Irrationality of π
Using Just Derivatives

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Abstract

The quest for an irrationality of pi proof that can be incorporated into an analysis (or a calculus) course is still extant. Ideally a proof would be well motivated and use in an interesting way the topics of such a course. In particular $e^{\pi i}$ should be used and the more easily algebraic of derivatives and integrals – i.e. derivatives. A further worthy goal is to use techniques that anticipate those needed for other irrationality and, maybe even, transcendence proofs. We claim to have found a candidate proof.

Introduction

Invariably irrationality proofs use *proof by contradiction*. The number in question is assumed to be rational and a contradiction is derived. Why does this work? It works because irrational numbers are always changing; their tails change. Assuming that they don't change, that all zeros or 9s occur, eventually the approximation implicit in an irrational number represented by a rational becomes large enough that it is manifest that the fixed assumption can't work: there's a contradiction.

A combination of polynomials with fixed roots and ever changing partial sums of series seem a likely avenue to an irrationality proof. This is especially true as series in the form of a power series or e^x or e^{ix} have partials that double as polynomials. Assuming the polynomial has a certain root and that the series for which the polynomial is a partial is also converging to this number should work to generate the schism mentioned. A natural candidate that embodies these ideas is Euler's famous formula:

$$e^{\pi i} - 1 = 0.$$

Derivatives of Polynomials

All polynomials are integer polynomials, z is a complex number, n and j are non-negative integers, and p is a prime number.

Definition 1. *Given a polynomial f(z), lowercase, the sum of all its derivatives is designated with F(z), uppercase.*

Example 1. *If $f(z) = cz^n$ then*

$$F(z) = \sum_{k=0}^{n} f^{(k)}(z) = cz^n + cnz^{n-1} + cn(n-1)z^{n-2} + \cdots + cn!.$$

Lemma 1. *If $f(z) = cz^n$, then*

$$F(0)e^z = F(z) + f(z)\sum_{k=1}^{\infty} \frac{z^k n!}{(n+k)!} \tag{1}$$

Proof. As $F(z) = c(z^n + nz^{n-1} + \cdots + n!)$, $F(0) = cn!$. Thus,

$$\begin{aligned} F(0)e^z &= cn!(1 + z/1 + z^2/2! + \cdots + z^n/n! + \dots) \\ &= cz^n + cnz^{(n-1)} + \cdots + cn! + cz^{n+1}/(n+1)! + \dots \\ &= F(z) + cz^n(z/(n+1) + z^2/(n+1)(n+2) + \dots) \\ &= F(z) + f(z)\sum_{k=1}^{\infty} \frac{z^k n!}{(n+k)!}, \end{aligned}$$

giving (1). ☐

Definition 2. *Let*

$$\delta_{n!} = \sum_{k=1}^{\infty} \frac{z^k n!}{(n+k)!}.$$

Lemma 2.

$$\lim_{p\to\infty} \frac{\delta_{n!}}{(p-1)!} = 0. \tag{2}$$

Proof. We have

$$\left|\frac{\delta_{n!}(z)}{(p-1)!}\right| = \left|\frac{z/(n+1) + z^2/(n+1)(n+2) + \dots e^z}{(p-1)!}\right| < \left|\frac{e^z}{(p-1)!}\right|$$

and

$$\lim_{n\to\infty}\left|\frac{e^z}{(p-1)!}\right| = 0.$$

This implies (2). □

Lemma 3. *If $F(z)$ is the sum of the derivatives of $f(z) = c_0 + c_1 z + \cdots + c_n z^n$, then*

$$F(0)e^z = F(z) + \sum_{k=0}^{n} c_k z^k \delta_{k!}(z). \tag{3}$$

Proof. Let $f_j(z) = c_j z^j$, for $0 \leq j \leq n$. Using the derivative of the sum is the sum of the derivatives,

$$F = \sum_{k=0}^{n}(f_0 + f_1 + \cdots + f_n)^{(k)} = F_0 + F_1 + \cdots + F_n,$$

where F_j is the sum of the derivatives of f_j. Using Lemma 1,

$$e^z F_j(0) = F_j(z) + f_j(z)\delta_{j!}(z) \tag{4}$$

and summing (4) from $j = 0$ to n, gives

$$e^z F(0) = F(z) + \sum_{j=0}^{n} f_j(z)\delta_{j!}(z).$$

This is (3). □

Definition 3. *If $f_j(z) = c_j z^j$, for $0 \leq j \leq n$, then define*

$$\epsilon_{n!}(f(z)) = \sum_{j=0}^{n} f_j(z)\delta_{j!}(z),$$

where

$$f(z) = \sum_{j=0}^{n} f_j(z).$$

Lemma 4.

$$\lim_{p\to\infty}\frac{\epsilon_{n!}(z)}{(p-1)!} = 0. \tag{5}$$

Proof. As $\delta_{j!}(z) < e^z$ for $j = 0, \ldots, n$,

$$\left|\frac{\epsilon_{n!}(z)}{(p-1)!}\right| = \left|\frac{\sum_{j=0}^{n} f_j(z)\delta_{j!}(z)}{(p-1)!}\right| \le e^{|z|}\sum_{j=0}^{n}\frac{|f_j(z)|}{(p-1)!}.$$

Then, noting

$$\sum_{j=0}^{n}|f_j(z)| \le c\sum_{j=0}^{n}|z^j| \le cn|z|^r, \tag{6}$$

where $c = \max\{|c_0|, |c_1|, \ldots, |c_n|\}$ and $|z|^r = \max\{|z|, |z|^2, \ldots, |z|^n\}$ and

$$\lim_{p\to\infty}\frac{cn|z|^r}{(p-1)!} = 0,$$

we arrive at (5). Note: r will not vary with n. □

Structuring Roots

There is a relationship between the roots of $f(z)$ and those of $F(z)$. This will enable us to structure the roots of polynomials and apply (3) using z values that are roots of $f(z)$. A pattern will emerge of the following form

$$0 = I + \epsilon$$

where I is a non-zero integer and ϵ is as small as we please: a contradiction.

Lemma 5. *If polynomial $f(z)$ has a root r of multiplicity p, then $f^{(k)}(r) = 0$ for $0 \le k \le p-1$ and each term of $f^{(k)}(r)$, $p \le k \le n$ is a multiple of $p!$.*

Proof. Suppose $r = 0$ then, for some n we have $f(z) = z^p(b_n z^n + \cdots + b_0)$. Now $f(z)$ has $b_0 z^p$ as its term with minimal exponent. Using the derivative operator, $D(z^n) = nz^{n-1}$, repeatedly, we see the 0 through $p-1$ derivatives of $f(z)$ will have a positive exponent of z in each term. This implies that $r = 0$ is a root for these derivatives. Using the product of p consecutive natural numbers is divisible by $p!$, terms of subsequent derivatives will be multiples of $p!$.

If $r \neq 0$, then $f(z) = (z-r)^p Q(z)$, for some polynomial $Q(z)$. Let $g(z) = f(z+r) = z^p Q(z+r)$. As $g^{(k)} = f^{(k)}$ for all k, $g^{(k)}(0) = f^{(k)}(r)$, and the $r = 0$ case applies. □

Lemma 6. *If a and b are two non-zero Gaussian integers, then there exist a large enough prime p such that*

$$\frac{|p!a + (p-1)!b|}{(p-1)!} > 1.$$

Proof. Suppose $a = a_1 + ia_2$ and $b = b_1 + ib_2$.

$$|p!a + (p-1)!b| = |p!(a_1 + ia_2) + (p-1)!(b_1 + ib_2)|$$

$$= (p-1)!|pa_1 + ipa_2 + b_1 + ib_2|$$
$$= (p-1)!|(pa_1 + b_1) + i(pa_2 + b_2)|$$
$$= (p-1)!\sqrt{(pa_1 + b_1)^2 + (pa_2 + b_2)^2}$$

The square root contains the sum of two positive or zero integers. Then as both a and b are non-zero Gaussian integers, letting 0 indicate a zero value for a real or complex component and a 1 indicate a non-zero component the possibilities are

$a_1 b_1|$ 00 10 01 11 forcing $a_2 b_2|$ 11 01 10 00.

The only possibility resulting in a zero sum $|pa + b|$ occurs with $b = -pa$ with $b \neq 0$. This is a 11 case. Assuming a_1 and b_1 are non-zero and $p > \max\{|b_1|\}$, then $p \nmid |b_1|$ and $(pa_1 + b_1)^2$ must be non-zero as, if it is zero then then $pa_1 + b_1 = 0$ and $pa_1 = -b_1$ and $p||b_1|$, a contradiction. So one or both summands are non-zero positive integers. As the square root of a number greater than 1 is greater than 1, the Lemma is established. □

Pi is Irrational

Theorem 1. *π is irrational.*

Proof. Suppose not. Then $e^{\pi i} = e^{ri}$ where r is a rational, say a/b. Modify the polynomial

$$z^{p-1}(z - ai/b)^p$$

to make it an integer polynomial:

$$f(z) = (bz)^{p-1}(bz - ai)^p.$$

Then, using Euler's formula and Lemma 1

$$0 = F(0)(e^{ri} + 1) = F(ri) + F(0) + \epsilon_{n!}(f(z)).$$

There is a prime p large enough that the left hand side of

$$\left|\frac{\epsilon_{n!}(f(z))}{(p-1)!}\right| = \left|\frac{F(ri) + F(0)}{(p-1)!}\right|$$

is less than one per Lemma 4 and the right hand side is greater than one per Lemma 6, a contradiction. □

Conclusion

This proof of the irrationality of π uses derivatives and limits at a level of a real analysis course based on Rudin or Apostol [1, 3]. It also anticipates proofs of the transcendence of e and π [2].

References

[1] Apostol, T. M. (1974). *Mathematical Analysis*, 2nd ed. Reading, Massachusetts: Addison-Wesley.

[2] Eymard, P., Lafon, J.-P. (2004). *The Number* π. Providence, RI: American Mathematical Society.

[3] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill.