Two-dimensional differenceless derivatives of the first order of accuracy

Y. Mahotin

e-mail: yuri.mahotin@yahoo.com

Abstract—In this article, we present a method of differenceless derivatives for the numerical differentiation of two-dimensional functions when a set of arbitrary points is given near the point where derivatives are calculated. The new algorithm can be used in various fields of science and technology.

Keywords: numerical derivatives, forward differences, central differences

INTRODUCTION

Numerical differentiation of two dimensional function is an interesting topic in the field of numerical analysis. Almost all methods for calculating two-dimensional derivatives are based on finite difference approximations and their varieties, built on rectangular grids and their modifications, and work well as long as the function values are available on this grid. What happens when the function values are not available on a rectangular grid or the rectangular grid does not exist? Usually, in such cases, such a problem is not considered, due to the impossibility of obtaining a solution. We recently developed an algorithm for differenceless derivatives in the case of one-dimensional functions [1]. In this paper, we applied the method of differenceless derivatives for two-dimensional functions. Most importantly, the new algorithm for estimating the first derivatives is impossible to replace if the values of the functions are given on a set of arbitrary, one might say random, points.

DIFFERENCELESS DERIVATIVES

Widespread popularity, as the main starting point, Taylor series are used to calculate derivatives. For a two-dimensional function $f(x_1, x_2)$ in a neighborhood of the point (x_1, x_2), the Fourier series can be written as:

$$f(x_1+h_1, x_2+h_2)=f(x_1, x_2)+\frac{\partial f(x_1, x_2)}{\partial x_1}h_1+\frac{\partial f(x_1, x_2)}{\partial x_2}h_2+ \\ \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2}\frac{h_1^2}{2!}+\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}h_1 h_2+\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2}\frac{h_2^2}{2!}+\mathrm{O}(h_1^3, h_2^3) \tag{1}$$

It should be noted that the application of the method of differenceless derivatives to expression (1) leads to a system of fifth-order equations, for which it is very difficult, almost impossible to obtain analytical solutions, and numerical methods will have to be used. But it is our task to study the application of differenceless derivatives for functions of two variables in order to better understand the possibilities of the new method, so we will limit ourselves to calculating only the first derivatives, the solutions of which can be obtained in an analytical form.

Consider a set of m arbitrary points (x_1+h_{1i}, x_2+h_{2i}) in a neighborhood of the point (x_1, x_2) schematically shown in Figure 1.

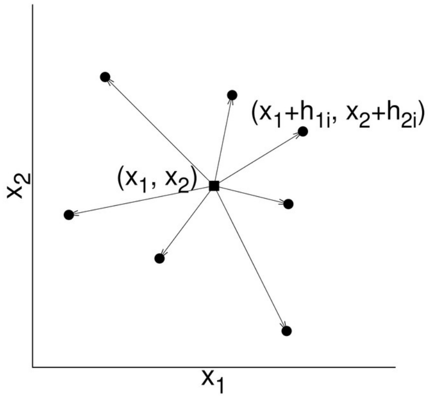

Figure 1

Note that our algorithm is not limited to seven points only, as shown in Figure 1, there is only one requirement $m \geqslant 2$. We will use a simplified formula (1), written in a format convenient for us:

$$h_{1i}\frac{\partial f(x_1,x_2)}{\partial x_1}+h_{2i}\frac{\partial f(x_1,x_2)}{\partial x_2}=$$
$$f(x_1+h_{1i},x_2+h_{2i})-f(x_1,x_2)$$

This expression must be correct for every value of $i=(1,m)$ with a precision of $\mathrm{O}(h_1^2,h_2^2)$. At first glance, it seems surprising that the accuracy of the calculation does not depend on the number of points used, usually with an increase in the number of points, the accuracy of the calculations improves as $\mathrm{O}(h_1^2,h_2^2)\rightarrow\mathrm{O}(h_1^3,h_2^3)\rightarrow\mathrm{O}(h_1^4,h_2^4)\ldots$. In our case, this is a consequence of the fact that we use only the first derivatives in expression (1), and this allows us to obtain an analytical solution, otherwise it is necessary to use numerical methods. It's really easy to verify that derivatives are computed in matrix form as:

$$\begin{pmatrix}\dfrac{\partial f(x_1,x_2)}{\partial x_1}\\ \dfrac{\partial f(x_1,x_2)}{\partial x_2}\end{pmatrix}=\begin{pmatrix}g_{10} & g_{11} & g_{12} & g_{13} & \cdots & g_{1m}\\ g_{20} & g_{21} & g_{22} & g_{23} & \cdots & g_{2m}\end{pmatrix}\begin{pmatrix}f(x_1,x_2)\\ f(x_1+h_{11},x_2+h_{21})\\ f(x_1+h_{12},x_2+h_{22})\\ f(x_1+h_{13},x_2+h_{23})\\ \cdots\\ f(x_1+h_{1m},x_2+h_{2m})\end{pmatrix} \qquad (2)$$

where:

$$\begin{aligned}g_{10}&=-\sum_{i=1}^{m}(a h_{1i}+b h_{2i}),\quad g_{1i}=a h_{1i}+b h_{2i},\quad i=1,m\\ g_{20}&=-\sum_{i=1}^{m}(b h_{1i}+c h_{2i}),\quad g_{2i}=b h_{1i}+c h_{2i},\quad i=1,m\end{aligned} \qquad (3)$$

$$a=\frac{s(h_{2i}^2)}{d},\quad b=\frac{-s(h_{1i}h_{2i})}{d},\quad c=\frac{s(h_{1i}^2)}{d},\quad d=s(h_{1i}^2)s(h_{2i}^2)-s(h_{1i}h_{2i})^2,\quad s(x_i)=\sum_{i=1}^{m}x_i \qquad (4)$$

Expression (2) for calculating derivatives confirms the name of the method - differenceless derivatives. Representing the solution in matrix form is very convenient for computer calculations, especially in the case of partial differential equations on an arbitrary mesh.

CONNECTION WITH FINITE DIFFERENCES

The fundamental question arises, what happens if expression (2) is applied to the set of points used in forward differences or central differences.

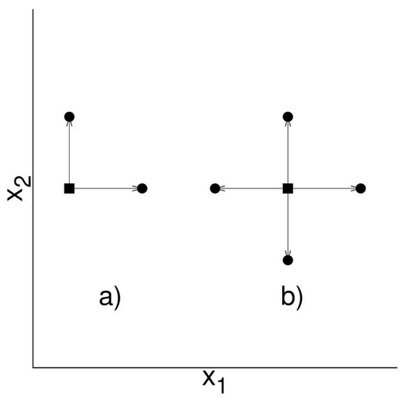

Figure 2

Consider the simplest case when two points with coordinates $(h, 0)$ and $(0, h)$ are given, shown schematically in Figure 2a. Substituting these coordinate values into expressions (4) and (3) we get: $d = h^4$, $a = \frac{1}{h^2}$, $b = 0$, $c = \frac{1}{h^2}$ and therefore $g_{10} = \frac{-1}{h}$, $g_{11} = \frac{1}{h}$, $g_{12} 0$, $g_{20} = \frac{-1}{h}$, $g_{21} = 0$ and $g_{22} = \frac{1}{h}$. Thus, expression (2) can be written in the following matrix form:

$$\begin{pmatrix} \frac{\partial f(x_1, x_2)}{\partial x_1} \\ \frac{\partial f(x_1, x_2)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{h} & \frac{1}{h} & 0 \\ \frac{-1}{h} & 0 & \frac{1}{h} \end{pmatrix} \begin{pmatrix} f(x_1, x_2) \\ f(x_1 + h, x_2 + 0) \\ f(x_1 + 0, x_2 + h) \end{pmatrix}$$

and if we transform these expressions, we get the well-known forward differences: $\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h}$ and $\frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{f(x_1, x_2 + h) - f(x_1, x_2)}{h}$. As follows from this example, differenceless derivatives include forward differences as a special case.

In the second example, we consider the four-point scheme shown in Figure 2b. The coordinates of the four points are $(0, h), (h, 0), (0, -h), (-h, 0)$, performing calculations similar to the above, it is easy to obtain matrix expressions for derivatives:

$$\begin{pmatrix} \frac{\partial f(x_1, x_2)}{\partial x_1} \\ \frac{\partial f(x_1, x_2)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2h} & 0 & \frac{-1}{2h} \\ 0 & \frac{1}{2h} & 0 & \frac{-1}{2h} & 0 \end{pmatrix} \begin{pmatrix} f(x_1, x_2) \\ f(x_1 + 0, x_2 + h) \\ f(x_1 + h, x_2 + 0) \\ f(x_1 + 0, x_2 - h) \\ f(x_1 - h, x_2 + 0) \end{pmatrix}$$

which are nothing more than central differences: $\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{f(x_1 + h, x_2) - f(x_1 - h, x_2)}{2h}$ and $\frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{f(x_1, x_2 + h) - f(x_1, x_2 - h)}{2h}$. As in the first example, we conclude: differenceless derivatives include central differences. This is a very good property, for example, when solving partial differential equations, the requirements for the mesh are reduced.

CONCLUSIONS

We have developed an algorithm - differenceless derivatives for two-dimensional functions. The algorithm allows calculating the first derivatives of the first order of accuracy of two-dimensional functions on a set of arbitrary, random points. The minimum number of required points is two, the maximum number is unlimited.

REFERENCES

1. Y. Mahotin, Estimation of derivatives by the method of differenceless derivatives, [3259] viXra:2304.0033, 2023