On the quantization of electric and magnetic charges and fluxes.
Axiomatic foundation of non-integrable phases.
Two natural speeds different from c

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Abstract. We present an axiomatic foundation of non-integrable phases of Schrödinger wave functions and use it for interpreting Dirac's 1931 pioneering article in terms of the electromagnetic 4-potential. The quantization of the electric charge in terms of e implies the quantization of the dielectric flux through closed surfaces $\bar{\Psi} := \oiint \vec{D} \cdot \mathrm{d}\vec{S}$ in terms of the 'Lagrangean' dielectric flux quantum $\Psi_D = e$. The quantization of the analogous magnetic monopole charge in terms of g implies the quantization of the magnetic flux through closed surfaces $\bar{\Phi} := \oiint \vec{B} \cdot \mathrm{d}\vec{S}$ in terms of the 'Diracian' magnetic induction flux quantum $\Phi_B = g = h/e$, and *vice versa*. Here, the question is raised, if the quantization of the magnetic charge (and hence field) in a given volume depends on the total electric charge in that volume. Furthermore, we have $\Phi_B/\Psi_D = g/e = h/e^2 = R_{\mathrm{K}}$, the von Klitzing constant, the basic resistance of the quantum Hall effect. R_{K} and the vacuum permittivity ε_0 and permeability μ_0, respectively, combine to two natural speed constants different from that of light in vacuum c.[1]

[1]This article is an extension of a talk presented (in German) before the Physical Society at Berlin, Febr. 15, 2023, `https://www.dpg-physik.de/veranstaltungen/2023/mhb-agsen-2023-02-15`.

1 Introduction

Why there are free electric but – according to all experiments done so far – no free magnetic charges? As a matter of fact, this question is not new.

A seminal step is due to Dirac's famous explorations of the non-integrable phases of wave functions[2] It has been largely extended in a second article[3].

The existence of an elementary magnetic charge implies the existence of an elementary electric charge, while the opposite is not true. As a matter of fact, this is still the only explanation for the existence of the elementary electric charge.

In particle physics, a magnetic monopole is a hypothetical elementary particle which consists of an isolated magnet with only one magnetic pole (a north pole without a south pole or *vice versa*).[4][5] A magnetic monopole would have a net 'magnetic charge'. Modern interest in the concept stems notably from the grand unified and superstring theories which predict their existence.[6][7]

Magnetism in bar magnets and electromagnets is not caused by magnetic monopoles. Some condensed matter systems contain quasi-particles which behave like effective (non-isolated) magnetic monopoles[8] or exhibit phenomena that are mathematically analogous to magnetic monopoles[9].

This article concentrates on the following four points.

1. The quantization of the electric charge implies the quantization of the dielectric flux through closed surfaces (Section 2).

2. An axiomatic foundation of the occurrence of non-integrable phases of Schrödinger wave functions is presented (Section 3).

3. That foundation is exploited for interpreting Dirac's quantization of the magnetic charge (magnetic monopoles) and the magnetic induction flux through closed surfaces in terms of the electromagnetic 4-potential from the very beginning (Section 4).

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[2] P. A. M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. Roy. Soc. A CXXXIII (1931) 60–72, `https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1931.0130` *aka* `https://doi.org/10.1098/rspa.1931.0130`

[3] P. A. M. Dirac, *The Theory of Magnetic Poles*, Phys. Rev. 74 (1948) 7, 817–830, `http://www.fisicafundamental.net/relicario/doc/Dirac-Poles.pdf` (12.09.2022)

[4] D. Hooper, *Dark Cosmos: In Search of Our Universe's Missing Mass and Energy*, Harper Collins 2009

[5] S. Eidelman *et al.* (Particle Data Group), Phys. Lett. B 592, 1 (2004), `http://pdg.lbl.gov` (07.08.2021)

[6] X.-G. Wen & E. Witten, *Electric and magnetic charges in superstring models*, Nuclear Physics B 261 (1985) 651–677

[7] S. R. Coleman, *The magnetic monopole 50 years later*, 1981 Int. School Subnucl. Phys., "Ettore Majorana", HUTP-82/A032, `https://lib-extopc.kek.jp/preprints/PDF/1982/8211/8211084.pdf` (28.08.2021)

[8] C. Castelnovo, R. Moessner & S. L. Sondhi, *Magnetic monopoles in spin ice*, Nature 451 (2008) 42–45, `https://arxiv.org/abs/0710.5515` (17.12.2022)

[9] M. W. Ray, E. Ruokokoski, S. Kandel, M. Möttönen & D. S. Hall, *Observation of Dirac monopoles in a synthetic magnetic field*, Nature 505 (7485): 657–660, `https://arxiv.org/abs/1408.3133`

4. The ratio of the elementary magnetic and (di)electric fluxes and charges equals the von Klitzing constant. The latter one combines with the vacuum permittivity and permeability to two natural speeds different from that of light *in vacuo* (Section 5).

Thus, the purpose of this article is also to stimulate research along the following two questions.

1. Can the search for magnetic charges (monopoles) be brought forward through considering the magnetic flux through *closed* surfaces?

2. Can the quantization of the electric flux be exploited like that of the magnetic flux in SQUIDs?

2 Elementary electric charge and dielectric flux quanta

For later use, we will sketch the relation between the discretization of field sources and fluxes through closed surfaces, using the electric field. In particular, we will use Gauss' law in its differential and integral forms as an example to stress this:

Theorem 1 (Gaussian theorem) *The amount of sources of a field (charge, mass) within a volume Ω equals the total flux of that field through any surface $\partial\Omega$ enclosing that volume.*

Specifically, Gauss' law[10] states that the scalar source of the dielectric displacement $\vec{D}$ is the density ρ_{e} of the free electric charges q_{e}.[11]

$$\nabla \cdot \vec{D} = \rho_{\mathrm{e}} \tag{1}$$

Its integral form reads

$$\iiint\limits_{\Omega} \rho_{\mathrm{e}}\, \mathrm{d}^3 r = q_{\mathrm{e}} = \iiint\limits_{\Omega} \nabla \cdot \vec{D}\, \mathrm{d}^3 r = \oint\limits_{\partial\Omega} \vec{D} \cdot \mathrm{d}^2 \vec{r} =: \bar{\Psi}, \tag{2}$$

[10] C. F. Gauss, *Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata*, Göttingen: Diederich 1813; in: *Werke*, Berlin · Heidelberg: Springer 1877, vol. V, p. 1. Gauss mentions Proposition XCI in Newton's *Principia* regarding the force exerted by a sphere on a point anywhere along an axis passing through the sphere. According to P. Duhem, *Leçons sur l'électricité et le magnétisme*, Vol. 1, Ch. 4, pp. 22–23, Gauss is preceded by J.-L. Lagrange, *Sur l'attraction des sphéroïdes elliptiques*, Nouv. Mém. l'Acad. Berlin, 1773, in: *Œvres de Lagrange*, T. 3, pp. 619–658, `https://gallica.bnf.fr/ark:/12148/bpt6k229222d/f620`, `https://gdz.sub.uni-goettingen.de/id/PPN308900308` (05.01.2023).

[11] J. C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Trans. Roy. Soc. CLV (1965), Pt. III (article accompanying the Dec. 8, 1864, presentation to the Royal Society), `https://en.wikisource.org/wiki/A_Dynamical_Theory_of_the_Electromagnetic_Field` (12.03.2020), Art. 68, eqs. (G); reprint (with a Preface and Introduction by Th. F. Torrance, 1982, and an Appreciation by A. Einstein, 1931) by Wipf and Stock, Eugene (OR) 1996

where $\bar{\Psi}$ denotes the dielectric flux through the closed surface $\overline{\partial\Omega}$. It reveals that the dielectric displacement $\vec{D}$ is also a flux density. If the surface lies outside the charge distribution, $\bar{\Psi}$ is independent of the surface, provided that Gauss-Ostrogradsky's divergence theorem used in eq. (2) applies.

Indeed, like the magnetic induction $\vec{B}$, the dielectric displacement $\vec{D}$ is an "area vector."[12] In contrast, the electric and magnetic field strengths $\vec{E}$ and $\vec{H}$ are "line vectors" (see fn. 20 for more details). Notice also that $\vec{B}$ and $\vec{D}$ are *extensive* quantities, while $\vec{E}$ and $\vec{H}$ are *intensive* ones.

According to Theorem 1, the volume Ω contains an integer number of free elementary particles of charge 0 or $\pm e$, $-e$ being the electron charge. Hence, q_e is discretized as

$$q_\mathrm{e} = N_{\mathrm{e}+}e + N_{\mathrm{e}-}(-e) = (N_{\mathrm{e}+} - N_{\mathrm{e}-})\, e, \tag{3}$$

where $N_{\mathrm{e}\pm}$ is the number of positively/negatively charged elementary particles. As a consequence, the dielectric flux through closed surfaces is *discretized*, too.

$$\bar{\Psi} = (N_{\mathrm{e}+} - N_{\mathrm{e}-})\, e = (N_{\mathrm{e}+} - N_{\mathrm{e}-})\, \Psi_D, \tag{4}$$

where Ψ_D is the 'Lagrangean'[10] elementary dielectric flux quantum,

$$\Psi_D = e. \tag{5}$$

3 Axiomatic foundation of non-integrable phases of Schrödinger wave functions

In this section, we will provide an *axiomatic* foundation of the occurrence of non-integrable phases for Schrödinger wave functions in terms of the electromagnetic 4-potential. It seems to us that this makes some subtleties of Dirac's approach[2] easier to grasp. Moreover, we are using SI units. This facilitates to understand that the factor of 2 in his eq. (9) is *not* related to the magnetic flux quantization in superconductors but to his use of Gaussian units.

3.1 Relationships between interactions and conserved quantities after and beyond Helmholtz

Thus, we will generalize Helmholtz's exploration of the relationship between mechanical forces and conservation of energy as follows.[13]

[12] G. Mie, *Lehrbuch der Elektrizität und des Magnetismus*, Stuttgart: Enke ^{2}1941, Art. 80, p. 104

[13] P. Enders, *Von der Klassischen zur Quantenphysik und zurück. Ein deduktiver Zugang* [From classical to quantum physics, and back. A deductive approach], Nova Acta Leopoldina 19 (2004) 54; for a detailed description, see *Classical Mechanics and Quantum Mechanics*, Sharjah, UAE: Bentham 2019, Ch. 11.

- For a point-like body, its momentum $\vec{p}(t)$ is a stationary-state function in the sense that it is time-*in*dependent in stationary (force-free) states, in which $\vec{p}(t) = \vec{p}_0 =$ const. Are there interactions (external forces) which leave the momentum unchanged? The answer is 'no'.

- Next, consider a mechanical system in a stationary state with total energy E. Are there interactions (external forces) which leave the amount of E unchanged? The answer is 'yes', given by forces of the form

$$-\nabla V(\vec{r}) + \vec{v} \times \vec{K}(t, \vec{r}, \vec{v}, \vec{a}, \ldots)\,. \tag{6}$$

  Here, $\vec{K}(t, \vec{r}, \vec{v}, \vec{a}, \ldots)$ is a rather arbitrary vector function of time t, position $\vec{r}$, velocity $\vec{v}$, acceleration $\vec{a}$, and higher-order time-derivatives of $\vec{r}$.[14] Its relevance reveals from this: It is compatible with canonical mechanics, iff $\vec{K}(t, \vec{r}, \vec{v}, \vec{a}, \ldots) = \nabla \times \vec{K}'(t, \vec{r})$. This leads to the magnetic Lorentz force. [15] In this case, the Hamiltonian $H(\vec{p}, \vec{r}, t) = H_0(\vec{p}, \vec{r}) = E =$ const. is a stationary-state function.

- For a quantum-mechanical system with wave function $\psi(\vec{r}, t)$ and Hamiltonian $H(\hat{\vec{p}}, \vec{r}, t)$, the expressions

$$|\psi(\vec{r}, t)|^2 \quad \text{and} \quad \langle \psi(\vec{r}, t)|H(\hat{\vec{p}}, \vec{r}, t)|\psi(\vec{r}, t)\rangle \tag{7}$$

  are stationary-state functions in the sense that they are time-independent in stationary states with energy E, where

$$|\psi(\vec{r}, t)|^2 = |\psi_E(\vec{r})|^2 \tag{8a}$$

$$\langle \psi(\vec{r}, t)|H(\hat{\vec{p}}, \vec{r}, t)|\psi(\vec{r}, t)\rangle = \langle \psi_E(\vec{r})|H_0(\hat{\vec{p}}, \vec{r})|\psi_E(\vec{r})\rangle = E = \text{const.} \tag{8b}$$

  Are there interactions which leave the stationary "weight function"[16] $|\psi_E(\vec{r})|^2$ and the energy E unchanged? The answer is 'yes' as will be shown in the next subsection.

3.2 Interactions leaving the time-*in*dependent weight function and the energy unchanged. (Ehrenberg-Siday-)Aharonov-Bohm effect

Obviously, the value of the stationary weight function $|\psi_E(\vec{r})|^2$ is not changed when $\psi_E(\vec{r}) =: \psi_{E,0}(\vec{r})$ is replaced by $\psi_{E,\beta}(\vec{r}) = \psi_{E,0}(\vec{r}) e^{i\beta_E(\vec{r})}$. Then, the value

[14] The second term is due to R. Lipschitz, *priv. commun. to Helmholtz*, in: H. v. Helmholtz, *Über die Erhaltung der Kraft*, Addendum 3 to the 1881 ed. in Ostwald's Klassiker 1; also in: *Wissenschaftliche Abhandlungen I*, p. 70. See also M. Planck, *Das Prinzip von der Erhaltung der Energie*, Leipzig · Berlin: Teubner ^{2}1908 (Wissenschaft und Hypothese VI), p. 182. Surprisingly enough, it is missing in all textbooks we are aware of.

[15] P. Enders, *Towards the Unity of Classical Physics*, Apeiron 16 (2009) 22–44, http://redshift.vif.com/JournalFiles/V16NO1PDF/V16N1END.pdf (13.09.2022)

[16] E. Schrödinger, *Quantisierung als Eigenwertproblem. Vierte Mitteilung*, Ann. Physik 81 (1926) 109–139, § 7, p. 135, https://gallica.bnf.fr/ark:/12148/bpt6k15383q/f117 (15.09.2022)

of $\langle\psi_E(\vec{r})|H_0(\hat{\vec{p}},\vec{r})|\psi_E(\vec{r})\rangle = E$ is also not changed when at once $H_0(\hat{\vec{p}},\vec{r})$ is replaced by $H_\beta = H_0(\hat{\vec{p}} - \hbar\,\nabla\beta_E(\vec{r}),\,\vec{r})$.

Now, the Hamiltonian $H_0(\hat{\vec{p}} - \hbar\nabla\beta_E(\vec{r}),\,\vec{r})$ is that of an electrical charge q_e in the external vector potential $\vec{A}(\vec{r}) = \hbar\,\nabla\beta(\vec{r})/q_\mathrm{e}$, where β is independent of E. This phase factor causes the (Ehrenberg-Siday-)Aharonov-Bohm effect[17]. There is no electromagnetic field connected with such a vector potential as $\vec{B} = \nabla\times\vec{A} \equiv \vec{0}$ and $\vec{E} = -\partial\vec{A}/\partial t \equiv \vec{0}$. For this, in the next subsection, we will generalize this approach such that arbitrary electromagnetic fields are dealt with. This will lead us to the non-integrable phases in Subsection 3.4.

3.3 Interactions leaving time-dependent weight function and Schrödinger equation unchanged (arbitrary electromagnetic fields). Gauge invariance

The results of the foregoing subsection can be generalized by means of the phase

$$\beta(\vec{r},\,t) = \frac{q_\mathrm{e}}{\hbar}\int_{\vec{r}_0}^{\vec{r}}\vec{A}(\vec{r}\,',\cdot)\cdot\mathrm{d}\vec{r}\,' - \frac{q_\mathrm{e}}{\hbar}\int_{t_0}^{t}\phi(\cdot,t')\,\mathrm{d}t' \tag{9}$$

(ϕ being the scalar potential). Here, $\vec{A}(\vec{r}\,',\cdot)$ will not be differentiated w.r.t. t and $\phi(\cdot,t')$ not w.r.t. $\vec{r}$. That means that

$$\frac{\partial\beta}{\partial t}(\vec{r},\,t) = -\frac{q_\mathrm{e}}{\hbar}\phi(\vec{r},t)\,,\quad \nabla\beta(\vec{r},\,t) = \frac{q_\mathrm{e}}{\hbar}\vec{A}(\vec{r},t)\,. \tag{10}$$

These formulas (10) correspond to Dirac's more general definitions (fn. 2, formulas after (3))

$$\kappa_x := \frac{\partial\beta}{\partial x},\quad \kappa_y := \frac{\partial\beta}{\partial y},\quad \kappa_z := \frac{\partial\beta}{\partial z},\quad \kappa_0 := \frac{\partial\beta}{\partial t}\,. \tag{11}$$

However, as Dirac does not point to another application than electromagnetism, we will continue using the electromagnetic potentials and fields. This is easier to grasp and agrees with our concentration onto magnetic charges *aka* monopoles.

Analogously to the foregoing subsection, the time-dependent weight function $|\psi(\vec{r},\,t)|^2$ is unchanged when $\psi(\vec{r},\,t)$ is multiplied by $e^{i\beta(\vec{r},\,t)}$, while now not the time-independent but the time-dependent Schrödinger equation remains unchanged as

$$\left[H(\hat{\vec{p}},\,\vec{r},\,t) - i\hbar\frac{\partial}{\partial t}\right]\psi(\vec{r},\,t) = \left[H(\hat{\vec{p}} - q_\mathrm{e}\vec{A},\,\vec{r},\,t) + q_\mathrm{e}\phi(\vec{r},t) - i\hbar\frac{\partial}{\partial t}\right]\psi(\vec{r},\,t)e^{i\beta(\vec{r},\,t)}\,. \tag{12}$$

[17] W. Ehrenberg & R. E. Siday, *The Refractive Index in Electron Optics and the Principles of Dynamics*, Proc. Phys. Soc. B 62 (1949) 8–21; Y. Aharonov & D. Bohm, *Significance of Electromagnetic Potentials in the Quantum Theory*, Phys. Rev. 115 (1959) 485–491; *Further considerations of electromagnetic potential in the quantum theory*, Phys. Rev. 123 (1961) 1511–1524

This fact is closely related to gauge invariance. If $\psi_\beta = e^{i\beta}\psi_0$ obeys any wave equation involving the energy-momentum operator $\hat{p}^\mu$ (14b), ψ_0 obeys the corresponding wave equation in which $\hat{p}^\mu$ is replaced with $\hat{p}^\mu - q_\mathrm{e} A^\mu$ (A^μ being the contravariant 4-potential (14a); cf. fn. 2, p. 65). If A^μ is regauged to $A'^\mu = A^\mu + \partial^\mu\chi$, β has to be changed to $\beta + \chi$.

3.4 Non-integrable phase

The phase β (9) is non-integrable, if

$$\frac{\partial^2\beta}{\partial x\partial y} - \frac{\partial^2\beta}{\partial y\partial x} \propto \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z \neq 0 \quad \text{etc.} \tag{13}$$

Dirac (fn. 2, p. 66) notes,

> "The connection between non-integrability of phase and the electromagnetic field given in this section is not new, being essentially just Weyl's Principle of Gauge Invariance in its modern form."[18])

That gauge invariance is displayed in eq. (12).

In what follows, it will turn out that the usage of 4-vectors and 4-tensors considerably shortens the notation, where we will use the signature $+---$.

$$A^\mu = \left(\tfrac{1}{c}\Phi,\, \vec{A}\right) = -\frac{\hbar}{q_e}\,\partial^\mu\beta = -\frac{\hbar}{q_e}\,\kappa^\mu\,, \tag{14a}$$

$$\hat{p}^\mu = \left(i\hbar\tfrac{\partial}{c\partial t}, -i\hbar\nabla\right) =: \left(\tfrac{1}{c}\hat{W}, \hat{\vec{p}}\right); \quad \mu = 0,1,2,3 \tag{14b}$$

Non-integrability implies that the change in phase round a closed curve s_μ,

$$\oint_{\partial S}\frac{\partial\beta}{\partial x_\mu}\,\mathrm{d}s_\mu = -\frac{q_e}{\hbar}\oint_{\partial S} A^\mu\,\mathrm{d}s_\mu = -\frac{q_e}{\hbar}\iint_S \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\mathrm{d}S_{\mu\nu} = -\frac{q_e}{\hbar}\iint_S F^{\mu\nu}\mathrm{d}S_{\mu\nu}\,, \tag{15}$$

may *not* vanish (cf. fn. 2, formula (4); $F^{\mu\nu}$ being the Faraday tensor). This corroborates an important conclusion by Dirac obtained from a much more complicated reasoning.

[18]Dirac refers to H. Weyl, *Elektron und Gravitation*, Zs. Physik 56 (1929) 330–352; reprint in: A. S. Blum & D. Rickles (eds.), *Quantum Gravity in the First Half of the Twentieth Century : A Sourcebook*, 2018 (Ed. Open Sources, Sources 10, MPI for the History of Science, Berlin), Ch. 12, `https://edition-open-sources.org/media/sources/10/14/sources10chap12.pdf` (08.09.2021). On p. 331, Weyl writes, "Es scheint mir darum dieses nicht aus der Spekulation, sondern aus der Erfahrung stammende neue Prinzip der Eichinvarianz zwingend darauf hinzuweisen, daß das elektrische Feld ein notwendiges Begleitphänomen nicht des Gravitationsfeldes, sondern des materiellen, durch $\bar{\Psi}$ dargestellten Wellenfeldes ist." En.: Therefore, this new principle of gauge invariance, which does not come from speculation, but from experience, seems to me to indicate compellingly that the electric field is a necessary accompanying phenomenon not of the gravitational field, but of the material wave field represented by $\bar{\Psi}$.

"The above result that the change in phase round a closed curve must be the same for all wave functions means that this change in phase must be something determined by the dynamical system itself (and perhaps also partly by the representation) and must be independent of which state of the system is considered. As our dynamical system is merely a simple particle, it appears that the non-integrability of phase must be connected with the field of force in which the particle moves." (fn. 2, p. 64)

Luckily, it is sufficient that the path lies completely in the $3d$ position space, $s_\mu = (0, -\vec{s})$ (fn. 2, p. 67). In this case, formula (15) simplifies to

$$\frac{q_e}{\hbar} \oint_{\partial S} \vec{A} \cdot \mathrm{d}\vec{s} = \frac{q_e}{\hbar} \iint_S (\nabla \times \vec{A}) \cdot \mathrm{d}\vec{S} = \frac{q_e}{\hbar} \iint_S \vec{B} \cdot \mathrm{d}\vec{S} = \frac{q_e}{\hbar} \Phi \,, \tag{16}$$

Φ being the magnetic flux through the surface S. As long as there are no singularities, this holds true for any surface S with boundary ∂S. Then, if the surface S is single-connected (what is not the case in the (Ehrenberg-Siday-)Aharonov-Bohm effect mentioned in Subsection 3.2), the magnetic induction flux through a closed surface $\bar{S}$ vanishes identically.

$$\bar{\Phi} := \oiint_{\bar{S}} \vec{B} \cdot \mathrm{d}\vec{S} \equiv 0 \tag{17}$$

4 Quantization of magnetic charge and induction flux through closed surfaces

4.1 Singularities of vanishing wave functions

Now, singularities occur when the wave function vanishes, since then its phase looses its meaning (although it enters the conditions of vanishing, of course). As this vanishing includes two conditions (the wave function is complex-valued), they occur along "nodal lines" (fn. 2, p. 67), nowadays called 'Dirac strings'. Again, it is sufficient to consider Dirac strings in 3d position space. If a Dirac string crosses the surface under consideration, the surface is no longer simply connected. As a consequence, the phase change along its boundary (16) is not necessarily small when the length of the boundary vanishes but close to $2\pi n$, where the integer n is a characteristic of that Dirac string. The sign of n indicates the direction in which it crosses the surface. If more than one Dirac string crosses the surface, their characteristic values n sum up as (cf. fn. 2, sum (8))

$$\frac{q_\mathrm{e}}{\hbar} \oint_{\partial S} \vec{A} \cdot \mathrm{d}\vec{r} = \frac{q_\mathrm{e}}{\hbar} \iint_S \vec{B} \cdot \mathrm{d}\vec{S} + 2\pi \sum n \,. \tag{18}$$

4.2 Magnetic flux quantum. Elementary magnetic and electric charges

For a closed surface, formula (18) yields not the identity (17) but

$$\oiint_{\bar{S}} \vec{B} \cdot \mathrm{d}\vec{S} + \frac{2\pi\hbar}{q_\mathrm{e}} \sum n = 0\,. \tag{19}$$

Here, the sum over n includes only those Dirac strings which end within the enclosed volume.

> "If $\sum n$ does not vanish, some nodal lines [Dirac strings] must have end points inside the closed surface, since a nodal line without such end point must cross the surface twice (at least) and will contribute equal and opposite amounts to $\sum n$ at the two points of crossing. The value of $\sum n$ for the closed surface will thus equal the sum of the values of n for all nodal lines having end points inside the surface. This sum must be the same for all wave functions.[19] Since this result applies to *any* closed surface, it follows *that the end points of nodal lines must be the same for all wave functions. These end points are then points of singularity in the electromagnetic field.*" (fn. 2, p. 68)

The flux through a small surface surrounding just *one* of that end points is

$$\bar{\Phi} := \oiint_{\bar{S}} \vec{B} \cdot \mathrm{d}\vec{S} = -\frac{2\pi\hbar}{q_\mathrm{e}}\, n\,. \tag{20}$$

In what follows, Dirac – without notice – confines himself to the case of a single electron, $q_e = -e$. As a consequence, there is an 'Diracian' elementary magnetic (induction) flux quantum

$$\Phi_B = \frac{h}{e} = 2\Phi_0\,. \tag{21}$$

Here, Φ_0 is the magnetic flux quantum in superconductors with Cooper pairs of electrons with total charge $-2e$. It corresponds to an elementary magnetic charge

$$g = \Phi_B = \frac{h}{e} \tag{22}$$

(see Section 2 for the relation between field sources and fluxes). In turn, there is an elementary electric charge e as

$$e = \frac{h}{g}\,. \tag{23}$$

[19] This is obvious for all cases in which the phase is determined by external quantities as in formula (9).

4.3 Question

More generally, according to eq. (20), the flux through a small surface surrounding just *one* of that end points is

$$\bar{\Phi} := \oiint_{\bar{S}} \vec{B} \cdot \mathrm{d}\vec{S} = -\frac{2\pi\hbar}{q_\mathrm{e}} n = -\frac{h}{\left(N_\mathrm{e}^+ - N_\mathrm{e}^-\right) e} n = -\frac{h}{e} \frac{n}{N_\mathrm{e}^+ - N_\mathrm{e}^-} \,. \tag{24}$$

Does the quantization of the magnetic charge (and hence field) in a given volume depend on the total electric charge in that volume?

4.4 The force between elementary magnetic and electric charges

The force between two electrons and two elementary monopoles, respectively, equal

$$F_e = \frac{e^2}{4\pi\varepsilon_0 r^2}; \quad F_g = \frac{g^2}{4\pi\mu_0 r^2} = \frac{1}{4\alpha^2} F_e \approx 4691 F_e, \tag{25}$$

where

$$\alpha := \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137} \tag{26}$$

is the fine structure constant. In view of $F_g \ggg F_e$, Dirac concludes, “that there must be some cause of dissimilarity between electricity and magnetism” (fn. 2, pp. 71f.). Notice that this dissimilarity concerns also the different geometric properties of the four electromagnetic field quantities[20] and the different PT symmetries of the electric and magnetic charge and current densities[21].

5 Von Klitzing constant and conductance quantum. Two natural velocities differing from that of light in vacuum

5.1 Von Klitzing constant and conductance quantum

The elementary charges $g = \Phi_B$ and $e = \Psi_D$ combine to

$$g \cdot e = \Phi_B \cdot \Psi_D = h \quad \text{and} \tag{27a}$$

$$\frac{g}{e} = \frac{\Phi_B}{\Psi_D} = \frac{h}{e^2} =: R_\mathrm{K} = \frac{2}{G_0} \,. \tag{27b}$$

[20] F. W. Hehl & Yu. N. Obukhov, *Foundations of Classical Electrodynamics. Charge, Flux, and Metric*, Basel: Birkhäuser 2003 (Progr. Math. Phys. 33)

[21] J. D. Jackson, *Classical Electrodynamics*, New York: Wiley ^{3}1998, Sect. 6.11. Here and in Sect. 6.12, an alternative representation as well as historical and review references are given.

Here, $R_\mathrm{K} \approx 25.8\,\mathrm{k}\Omega$ is the von Klitzing constant which governs the Hall resistance in the integer quantum Hall effect[22]. G_0 is the conductance quantum[23], where the factor of 2 is due to the spin degeneracy of the electron states involved.

That means that R_K and G_0 represent not only a relation between voltage and current but also between elementary electric e and magnetic charges g as well as between the corresponding elementary dielectric Ψ_D and induction fluxes Φ_B.

5.2 Space-time and electro-magnetic relationships in vacuum permittivity and permeability. On the fine-structure constant

Similarly to g and e, the vacuum permeability μ_0 and permittivity ε_0 combine to two important natural constants. (i),

$$\mu_0\,\varepsilon_0 = 1/c_0^2\,, \tag{28a}$$

where c_0 is the speed of light in vacuum. Through its dimension 'length/time', c_0 expresses the space-time relationship contained in μ_0 and ε_0. (ii),

$$\frac{\mu_0}{\varepsilon_0} = Z_0^2, \tag{28b}$$

where Z_0 is the impedance of free space. Through its dimension 'voltage/current', Z_0 exhibits the electro-magnetic relationship in μ_0 and ε_0.[24]

The ratio of the impedance of free space Z_0 (28b) and the von Klitzing constant R_K (27b) equals just twice the fine-structure constant α (26).

$$\frac{Z_0}{R_\mathrm{K}} = 2\alpha \tag{29}$$

Recall that $(2\alpha)^2$ equals the ratio of the forces between two elementary electrical and magnetic charges, respectively, see formula (25).

5.3 Two natural speeds differing from that of light in vacuum

In vacuum, one can define an elementary electrical flux $\Psi_E := \Psi_D/\varepsilon_0$. The ratio of the elementary electrical and induction fluxes yields a natural speed being

[22]K. v. Klitzing, G. Dorda & M. Pepper, *New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance*, Phys. Rev. Lett. 45 (1980) 6, 494–497

[23]B. J. van Wees, H. van Houten, C. W. J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. van der Marel & C. T. Foxon, *Quantized Conductance of Point Contacts in a Two-Dimensional Electron Gas*, Phys. Rev. Lett. 60 (1988) 9, 848-–850, https://scholarlypublications.universiteitleiden.nl/handle/1887/3316 (15.09.2022)

[24]This is elaborated in much more detail in R. Germer, *Die abzählbare Physik* [Countable Physics], https://de.wikibooks.org/wiki/Die_abzÃďhlbare_Physik, Ch. 6.

much smaller than the speed of light in vacuum c_0.

$$c_< := \frac{\Psi_E}{\Phi_B} = \frac{\Psi_D}{\Phi_B\,\varepsilon_0} = \frac{1}{R_\mathrm{K}\,\varepsilon_0} = 2\alpha c_0 \approx \frac{1}{68.5}\,c_0 \tag{30}$$

Analogously, one can define an elementary magnetic (field strength) flux $\Phi_H := \Psi_D/\mu_0$. The ratio of the elementary magnetic and dielectric fluxes yields a natural speed being much larger than the speed of light in vacuum c_0.

$$c_> := \frac{\Phi_H}{\Psi_D} = \frac{\Phi_B}{\mu_0\Psi_D} = \frac{R_\mathrm{K}}{\mu_0} = \frac{1}{2\alpha}\,c_0 \approx 68.5\,c_0 \tag{31}$$

$c_>$ is of the order of the expansion speed during the inflation phase of the universe required by Barrow[25] for making the monopoles vanishing.

In that two velocities, the pairing of electromagnetic field quantities is $(\vec{E},\vec{B})$ and $(\vec{H},\vec{D})$ as in the Maxwell equations. That pairing is in contrast to their pairing on p. 4.[26]

In bypassing we note that the product formula

$$c_<\,c_> = c_0^2 \tag{32}$$

resembles the relation

$$c_\mathrm{phase} c_\mathrm{group} = c_0^2 \tag{33}$$

for the phase and group velocities of a matter wave in de Broglie's imagination of electron waves in Bohr orbitals[27].

5.4 Occurrence of $c_<$ and $c_>$ in circuits. On the meaning of R_K and Z_0

$c_<$ and $c_>$ also occur in circuits with condensers, inductances, and resistors as follows.

The capacity of a condenser consisting of two plates of area ℓ_C^2 in distance ℓ_C in vacuum equals $C = \varepsilon_0\,\ell_C$. The combination of that condenser with a resistor of value R_K exhibits the time constant $\tau_{C,\mathrm{K}} = R_\mathrm{K}\,C$. Dividing the characteristic length ℓ_C by the time constant $\tau_{C,\mathrm{K}}$ yields the characteristic speed

$$c_{C,\mathrm{K}} := \frac{\ell_C}{\tau_{C,\mathrm{K}}} = \frac{C}{\varepsilon_0}\,\frac{1}{R_\mathrm{K}\,C} = \frac{1}{\varepsilon_0 R_\mathrm{K}} = c_<\,. \tag{34}$$

Analogously, for an inductance L, there is a characteristic length $\ell_L := L/\mu_0$. The combination of an inductance L with a resistor of value R_K exhibits the

[25] J. Barrow, *Der Ursprung des Universums*, Bertelsmann 1998

[26] See also W. Pauli, *Electrodynamics* (Pauli Lectures on Physics 1, ed. by Ch. P. Enz, transl. by S. Margulies & H. R. Lewis, Foreword by V. F. Weisskopf), Cambridge (MA): MIT Press 1973; reprint: New York: Dover 2000

[27] L. de Broglie, *Recherches sur la théorie des quanta*, Thèsis, Paris 1924; Ann. Physique [10] III (1925) 22–128, Ch. I, `tel.archives-ouvertes.fr/tel-00006807` (04.04.2012); reprint: Ann. Fond. Louis de Broglie 17 (1992) 1, 1–109

time constant $\tau_{L,\mathrm{K}} = L/R_\mathrm{K}$. Dividing the characteristic length ℓ_L by the time constant $\tau_{L,\mathrm{K}}$ yields the characteristic speed

$$c_{L,\mathrm{K}} := \frac{\ell_L}{\tau_{L,\mathrm{K}}} = \frac{L}{\mu_0}\,\frac{R_\mathrm{K}}{L} = \frac{R_\mathrm{K}}{\mu_0} = c_> \,. \tag{35}$$

When using the vacuum impedance Z_0 rather than the von Klitzing resistance R_K, the resulting characteristic speeds are that of light in vacuum c. This suggests R_K to correspond to localized phenomena, while Z_0 is characteristic for the (delocalized) propagation of waves.

The relationship between spatial and temporal aspects in the speed of wave propagation V is also reflected in d'Alembert's wave equation,

$$V^2 = \frac{\partial^2 Y/\partial t^2}{\partial^2 Y/\partial x^2}\,. \tag{36}$$

The numerator corresponds to an acceleration, the denominator – to a curvature; both being in equilibrium to another. In sound waves, the wave equation results from the equilibrium between the force densities of inertia (resistance against changing the temporal aspect of motion) and elasticity (resistance against changing positions).

6 Summary and conclusions

Generalizing Helmholtz's and Lipschitz's explorations of the relation between forces and energies[14], we have presented an axiomatic foundation of Dirac's phase factor $e^{i\beta}$ with *non*-integrable phase β for Schrödinger wave functions. The phase factors (not the phases themselves) *uniquely* determine the electromagnetic field.[28] Having that in mind, Dirac's 1931 pioneering approach[2] is interpreted in terms of the electromagnetic 4-potential from the very beginning. If the existence of the Diracian induction flux quantum $\Phi_B = h/e$ and the elementary magnetic charge (monopole) $g = \Phi_B$ is due to a wave-mechanical effect, the discretization of magnetic induction flux and magnetic charge is actually a quantization.

g and e combine to Planck's constant $h = g\,e$ and von Klitzing's constant $R_\mathrm{K} = g/e$.

Magnetic induction flux quantization is best known from superconductors. However, that does not refer to the flux through a closed surface. As a consequence, the actual flux quantum can deviate from $\Phi_0 = h/2e$.

Another example of magnetic induction flux quantization through an open surface occurs in the Landau levels. For an electron, the flux through the surface enclosed by the corresponding classical cyclotron orbit of the Landau level № n equals $(h/e)(n+\frac{1}{2}) = \Phi_B(n+\frac{1}{2})$. As there is only one electron involved, Φ_B rather than Φ_0 appears.

[28] T. T. Wu & C. N. Yang, *Concept of nonintegrable phase factors and global formulation of gauge fields*, Phys. Rev. D 12 (1975) 3845–3857; D. J. Gross, *Gauge Theory – Past, Present, and Future?*, Chin. J. Phys. 30 (1992) 955–972

While the magnetic induction $\vec{B}$ is also called 'magnetic flux density', its electric analogue, the dielectric displacement $\vec{D}$, is usually *not* called 'dielectric flux density'. However, Gauss' law in integral form justifies this alias. Due to the discretization resp. quantization of the free electric charges as entire multiples of the electron charge e, the dielectric flux through a closed surface is discretized resp. quantized, too. It is an entire multiple of the Lagrangian dielectric flux quantum $\Psi_D = e$.

In vacuum, additionally to the elementary flux quanta of magnetic induction, $\Phi_B = g$, and dielectric displacement, $\Psi_D = e$, elementary magnetic, $\Phi_H := \Phi_B/\mu_0 = g/\mu_0$, and electric, $\Psi_E := \Psi_D/\varepsilon_0 = e/\varepsilon_0$, flux quanta can be defined. This leads to two natural speeds which are much smaller resp. larger than that of light in vacuum by a factor of twice the fine structure constant, $2\alpha \approx 1/68.5$.

The factor 2α also equals the ratio (29) of the impedance of free space and the von Klitzing constant. Its square equals the ratio of the forces between two elementary electrical and magnetic charges, respectively, see formula (25).

Possibly, there are *two* speeds related to the time it takes to induce the electric field by a vortex of the magnetic fields, and *vice versa.* $c_> = c/2\alpha \approx 68.5\,c$ may be related to the enhanced speed of light during the inflationary development of our universe when compared with its nowadays value c[25].

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