{2^{k}\cdot n+38523}$ O_{2a} : y = $2^{k-1}\cdot n+19261$ → f(y) = $3^{2}\cdot 2^{k-3}\cdot n+43339$ O_{2a} : y = $3^{2}\cdot 2^{k-4}\cdot n+21669$ →
f(y) = $3^{4}\cdot 2^{k-6}\cdot n+48757$ O_1 : x = $3^{4}\cdot 2^{k-7}\cdot n+24378$ → f(x) = $3^{5}\cdot 2^{k-8}\cdot n+36568 < 2^{k}\cdot n+38523$ O_{2a}
k = 8 → $\mathbf{3^{5}\cdot n+36568 < 2^{8}\cdot n+38523}$ O_{2a} : N(s) = **13** ~ 243n+118 < 256n+123 O_{2a} : n = 150

32n-1 = 77023 O_{3a}
$\mathbf{2^{k}\cdot n+77023}$ O_{3a} : z = $2^{k-1}\cdot n+38511$ → f(z) = $3^{3}\cdot 2^{k-3}\cdot n+259955$ O_2^* : y = $3^{3}\cdot 2^{k-4}\cdot n+129977$ →
f(y) = $3^{5}\cdot 2^{k-6}\cdot n+292450$ → $3^{5}\cdot 2^{k-7}\cdot n+146225$ O_1 : x = $3^{5}\cdot 2^{k-8}\cdot n+73112$ →
f(x) = $3^{6}\cdot 2^{k-9}\cdot n+109669$ O_1 : x = $3^{6}\cdot 2^{k-10}\cdot n+54834$ → f(x) = $3^{7}\cdot 2^{k-11}\cdot n+82252$ →
$3^{7}\cdot 2^{k-12}\cdot n+41126 < 2^{k}\cdot n+77023$ O_{3a}
k = 12 → $\mathbf{3^{7}\cdot n+41126 < 2^{12}\cdot n+77023}$ O_{3a} : N(s) = **19** ~ 2187n+1760 < 4096n+3295 O_{3a} : n = 18

32n+7 = 77031 O_{3b}
$\mathbf{2^{k}\cdot n+77031}$ O_{3b} : z = $2^{k-1}\cdot n+38515$ → f(z) = $3^{3}\cdot 2^{k-3}\cdot n+259982$ → $3^{3}\cdot 2^{k-4}\cdot n+129991$ O_{3b} :
z = $3^{3}\cdot 2^{k-5}\cdot n+64995$ → f(z) = $3^{6}\cdot 2^{k-7}\cdot n+438722$ → $3^{6}\cdot 2^{k-8}\cdot n+219361$ O_1 : x = $3^{6}\cdot 2^{k-9}\cdot n+109680$ →
f(x) = $3^{7}\cdot 2^{k-10}\cdot n+164521$ O_1 : x = $3^{7}\cdot 2^{k-11}\cdot n+82260$ → f(x) = $3^{8}\cdot 2^{k-12}\cdot n+123391$ O_{3a} :
z = $3^{8}\cdot 2^{k-13}\cdot n+61695$ → f(z) = $3^{11}\cdot 2^{k-15}\cdot n+416447$ O_{3a} : z = $3^{11}\cdot 2^{k-16}\cdot n+208223$ →
f(z) = $3^{14}\cdot 2^{k-18}\cdot n+1405511$ O_{3b} : z = $3^{14}\cdot 2^{k-19}\cdot n+702755$ → f(z) = $3^{17}\cdot 2^{k-21}\cdot n+4743602$ →
$3^{17}\cdot 2^{k-22}\cdot n+2371801$ O_1 : x = $3^{17}\cdot 2^{k-23}\cdot n+1185900$ → f(x) = $3^{18}\cdot 2^{k-24}\cdot n+1778851$ O_2^* :
y = $3^{18}\cdot 2^{k-25}\cdot n+889425$ → f(y) = $3^{20}\cdot 2^{k-27}\cdot n+2001208$ → $3^{20}\cdot 2^{k-28}\cdot n+1000604$ →
$3^{20}\cdot 2^{k-29}\cdot n+500302$ → $3^{20}\cdot 2^{k-30}\cdot n+250151$ O_{3b} : z = $3^{20}\cdot 2^{k-31}\cdot n+125075$ →
f(z) = $3^{23}\cdot 2^{k-33}\cdot n+844262$ → $3^{23}\cdot 2^{k-34}\cdot n+422131$ O_2^* : y = $3^{23}\cdot 2^{k-35}\cdot n+211065$ →
f(y) = $3^{25}\cdot 2^{k-37}\cdot n+474898$ → $3^{25}\cdot 2^{k-38}\cdot n+237449$ O_1 : x = $3^{25}\cdot 2^{k-39}\cdot n+118724$ →
f(x) = $3^{26}\cdot 2^{k-40}\cdot n+178087$ O_{3b} : z = $3^{26}\cdot 2^{k-41}\cdot n+189043$ → f(z) = $3^{29}\cdot 2^{k-43}\cdot n+601046$ →
$3^{29}\cdot 2^{k-44}\cdot n+300523$ O_{2a} : y = $3^{29}\cdot 2^{k-45}\cdot n+150261$ → f(y) = $3^{31}\cdot 2^{k-47}\cdot n+338089$ O_1 :
x = $3^{31}\cdot 2^{k-48}\cdot n+169044$ → f(x) = $3^{32}\cdot 2^{k-49}\cdot n+253567$ O_{3a} : z = $3^{32}\cdot 2^{k-50}\cdot n+126783$ →
f(z) = $3^{35}\cdot 2^{k-52}\cdot n+855791$ O_{3c} : z = $3^{35}\cdot 2^{k-53}\cdot n+855791$ → f(z) = $3^{38}\cdot 2^{k-55}\cdot n+2888297$ O_1 :
x = $3^{38}\cdot 2^{k-56}\cdot n+1444148$ → f(x) = $3^{39}\cdot 2^{k-57}\cdot n+2166223$ O_{3c} : z = $3^{39}\cdot 2^{k-58}\cdot n+1083111$ →
f(z) = $3^{42}\cdot 2^{k-60}\cdot n+7311005$ O_1 : x = $3^{42}\cdot 2^{k-61}\cdot n+3655502$ → f(x) = $3^{43}\cdot 2^{k-62}\cdot n+5483254$ →
$3^{43}\cdot 2^{k-63}\cdot n+2741627$ O_{2a} : y = $3^{43}\cdot 2^{k-64}\cdot n+1370813$ → f(y) = $3^{45}\cdot 2^{k-66}\cdot n+3084331$ O_{2a} :
y = $3^{45}\cdot 2^{k-67}\cdot n+1542165$ → f(y) = $3^{47}\cdot 2^{k-69}\cdot n+3469873$ O_1 : x = $3^{47}\cdot 2^{k-70}\cdot n+1734936$ →

$f(x) = 3^{48}\cdot2^{k-71}\cdot n+2602405\ O_1 : x = 3^{48}\cdot2^{k-72}\cdot n+1301202 \rightarrow f(x) = 3^{49}\cdot2^{k-73}\cdot n+1951804 \rightarrow$
$3^{49}\cdot2^{k-74}\cdot n+975902 \rightarrow 3^{49}\cdot2^{k-75}\cdot n+487951\ O_{3b} : z = 3^{49}\cdot2^{k-76}\cdot n+243975 \rightarrow$
$f(z) = 3^{52}\cdot2^{k-78}\cdot n+1646837\ O_1 : x = 3^{52}\cdot2^{k-79}\cdot n+823418 \rightarrow f(x) = 3^{53}\cdot2^{k-80}\cdot n+1235128 \rightarrow$
$3^{53}\cdot2^{k-81}\cdot n+617564 \rightarrow 3^{53}\cdot2^{k-82}\cdot n+308782 \rightarrow 3^{53}\cdot2^{k-83}\cdot n+154391\ O_3^* : z = 3^{53}\cdot2^{k-84}\cdot n+77195 \rightarrow$
$f(z) = 3^{56}\cdot2^{k-86}\cdot n+521072 \rightarrow 3^{56}\cdot2^{k-87}\cdot n+260536 \rightarrow 3^{56}\cdot2^{k-88}\cdot n+130268 \rightarrow$
$3^{56}\cdot2^{k-89}\cdot n+65134 < 2^k\cdot n+77031\ O_{3b}$
k = 89 → $\mathbf{3^{56}\cdot n+65134 < 2^{89}\cdot n+77031}$ O_{3b} : N(s) = **145**

32n+15 = 77039 O_{3c}
$\mathbf{2^k\cdot n+77039}$ $O_{3c} : z = 2^{k-1}\cdot n+38519 \rightarrow f(z) = 3^3\cdot2^{k-3}\cdot n+260009\ O_1 : x = 3^3\cdot2^{k-4}\cdot n+130004 \rightarrow$
$f(x) = 3^4\cdot2^{k-5}\cdot n+195007\ O_{3a} : z = 3^4\cdot2^{k-6}\cdot n+97503 \rightarrow f(z) = 3^7\cdot2^{k-8}\cdot n+658151\ O_{3b} :$
$z = 3^7\cdot2^{k-9}\cdot n+329075 \rightarrow f(z) = 3^{10}\cdot2^{k-11}\cdot n+2221262 \rightarrow 3^{10}\cdot2^{k-12}\cdot n+1110631\ O_{3b} :$
$z = 3^{10}\cdot2^{k-13}\cdot n+555315 \rightarrow f(z) = 3^{13}\cdot2^{k-15}\cdot n+3748382 \rightarrow 3^{13}\cdot2^{k-16}\cdot n+1874191\ O_{3c} :$
$z = 3^{13}\cdot2^{k-17}\cdot n+937095 \rightarrow f(z) = 3^{16}\cdot2^{k-19}\cdot n+6325397\ O_1 : x = 3^{16}\cdot2^{k-20}\cdot n+3162698 \rightarrow$
$f(x) = 3^{17}\cdot2^{k-21}\cdot n+4744048 \rightarrow 3^{17}\cdot2^{k-22}\cdot n+2372024 \rightarrow 3^{17}\cdot2^{k-23}\cdot n+1186012 \rightarrow$
$3^{17}\cdot2^{k-24}\cdot n+593006 \rightarrow 3^{17}\cdot2^{k-25}\cdot n+296503\ O_3^* : z = 3^{17}\cdot2^{k-26}\cdot n+148251 \rightarrow$
$f(z) = 3^{20}\cdot2^{k-28}\cdot n+1000700 \rightarrow 3^{20}\cdot2^{k-29}\cdot n+500350 \rightarrow 3^{20}\cdot2^{k-30}\cdot n+250175\ O_{3a} :$
$z = 3^{20}\cdot2^{k-31}\cdot n+125087 \rightarrow f(z) = 3^{23}\cdot2^{k-33}\cdot n+844343\ O_3^* : z = 3^{23}\cdot2^{k-34}\cdot n+422171 \rightarrow$
$f(z) = 3^{26}\cdot2^{k-36}\cdot n+2849660 \rightarrow 3^{26}\cdot2^{k-37}\cdot n+1424830 \rightarrow 3^{26}\cdot2^{k-38}\cdot n+712415\ O_{3a} :$
$z = 3^{26}\cdot2^{k-39}\cdot n+356207 \rightarrow f(z) = 3^{29}\cdot2^{k-41}\cdot n+2404403\ O_2^* : y = 3^{29}\cdot2^{k-42}\cdot n+1202201 \rightarrow$
$f(y) = 3^{31}\cdot2^{k-44}\cdot n+2704954 \rightarrow 3^{31}\cdot2^{k-45}\cdot n+1352477\ O_1 : x = 3^{31}\cdot2^{k-46}\cdot n+676238 \rightarrow$
$f(x) = 3^{32}\cdot2^{k-47}\cdot n+1014358 \rightarrow 3^{32}\cdot2^{k-48}\cdot n+507179\ O_{2a} : y = 3^{32}\cdot2^{k-49}\cdot n+253589 \rightarrow$
$f(y) = 3^{34}\cdot2^{k-51}\cdot n+570577\ O_1 : x = 3^{34}\cdot2^{k-52}\cdot n+285288 \rightarrow f(x) = 3^{35}\cdot2^{k-53}\cdot n+427933\ O_1 :$
$x = 3^{35}\cdot2^{k-54}\cdot n+213966 \rightarrow f(x) = 3^{36}\cdot2^{k-55}\cdot n+320950 \rightarrow 3^{36}\cdot2^{k-56}\cdot n+160475\ O_{2a} :$
$y = 3^{36}\cdot2^{k-57}\cdot n+80237 \rightarrow f(y) = 3^{38}\cdot2^{k-59}\cdot n+180535\ O_3^* : z = 3^{38}\cdot2^{k-60}\cdot n+90267 \rightarrow$
$f(z) = 3^{41}\cdot2^{k-62}\cdot n+609308 \rightarrow 3^{41}\cdot2^{k-63}\cdot n+304654 \rightarrow 3^{41}\cdot2^{k-64}\cdot n+152327\ O_{3b} :$
$z = 3^{41}\cdot2^{k-65}\cdot n+76163 \rightarrow f(z) = 3^{44}\cdot2^{k-67}\cdot n+514106 \rightarrow 3^{44}\cdot2^{k-68}\cdot n+257053\ O_1 :$
$x = 3^{44}\cdot2^{k-69}\cdot n+128256 \rightarrow f(x) = 3^{45}\cdot2^{k-70}\cdot n+192790 \rightarrow 3^{45}\cdot2^{k-71}\cdot n+96395\ O_{2a} :$
$y = 3^{45}\cdot2^{k-72}\cdot n+48197 \rightarrow f(y) = 3^{47}\cdot2^{k-74}\cdot n+108445\ O_1 : x = 3^{47}\cdot2^{k-75}\cdot n+54222 \rightarrow$
$f(x) = 3^{48}\cdot2^{k-76}\cdot n+81334 \rightarrow 3^{48}\cdot2^{k-77}\cdot n+40667 < 2^k\cdot n+77039\ O_{3c}$
k = 77 → $\mathbf{3^{48}\cdot n+40667}$ < $\mathbf{2^{77}\cdot n+77039}$ O_{3c} : N(s) = **125**

3.3.3. Two examples completion General List

By Theorem of Independence and formulas derived from Theorem 2n+1, we research cycles of links N(s) ≤ 21. If N(s) > 21 we don't write the corresponding cycle.

N(s) = **16** : 729n+362 < 1024n+507 O_{2a} ; 729n+410 < 1024n+575 O_{3a}

p : 507 < *p* < 575 = {575mod4 - $p \in O_2^* \cup p \in O_3^* \cup p \sim$ N(s) = 8 $\cup\ p \sim$ N(s) = 11 $\cup\ p \sim$ N(s) = 13} = {511; 539; 543; 559}
p = 511 → $2^k\cdot n+511\ O_{3a}$: N(s) > 21
p = 539 → $2^k\cdot n+539\ O_{2a} : y = 2^{k-1}\cdot n+269 \rightarrow f(y) = 3^2\cdot2^{k-3}\cdot n+607\ O_{3a} : z = 3^2\cdot2^{k-4}\cdot n+303 \rightarrow$
$f(z) = 3^5\cdot2^{k-6}\cdot n+2051\ O_2^* : y = 3^5\cdot2^{k-7}\cdot n+1025\ O_1 \rightarrow f(y) = 3^7\cdot2^{k-9}\cdot n+2308 \rightarrow$
$3^7\cdot2^{k-10}\cdot n+1154 \rightarrow 3^7\cdot2^{k-11}\cdot n+577\ O_1 : x = 3^7\cdot2^{k-12}\cdot n+288 \rightarrow f(x) = 3^8\cdot2^{k-13}\cdot n+433 < 2^k\cdot n+539\ O_{2a}$
k = 13 → $\mathbf{3^8\cdot n+433 < 2^{13}\cdot n+539}$ O_{2a} : to add General list N(s) = **21**
p = 543 → $2^k\cdot n+543\ O_{3a} : z = 2^{k-1}\cdot n+271 \rightarrow f(z) = 3^3\cdot2^{k-3}\cdot n+1835\ O_{2a} : y = 3^3\cdot2^{k-4}\cdot n+917 \rightarrow$
$f(y) = 3^5\cdot2^{k-6}\cdot n+2065\ O_1 : x = 3^5\cdot2^{k-7}\cdot n+1032 \rightarrow f(x) = 3^6\cdot2^{k-8}\cdot n+1549\ O_1 :$
$x = 3^6\cdot2^{k-9}\cdot n+774 \rightarrow f(x) = 3^7\cdot2^{k-10}\cdot n+1162 \rightarrow 3^7\cdot2^{k-11}\cdot n+581\ O_1 : x = 3^7\cdot2^{k-12}\cdot n+290 \rightarrow$
$f(x) = 3^8\cdot2^{k-13}\cdot n+436 < 2^k\cdot n+543\ O_{3a}$
k = 13 → $\mathbf{3^8\cdot n+436 < 2^{13}\cdot n+543}$ O_{3a} : to add General list N(s) = **21**

$p = 559 \rightarrow 2^k \cdot n+559\ O_{3c} : N(s) > 21$

$N(s) = \mathbf{19} : 3^7 \cdot n+910 < 2^{12} \cdot n+1703\ O_{3b}\ ;\ 3^7 \cdot n+955 < 2^{12} \cdot n+1787\ O_{2a}$

$p : 1703 < p < 1787 = \{1787\text{mod}4 - p \in O_2^* \cup p \in O_3^* \cup p \sim N(s) = 8 \cup p \sim N(s) = 11 \cup p \sim N(s) = 13 \cup p \sim N(s) = 16\} = \{1727; 1743; 1775\}$
$p = 1727 : \mathbf{2^k \cdot n+1727}\ O_{3a} : z = 2^{k-1} \cdot n+863 \rightarrow f(z) = 3^3 \cdot 2^{k-3} \cdot n+5831\ O_{3b} : z = 3^3 \cdot 2^{k-4} \cdot n+2915 \rightarrow f(z) = 3^6 \cdot 2^{k-6} \cdot n+19682 \rightarrow 3^6 \cdot 2^{k-7} \cdot n+9841\ O_1 : x = 3^6 \cdot 2^{k-8} \cdot n+4920 \rightarrow f(x) = 3^7 \cdot 2^{k-9} \cdot n+7381\ O_1 :$
$x = 3^7 \cdot 2^{k-10} \cdot n+3690 \rightarrow f(x) = 3^8 \cdot 2^{k-11} \cdot n+5536 \rightarrow 3^8 \cdot 2^{k-12} \cdot n+2768 \rightarrow$
$3^8 \cdot 2^{k-13} \cdot n+1384 < 2^k \cdot n+1727\ O_{3a}$
$k = 13 \rightarrow \mathbf{3^8 \cdot n+1384 < 2^{13} \cdot n+1727}\ O_{3a}$: to add General list $N(s) = \mathbf{21}$
$p = 1743 \rightarrow 2^k \cdot n+1743\ O_{3c} : N(s) > 21$
$p = 1775 \rightarrow 2^k \cdot n+1775\ O_{3c} : N(s) > 21$

3.4. Big Bang and Big Crunch

Maybe the cycles of the links of the SC (CC) represents a possible kind of law on expansion of Cosmos called BIG BANG. Starting from 1 we reach very high and very large horizons. As we saw BIG CRUNCH is always possible, but BIG BANG has no End.

BIG BANG :

$1 \rightarrow 2 \rightarrow 4 \rightarrow 16 \rightarrow 32 \rightarrow 128 \rightarrow 256 \rightarrow 1024 \rightarrow 4096 \rightarrow 8192 \rightarrow 2^{15} \rightarrow 2^{16} \rightarrow 2^{18} \rightarrow \ldots \rightarrow 2^{21} \rightarrow 2^{23} \rightarrow 2^{24} \rightarrow 2^{26} \rightarrow 2^{27} \rightarrow 2^{29} \rightarrow 2^{31} \rightarrow \ldots \rightarrow 2^{35} \rightarrow \ldots \rightarrow 2^{40} \rightarrow 2^{42} \rightarrow \ldots \rightarrow 2^{45} \rightarrow \ldots \rightarrow 2^{51} \rightarrow \ldots \rightarrow 2^{54} \rightarrow 2^{56} \rightarrow \ldots \rightarrow 2^{59} \rightarrow \ldots \rightarrow 2^{77} \rightarrow \ldots \rightarrow 2^{89} \rightarrow \ldots \rightarrow 2^k \rightarrow \ldots \rightarrow \infty$

Where … are occupied by 2^k of the cycles of links to us still unknown.
Increment of k = 1 or 2.

BIG CRUNCH :

$3^h \rightarrow 3^{h-1} \rightarrow 3^{h-2} \rightarrow 3^{h-3} \rightarrow \ldots \rightarrow 3^{37} \rightarrow 3^{36} \rightarrow 3^{35} \rightarrow 3^{34} \rightarrow 3^{33} \rightarrow \ldots \rightarrow 3^{20} \rightarrow 3^{19} \rightarrow 3^{18} \rightarrow \ldots \rightarrow 3^{12} \rightarrow 3^{11} \rightarrow 3^{10} \rightarrow 3^9 \rightarrow 6561 \rightarrow 2187 \rightarrow 729 \rightarrow 243 \rightarrow 81 \rightarrow 27 \rightarrow 9 \rightarrow 3 \rightarrow 1$

Decrease of h always 1 .

3.5. Final Proof

We have seen that SC is a sort of *Circle Quadrature*, and this paper proves that is not possible to arrive 100% coverage of ℕ by binomial inequalities. So, CC is not completely demonstrable, but we can claim that its initial statement is true, in fact:

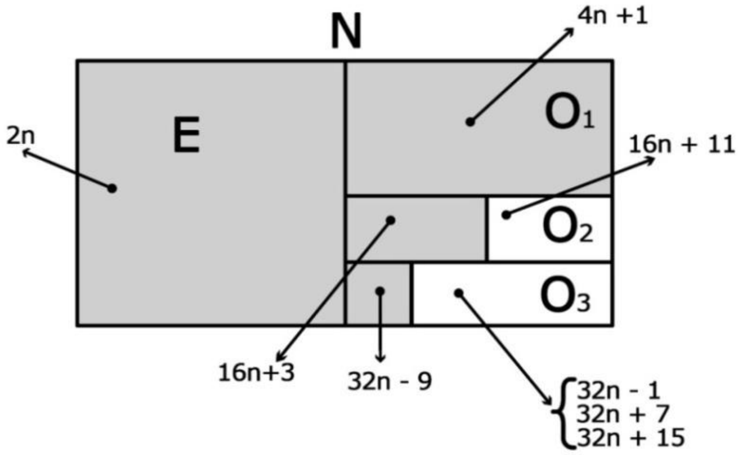
$$\text{For any } \Theta(m) = 2^k \cdot n+p \text{ ; there is } \Theta(l) = 3^h \cdot n+q : \begin{cases} h < k \\ q < p \end{cases} : \ \Theta(l) < \Theta(m)$$

This is guaranteed from Theorem 2n+1 and its addition as well as Theorem of Independence.

Hence there exists $a_n \in S(2n+1) : a_n < 2n+1$ after $N(s) = h + k$ steps.
Since exists $a_n \in S(2n) : a_n < 2n$ after N(s) = one step : $2n \rightarrow n$. Being $E \cup O = \mathbb{N}$.
So every $n \in \mathbb{N}$ falls down to 1 !!! → QED

3.6. Conclusion

Odd generating numbers in O_1 becomes less than itself after one application of Theorem 2n+1. So also it is for odd generating numbers in O_2* of type 16n+3. So also it is for odd generating numbers in O_3* of type 32n-9 ~ 32n+23. For the others odd generating numbers in $O_2 – O_2$* of type 16n+11 O_{2a} and in O_3 - O_3* of type 32n-1 ~ 32n+31 O_{3a} ; 32n+7 O_{3b} ; 32n+15 O_{3c} ; Theorem 2n+1 must be iterated two, three, or more times, or very many times, until the odd generating number becomes lesser than itself. In this way the blanks in the following pattern they fill more and more with grey almost to complete the set ℕ.



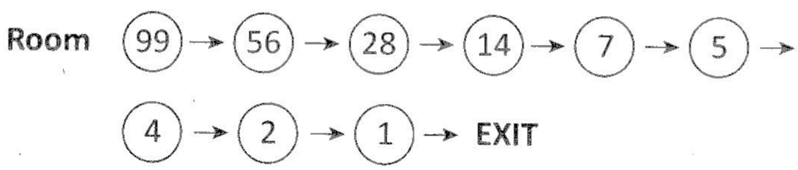
4. Appendices

The following appendices are added because Collatz Conjecture could be taught in High Schools successfully and student interest.

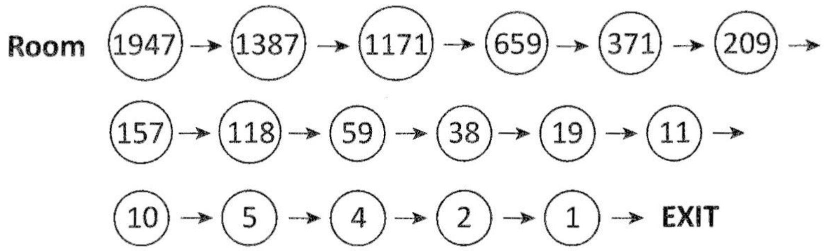
4.1. The labyrinth of Syracuse

We have seen how the endless successions generated by algorithm of Collatz are an inextricable labyrinth of numbered rooms, in which it seems impossible to find the exit; but the applications of Theorem 2n+1 and Independence Theorem allow us to find always the exit, standing in any room of this maze that we will call: Labyrinth of Syracuse.

Strategy to exit labyrinth of Syracuse as we stand in room number 99.



Strategy to exit labyrinth of Syracuse as we stand in room number 1947.

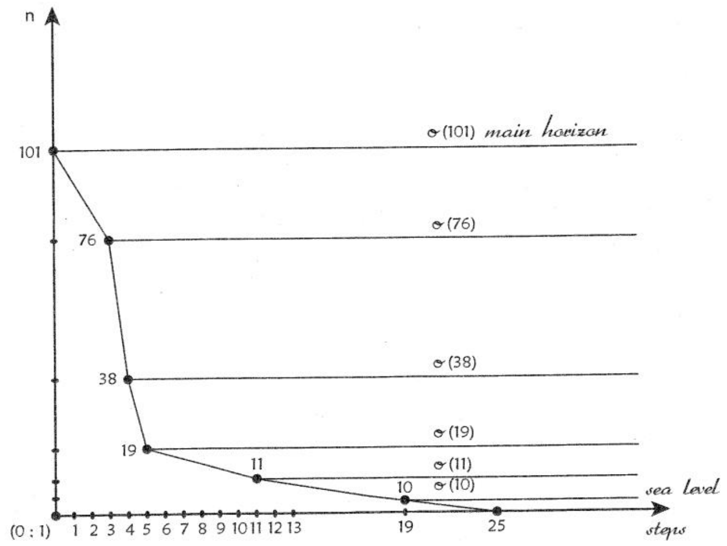


Main observations

In all cycles of links, the final cycle (last five elements = four steps) to exit from labyrinth of Syracuse is: **{10; 5; 4; 2; 1}**; or **{7; 5; 4; 2; 1}**; unless in the cycle appears an even number like that of kind 2^p, or an odd number of type $(4^p-1)/3$. In this case the subsequent steps are evaluable *a priori*, and the last five element are: $16 \to 8 \to 4 \to 2 \to 1$ = **{16; 8; 4; 2; 1}**.
Another final cycle in cycles of links is **{12; 6; 3; 2; 1}.** This cycle is anomalous, since it appears only if the generating numbers are of type $n = 3{\cdot}2^p \in I(3) = \{3; 6; 12; 24; 48; 96; 192; \dots; 3{\cdot}2^p\}$. The $n \in I(3)$ do not appear in any other iterative cycle. Not in those generated by even numbers, in fact: if $n \in E$, then $n \to n/2$, if $n/2 = 3{\cdot}2^p$: $n = 2{\cdot}3{\cdot}2^p = 3{\cdot}2^{p+1} \in I(3)$. Not in those generated by odd numbers; in fact if $n \in O$ then $n \to 3n+1$, if $3n+1 = 3{\cdot}2^p$: $n = (3{\cdot}2^p-1)/3 = 2^p-1/3$ which is impossible. The only odd number in I(3) is 3, which generates the sequence: **3** $\to 10 \to 5 \to 16 \to 8 \to 4 \to$ **2** < 3. In the labyrinth of Syracuse this anomalous cycle, up value 3, can be considered as an infinite sequence of even rooms, numbered one double of the other, aligned in a long hallway perfectly straight that is not connected to any other room of the maze. In the metaphor of the descent to the sea described in the following paragraph, instead, it can be interpreted in such a river that flows into the sea without having any tributaries or being affluent. For $n \geq 3$ all sequences generated by $n = 3{\cdot}2^p \in I(3)$ are monotone decreasing such that $a_n = a_{n-1}/2$. If, for example, the generating number is $n = 3{\cdot}2^{10} = 3072$ we have S(3072) = {3072; 1536; 768; 384; 192; 96; 48; 24; 12; 6; 3}.

4.2. Descent to the sea

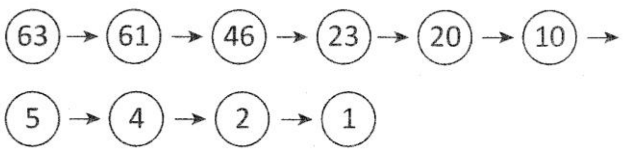
Instead of using the metaphor of the labyrinth, we use that of the descent to the sea, the examples are translated into graphs. It should be borne in mind that the final cycle {10; 5; 4; 2; 1} is done in 6 steps, since $\Theta(5)$ passes to $\Theta(4)$ in three steps, while the final cycle {7; 5; 4; 2; 1} is accomplished in 16 steps, since $\Theta(7)$ passes to $\Theta(5)$ in 11 steps and, as we already said, $\Theta(5)$ passes to $\Theta(4)$ in three steps. For convenience these end cycles are represented by a single line section, one step.
Graph of the descent to the sea from the main horizon $\Theta(m) = 101$. This arrives at the sea in 6 steps instead of 25.



In this way we can arrive at the sea level from highest heights, greatly reducing the number of steps and remaining always below the main horizon.

4.3. The skyscraper

If we wanted to get off on the first floor of a very high skyscraper finding ourselves on the 63rd floor, we should follow a clear strategy in pressing the buttons of the elevator, otherwise we will continue to go down and climb up again until to reach the 9232rd floor. Instead, using the method of the cycles of links, we know that we have to press the button 61 of the lift control panel. And then the button 46. And then the button 23. Since we already know the cycle of the links of number 23, we will get off on the first floor (1) with ease and from there, with the stairs, we will reach the ground floor. If we do not follow this strategy we would be forced to stay locked in the elevator, going up and down, for 107 floors. This is the summarized strategy.



References / Bibliography

[**1**] Zucchini R. *The logical solution Syracuse conjecture* (February 2022); Amazon on demand, Mnamon Milano Italy