# A PROBABILISTIC PROOF OF THE MULTINOMIAL THEOREM FOLLOWING THE NUMBER $A_n^p$

Atsu DEKPE

May 1, 2023

Abstract:
In this note, we give an alternate proof of the multinomial theorem following the number $A_n^p$ using probabilistic approach. Although the multinomial theorem following the number $A_n^p$ is basically a combinatorial result, our proof may be simple for a student familiar with only basic probability concepts.

## 1 INTRODUCTION

The following multinomial theorem based on number $n^p$ (development based on a power of a number) is an important result with many applications in mathematics statistics and computations. The theorem states as follows:

Theorem 1
Let n and m be non-zero natural numbers, $x_1, x_2, ..., x_m$ real numbers:

$$(x_1 + x_2 + ... + x_m)^n = \sum_{\sum_{i=1}^{m} k_i = n} \frac{n!}{k_1!k_2!...k_m!} x_1^{k_1} x_2^{k_2} ... x_m^{k_m} (1.1)$$

where $k_i$ are natural integers.
For readers interested, they can see [5] for further interpretation. Recently, we have published another type of multinomial theorem based on numbers $A_n^p$ and given some applications in the case of binomials. (see [1] for more details). In each case, the first demonstrations are based on a proof by induction using the binomial formula. A. Rosalsky proposed a probabilistic approach to this proof in the case of binomials [3] which will be generalized to the multinomial theorem by Kuldeep Kumar Kataria [4]. An urn contains $x_1$ balls numbered 1, $x_2$ balls numbered 2, ..., $x_m$ balls numbered m, such that the total number of balls is $N = \sum_{i=1}^{m} x_i$. Consider an experiment where we draw a ball from the urn without

replacement, and note the number on it each time. By repeating this experiment n times. The probability mass function of the variables $X_1, X_2, ..., X_m$ is :

$$P[X_1 = k_1, X_2 = k_2, ..., X_m = k_m] = n! \prod_{i=1}^{m} \frac{P_i^{k_j}}{k_j!} (1.2)$$

where $\sum_{i=1}^{m} k_i = n$ and $P_i^{k_j}$ the probability of having the balls numbered i $k_j$ times.
From (1.2) we have :

$$1 = \sum_{\sum_{i=1}^{m} k_i = n} n! \prod_{i=1}^{m} \frac{P_i^{k_j}}{k_j!} (1.3)$$

Next we will establish and prove the multinomial theorem following the number $A_n^p$.

## 2 A probabilistic proof of the multinomial theorem following the number $A_n^p$

Theorem 2
Let m and n be two non-zero natural numbers and $x_1, x_2, ..., x_m$ natural numbers. Then,

$$A^n_{(x_1+x_2+...+x_m)} = \sum_{\sum_{i=1}^{m} k_i = n} \frac{n!}{k_1!k_2!...k_m!} A_{x_1}^{k_1} A_{x_2}^{k_2} ... A_{x_m}^{k_m}$$

Where $k_i$ are non negative integers, $k_i \leq x_i$ and $A_n^{k_i} = k_i! \, C_n^{k_i} = k_i! \binom{n}{k_i}$
Proof : Let us consider :

$$A^n_{(x_1+x_2+...+x_m)} = (x_1+x_2+...+x_m)(x_1+x_2+...+x_{m-1})...(x_1+x_2+...+x_{m-n+1})$$

Using the distributivity property without resuming the number on the right side of the equation, it follows that for any natural numbers $x_i$ we have :

$$A^n_{(x_1+x_2+...+x_m)} = \sum_{\sum_{i=1}^{m} k_i = n} C_n^{k_1,k_2,...,k_m} A_{x_1}^{k_1} A_{x_2}^{k_2} ... A_{x_m}^{k_m} (2.1)$$

where $C_n^{k_1,k_2,...,k_m}$ are positive integers and $k_i$ are non negative integers satisfaying $\sum_{i=1}^{m} k_i = n$. We just need to show that.

$$C_n^{k_1,k_2,...,k_m} = \frac{n!}{k_1!k_2!...k_m!} (2.2)$$

We have $n \geq x_i$ for i=1,2,...,m Let's put :

$$P_j(i) = \frac{x_i^j}{x_1 + x_2 + ... + x_{m-j+1}} (2.3)$$

Where $x_i^j$ is the remaining number of the balls numbered i before the jth draw. $0 \leq P_j \leq 1$, substituting (2.3) in (1.3) we obtain

$$A^n_{(x_1+x_2+...+x_m)} = \sum_{\sum_{i=1}^m k_i = n} \frac{n!}{k_1!k_2!...k_m!} A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m} (2.4)$$

finally the subtraction of (2.4) from (2.1) gives :

$$\sum_{\sum_{i=1}^m k_i = n} ( C_n^{k_1,k_2,...,k_m} - \frac{n!}{k_1!k_2!...k_m!}) A^{k_1}_{x_1} A^{k_2}_{x_2} ... A^{k_m}_{x_m} = 0 (2.5); A^{k_i}_{x_i} \geq 0$$

since (2.5) is a zero polynomial in m variables, (2.2) follows and the proof is complete.

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