New proof of Pythagorean Theorem

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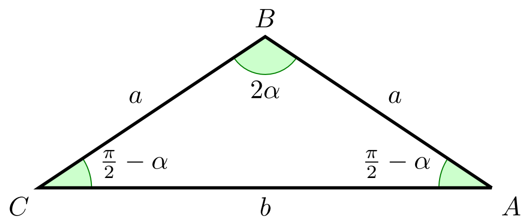
Abstract

We found a new proof of Pythagorean Theorem by using trigonometry. We induced double angle formula of sine and cosine functions in non-circular way.

1 Deriving double formula of sine and cosine



By law of sines [1],

$$\frac{b}{\sin 2\alpha} = \frac{a}{\sin\left(\dfrac{\pi}{2} - \alpha\right)} = \frac{a}{\cos\alpha}.$$

For $\overline{AC}$, it holds

$$b = 2 \cdot a\cos\left(\frac{\pi}{2} - \alpha\right) = 2a\sin\alpha.$$

Therefore, $\dfrac{2a\sin\alpha}{\sin 2\alpha} = \dfrac{a}{\cos\alpha}$ gives

$$\sin 2\alpha = 2\sin\alpha\cos\alpha.$$

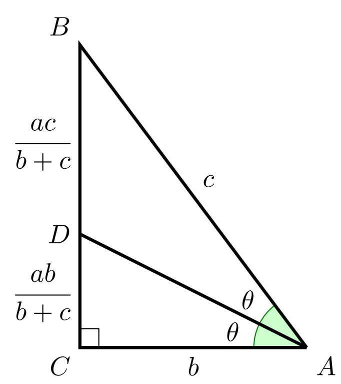
For $\overline{AB}$, it holds

$$a = b\cos\left(\frac{\pi}{2} - \alpha\right) + a\cos 2\alpha = b\sin\alpha + a\cos 2\alpha.$$

Putting $b = 2a\sin\alpha$ gives

$$a = 2a\sin^2\alpha + a\cos 2\alpha \Rightarrow \cos 2\alpha = 1 - 2\sin^2\alpha.$$

2 Proof of Pythagorean Theorem



Let $\overline{AD}$ be a bisector of $\angle BAC$. By Angle bisector theorem [2], it holds $\frac{\overline{CD}}{\overline{DB}} = \frac{\overline{AC}}{\overline{AB}}$. Therefore, $\overline{CD} = \frac{b}{c} \cdot \overline{DB}$. Since $\overline{CD} + \overline{DB} = a$, we have $\left(1 + \frac{b}{c}\right) \overline{DB} = a$, i.e. $\overline{DB} = \frac{ac}{b+c}$, $\overline{CD} = \frac{ab}{b+c}$.

Meanwhile, we have $\sin 2\theta = \frac{a}{c}$, $\cos 2\theta = \frac{b}{c}$, $\tan\theta = \frac{\frac{ab}{b+c}}{b} = \frac{a}{b+c}$.

From $\cos 2\theta = 1 - 2\sin^2\theta$, we have $\sin\theta = \sqrt{\frac{c-b}{2c}}$.

From $\sin 2\theta = 2\sin\theta\cos\theta$, we have $\tan\theta = \frac{\sin\theta}{cos\theta} = \frac{2\sin^2\theta}{\sin 2\theta} = \frac{\frac{c-b}{c}}{\frac{a}{c}} = \frac{c-b}{a}$.

Therefore, $\frac{a}{b+c} = \frac{c-b}{a}$, it completes $a^2 + b^2 = c^2$.

References

[1] Harold Scott Macdonald Coxeter and Samuel L Greitzer. *Geometry revisited*, volume 19. Maa, 1967.

[2] Alfred S Posamentier. *Advanced Euclidian Geometry: Excursions for Students and Teachers*. Springer Science & Business Media, 2002.