A domain wall model

Matthew Stephenson[*]
Stanford University

Abstract

We consider a general domain wall model with vanishingly small temperature. Under certain conditions, it is very likely that we can have a stable domain wall structure on the horizon, in the limit of the temperature, $T \to 0$.

I. DOMAIN WALL CONSTRUCTION

Let us us begin by arguing why an effective (probe) action of the type

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}e^{-2\eta\phi}\left(\partial_\mu a\right)^2 + V(\phi, a) + \frac{1}{e^2}Z(\phi, a)F^2\right] \tag{1}$$

can appear very generically from supergravity. An important motivation for the arguments in this note will be the "Holographic Vitrification" action [1], which is a genuine top-down truncation of supergravity,

$$S = \frac{1}{8\pi}\int d^4x\sqrt{-g}\left[\frac{1}{2\ell_p^2}R - \frac{3}{4\ell_p^2}\frac{(\partial x)^2 + (\partial y)^2}{y^2} - V(x,y) - G_{IJ}(x,y)F^I_{\mu\nu}F^{J\mu\nu} - \Theta_{IJ}(x,y)F^I_{\mu\nu}\tilde{F}^{J\mu\nu}\right]. \tag{2}$$

The kinetic term

The kinetic term $\frac{(\partial x)^2+(\partial y)^2}{y^2}$ appears very generically in supergravity actions and the form $\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^{-2\eta\phi}\left(\partial_\mu a\right)^2$ follows from it via simple field re-definitions $x \sim e^a$, $y \sim e^\phi$. We will refer to ϕ and a as the dilaton and the axion, respectively. The fact that ϕ suppresses the kinetic term for a will be essential in our argument for the stability of domain walls.

The potential and domain wall formation

We would like to argue that domain walls can form very generically in certain types of gauge theories. Typically, a cosine potential is generated in an effective field theory through non-perturbative effects, such as for example the gaugino condensation, or the presence of instantons. We will imagine that our bulk theory has instantons and that we are looking only at its low-energy effective action, with QCD being the prototypical example of this.[1]

---

* matthewjstephenson@stanford.edu

[1] In QCD, an extra $U(1)_{PQ}$ symmetry is introduced, which gives rise to the dynamical axion field. Its effective action has a cosine potential, as argued for example in [2].

Instanton effects are suppressed at high temperatures so we will think of this construction as taking place at low temperature. Instantons lead to a periodic vacuum structure. The lowest order approximation is a potential of the type

$$V(\phi, a) = \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] + \dots . \tag{3}$$

The vacua of the axion are then given by $a = 2\pi n m_a/\sqrt{\lambda}$. Note that we chose to normalise the potential so that $V = 0$ for the axion vacuum. Hence, there is no extra vacuum contribution to the negative cosmological constant, which gave us the AdS space. Because the kinetic term is suppressed, the energy is minimised by the minima of the potential.

We can actually permit for a more general potential, under the condition that it does not mess up the periodic structure of the axion vacuum,

$$V(\phi, a) = \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] + V_2(\phi) + V_3(\phi)V_4(a). \tag{4}$$

To be more precise about the dilaton, we assume that its solution takes the form

$$\phi(r \to \infty, x) \to \infty_+, \tag{5}$$

in AdS space as $T \to 0$,

$$ds^2 = \frac{1}{r^2}\left(-dt^2 + dr^2 + dx^2 + dy^2\right). \tag{6}$$

Then the axion kinetic term is completely suppressed in the near-horizon limit of $r \to \infty$.

As for the x-dependent behaviour of ϕ, we can follow [3] to argue that for thin domain walls of a, all of its energy density is stored in the wall. Furthermore, ϕ must be continuous but non-differentiable at the wall w.r.t. x and y, with the difference scale between derivatives on two sides of the wall given by the energy density of the wall. Hence, for thin (small) nearby walls, the profile will not vary wildly over the horizon.

Coupling to the Maxwell field

The equation of motion for the axion field is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[e^{-2\eta\phi}\sqrt{-g}g^{\mu\nu}\partial_\nu a\right] - \partial_a V - \frac{1}{e^2}\partial_a Z F^2 = 0. \tag{7}$$

As long as the dilaton behaves as in Eq. (5), with $\eta > 0$, the kinetic term goes to zero in the limit of $r \to \infty$ and we find

$$\partial_a V = -\frac{1}{e^2}\partial_a Z F^2. \tag{8}$$

For some generic electric field flowing on the horizon (transistors), we find that $\partial_a Z = 0$ in the regions of the axion vacuum on the horizon ($\partial_a V = 0$). Z is thus extremised w.r.t a when

$a = 2\pi n m_a/\sqrt{\lambda}$.

We now have several (bottom-up) types of choices we can make for Z:

• For the first choice we can assume that it's more likely for a to have $n = 0$ than $n > 0$ in the pockets of vacuum. A good choice of Z for such a scenario might be

$$Z(\phi, a) = \frac{1}{2}\tilde{a}\left[\tilde{a} - \sin\left(\tilde{a}\right)\right]\mathcal{Z}_\phi(\phi), \tag{9}$$

where we have defined a dimensionless

$$\tilde{a} \equiv \frac{\sqrt{\lambda}}{m_a} a. \tag{10}$$

Z has the property that $\partial_a Z\big|_{\tilde{a}=2\pi n} = 0$ and

$$Z(\phi, \tilde{a} = 2\pi n) = 2\pi^2 n^2 \mathcal{Z}_\phi(\phi). \tag{11}$$

• The second "conductor-insulator" choice can be made as

$$Z(\phi, a) = \frac{1}{2}\left[1 - \cos\left(\frac{\tilde{a}}{2}\right)\right]\mathcal{Z}_\phi(\phi), \tag{12}$$

which has $\partial_a Z\big|_{\tilde{a}=2\pi n} = 0$ and

$$Z(\phi, \tilde{a} = 2\pi n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathcal{Z}_\phi(\phi) & \text{if } n \text{ is odd.} \end{cases} \tag{13}$$

• The third choice is the most brutally insulating,

$$Z(\phi, a) = \frac{1}{2}\left[1 - \cos\left(\tilde{a}\right)\right]\mathcal{Z}_\phi(\phi), \tag{14}$$

which has $\partial_a Z\big|_{\tilde{a}=2\pi n} = 0$ and

$$Z(\phi, \tilde{a} = 2\pi n) = 0. \tag{15}$$

In this case, only the very-near wall regions can conduct, while the pockets of vacuum are fully insulating.

Let us now analyse the dilaton's equation of motion,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right] + \eta e^{-2\eta\phi}\left(\partial_\mu a \partial^\mu a\right) - \partial_\phi V - \frac{1}{e^2}\partial_\phi Z F^2 = 0, \tag{16}$$

in the limit of $r \to \infty$ in pure AdS, hence $e^{-2\eta\phi}\left(\partial_\mu a \partial^\mu a\right)$ is again completely suppressed by the dilaton. We also assume that $\phi(x)$ is slowly varying (as argued above) and static. We find

$$r^2\partial_r^2\phi - 2r\partial_r\phi - \partial_\phi V - \frac{1}{e^2}\mathcal{Z}_a\partial_\phi\mathcal{Z}_\phi F^2 = 0. \tag{17}$$

The simplest (EFT) choice we can make is $\partial_\phi \mathcal{Z}_\phi = 0$ (set $\mathcal{Z}_\phi = 1$) and write

$$V(\phi, a) = \frac{1}{2} m_\phi^2 \phi^2 + \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a} a\right)\right]. \tag{18}$$

Hence,

$$\phi(r \to \infty, x) \to r^{\frac{3}{2}+\nu} C_1(x) + r^{\frac{3}{2}-\nu} C_2(x), \qquad \nu = \sqrt{\frac{9}{4} + m^2}, \tag{19}$$

which gives us a solution consistent with everything above.

The final action

The simplest action that seems to have the right properties is thus

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} e^{-2\eta\phi}(\partial_\mu a)^2 + \frac{1}{2} m_\phi^2 \phi^2 + \frac{m_a^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a} a\right)\right] + \frac{Z(a)}{4e^2} F^2\right], \tag{20}$$

with two simple choices, i.e. the conductor-insulator and the insulator, *or many other choices*,

$$Z(a) = \frac{1}{2}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a}\frac{a}{2}\right)\right], \tag{21}$$

$$Z(a) = \frac{1}{2}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a} a\right)\right]. \tag{22}$$

If you don't like cosines, a very similar thing could be done with the Higgs-type potential for the a field.

II. FIRST-ORDER DISCUSSION OF WEAK DISORDER IN OUR AXION-DILATON MODEL

Use the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}e^{-2\eta\phi}\left(\partial_\mu a\right)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{m_a^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right] + \frac{Z(a)}{4e^2}F^2\right], \tag{23}$$

with

$$Z(a) = \frac{1}{2}\left[1+\cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right], \tag{24}$$

chosen so that at $a = 0$, $Z = 1$ and the system is a conductor.

Let us consider weak disorder, parametrised by ε and write the expansions for the two scalars as

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots \tag{25}$$

$$a = \varepsilon a_1 + \varepsilon^2 a_2 + \dots, \tag{26}$$

so that the axion is the field driving the disorder.

The three equation of motions,

$$r^4\partial_\mu\left[e^{-2\eta\phi}r^{-4}g^{\mu\nu}\partial_\nu a\right] - \frac{m_a^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}a}{m_a}\right) + \frac{1}{4e^2}\frac{\sqrt{\lambda}}{2m_a}\sin\left(\frac{\sqrt{\lambda}a}{m_a}\right)F^2 = 0, \tag{27}$$

$$r^4\partial_\mu\left[r^{-4}g^{\mu\nu}\partial_\nu\phi\right] + \eta e^{-2\eta\phi}g^{\mu\nu}\partial_\mu a\partial_\nu a - m_\phi^2\phi = 0, \tag{28}$$

$$\frac{1}{2}\partial_\mu\left[\left(1+\cos\left(\frac{\sqrt{\lambda}}{m_a}a\right)\right)r^{-4}F^{\mu\nu}\right] = 0. \tag{29}$$

can be expanded in ε.

From Eq. (28) we see that ϵ-dependent disorder only couples ϕ_2 to a_1. To leading order, Eqs. (27) and (29) give

$$r^4\partial_\mu\left[e^{-2\eta\phi_0}r^{-4}g^{\mu\nu}\partial_\nu a_1\right] - m_a^2 a_1 - \frac{\sqrt{\lambda}}{4e^2m_a}a_1F^2 = 0, \tag{30}$$

$$\partial_\mu\left[\left(1-\frac{\lambda}{4m_a^2}a_1^2\right)r^{-4}F^{\mu\nu}\right] = 0. \tag{31}$$

Now, we can use the bound

$$\frac{1}{e^2\mathbb{E}\left[1/Z\right]} \le \sigma \le \frac{\mathbb{E}\left[Z\right]}{e^2} \tag{32}$$

to see that

$$\frac{1}{e^2\left(1+\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)} \le \sigma \le \frac{\left(1-\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)}{e^2} \tag{33}$$

$$\frac{\left(1-\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)}{e^2} \le \sigma \le \frac{\left(1-\frac{\lambda}{4m_a^2}\varepsilon^2\mathbb{E}\left[a_1^2\right]\right)}{e^2}, \tag{34}$$

hence the two inequalities give an exact equality,

$$\sigma = \frac{1}{e^2} - \varepsilon^2 \frac{\lambda}{4e^2 m_a^2} \mathbb{E}\left[a_1^2\right] \tag{35}$$

We can go further and write

$$Z(a) = 1 - \varepsilon^2 \frac{\lambda a_1^2}{4m_a^2} - \varepsilon^3 \frac{\lambda a_1 a_2}{2m_a^2} + \varepsilon^4 \frac{\lambda^2 a_1^4 - 12m_a^2 \lambda a_2^2 - 24 m_a^2 \lambda a_1 a_3}{48 m_a^4} + \mathcal{O}(\varepsilon^5), \tag{36}$$

$$\mathbb{E}\left[Z(a)\right] = 1 - \varepsilon^2 \frac{\lambda \mathbb{E}\left[a_1^2\right]}{4m_a^2} - \varepsilon^3 \frac{\lambda \mathbb{E}\left[a_1 a_2\right]}{2m_a^2} + \varepsilon^4 \frac{\lambda^2 \mathbb{E}\left[a_1^4\right] - 12m_a^2 \lambda \mathbb{E}\left[a_2^2\right] - 24 m_a^2 \lambda \mathbb{E}\left[a_1 a_3\right]}{48 m_a^4} + \mathcal{O}(\varepsilon^5), \tag{37}$$

and

$$1/Z(a) = 1 + \varepsilon^2 \frac{\lambda a_1^2}{4m_a^2} + \varepsilon^3 \frac{\lambda a_1 a_2}{2m_a^2} + \varepsilon^4 \frac{\lambda^2 a_1^4 + 6m_a^2 \lambda a_2^2 + 12 m_a^2 \lambda a_1 a_3}{24 m_a^4} + \mathcal{O}(\varepsilon^5), \tag{38}$$

$$\mathbb{E}\left[1/Z(a)\right] = 1 + \varepsilon^2 \frac{\lambda \mathbb{E}\left[a_1^2\right]}{4m_a^2} + \varepsilon^3 \frac{\lambda \mathbb{E}\left[a_1 a_2\right]}{2m_a^2} + \varepsilon^4 \frac{\lambda^2 \mathbb{E}\left[a_1^4\right] + 6m_a^2 \lambda \mathbb{E}\left[a_2^2\right] + 12 m_a^2 \lambda \mathbb{E}\left[a_1 a_3\right]}{24 m_a^4} + \mathcal{O}(\varepsilon^5), \tag{39}$$

hence

$$\begin{aligned}\frac{1}{\mathbb{E}\left[1/Z(a)\right]} = {} & 1 - \varepsilon^2 \frac{\lambda \mathbb{E}\left[a_1^2\right]}{4m_a^2} - \varepsilon^3 \frac{\lambda \mathbb{E}\left[a_1 a_2\right]}{2m_a^2} \\ & + \varepsilon^4 \frac{3\lambda^2 \mathbb{E}\left[a_1^2\right]^2 - 2\lambda^2 \mathbb{E}\left[a_1^4\right] - 12m_a^2 \lambda \mathbb{E}\left[a_2^2\right] - 24 m_a^2 \lambda \mathbb{E}\left[a_1 a_3\right]}{48 m_a^4} + \mathcal{O}(\varepsilon^5).\end{aligned} \tag{40}$$

We find that the conductivity is given by

$$\sigma = \frac{1}{e^2} - \varepsilon^2 \frac{\lambda \mathbb{E}\left[a_1^2\right]}{4e^2 m_a^2} - \varepsilon^3 \frac{\lambda \mathbb{E}\left[a_1 a_2\right]}{2e^2 m_a^2} + \varepsilon^4 \frac{\lambda^2 \mathbb{E}\left[a_1^4\right] - 12m_a^2 \lambda \mathbb{E}\left[a_2^2\right] - 24 m_a^2 \lambda \mathbb{E}\left[a_1 a_3\right]}{48 e^2 m_a^4} - \varepsilon^4 \tilde{\sigma}_4, \tag{41}$$

$$0 \le \tilde{\sigma}_4 \le \frac{\lambda^2}{16 e^2 m^4} \mathrm{Var}\left[a_1^2\right], \tag{42}$$

where Var [] is the variance, which can be computed from the distribution of a.

However, because ϕ_0 diverges, Eq. (30) tells us that $a_1 = 0$ and so $\sigma = 1/e^2$ to leading order.

III. GENERAL DISCUSSION OF WEAK DISORDER

Let us define

$$\sigma = \frac{1}{e^2} \mathbb{E}\left[Z\right] - \tilde{\sigma}, \tag{43}$$

so that the bounds give

$$0 \le \tilde{\sigma} \le \frac{1}{e^2} \left(\mathbb{E}\left[Z\right] - \frac{1}{\mathbb{E}\left[1/Z\right]} \right). \tag{44}$$

Further define

$$Z = 1 - \mathcal{Z}, \tag{45}$$

where $\mathcal{Z} = \mathcal{O}(\varepsilon)$. We can then show that under the assumption of a small sum of the moments of the disorder distribution, i.e. $|\sum_{n=1}^{\infty} \mathbb{E}[\mathcal{Z}^n]| < 1$,

$$0 \leq e^2 \tilde{\sigma} \leq \sum_{m=1}^{\infty} (-1)^{m-1} \left(\sum_{n=1}^{\infty} \mathbb{E}[\mathcal{Z}^n] \right)^m - \mathbb{E}[\mathcal{Z}] = \mathbb{E}\left[\mathcal{Z}^2\right] - \mathbb{E}[\mathcal{Z}]^2 + \mathcal{O}\left(\varepsilon^3\right). \tag{46}$$

In our example with Z specified in Eq. (24), we have

$$\mathcal{Z} = \frac{1}{2}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m_a} a\right)\right] = \varepsilon^2 \frac{\lambda a_1^2}{4 m_a^2} + \dots . \tag{47}$$

We can thus confirm the above result obtained in Eq. (42)

$$0 \leq \tilde{\sigma} \leq \frac{\lambda^2 \varepsilon^4}{16 e^2 m_a^2} \left(\mathbb{E}\left[a_1^4\right] - \mathbb{E}\left[a_1^2\right]^2 \right) + \dots = \frac{\lambda^2 \varepsilon^4}{16 e^2 m^4} \mathrm{Var}\left[a_1^2\right] + \dots . \tag{48}$$

The final statement is that at weak disorder, the leading-order correction to $\sigma = 1/e^2$ is given by $\mathbb{E}[Z-1]/e^2$ and the sub-leading correction is purely negative and bounded by the variance of $(Z-1)^2/e^2$.

IV. "DISORDER-DRIVEN METAL-INSULATOR TRANSITION"

Claim: *An axion-dilaton model gives a perfect conductor at weak disorder and can only become an insulator in the presence of strong disorder (up to possibly some even more fine-tuned setups).*

Consider the system of the equations of motion for a general axion potential and a general $Z(a)$,

$$r^4 \partial_\mu \left[e^{-2\eta\phi} r^{-4} g^{\mu\nu} \partial_\nu a \right] - \partial_a V - \partial_a Z F^2 = 0, \tag{49}$$

$$r^4 \partial_\mu \left[r^{-4} g^{\mu\nu} \partial_\nu \phi \right] + \eta e^{-2\eta\phi} g^{\mu\nu} \partial_\mu a \partial_\nu a - m_\phi^2 \phi = 0, \tag{50}$$

$$\partial_\mu \left[Z r^{-4} F^{\mu\nu} \right] = 0. \tag{51}$$

At weak disorder, which we measure with $\varepsilon \ll 1$, we write the axion part of the axio-diaton $\tau = a + i e^{-\phi}$ as

$$a = \varepsilon a_1 + \varepsilon^2 a_2 + \dots, \tag{52}$$

and allow for the dilaton to have an $\mathcal{O}(\varepsilon^0)$ piece,

$$\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \dots . \tag{53}$$

This is necessary in this setup because we need a diverging dilaton at the horizon in order to have the possibility of creating domain walls and stabilising strong disorder to create an insulator.

Let us first study Eq. (50), which gives

$$
\begin{aligned}
&\varepsilon^0 \left\{ r^4 \partial_\mu \left[r^{-4} g^{\mu\nu} \partial_\nu \phi_0 \right] - m_\phi^2 \phi_0 \right\} \\
+&\varepsilon^1 \left\{ r^4 \partial_\mu \left[r^{-4} g^{\mu\nu} \partial_\nu \phi_1 \right] - m_\phi^2 \phi_1 \right\} \\
+&\varepsilon^2 \left\{ r^4 \partial_\mu \left[r^{-4} g^{\mu\nu} \partial_\nu \phi_2 \right] + \eta e^{-2\eta\phi_0} g^{\mu\nu} \partial_\mu a_1 \partial_\nu a_1 - m_\phi^2 \phi_2 \right\} + \ldots = 0.
\end{aligned} \tag{54}
$$

Assuming that the background is that of AdS-Schwarzschild$_4$ (with boundary at $r = 0$), we can solve for ϕ_0,

$$
\phi_0(r) = A_0 \left(\frac{r}{r_0} \right)^{\frac{3}{2}-\nu} {}_2F_1 \left[\frac{1}{2} - \frac{\nu}{3}, \frac{1}{2} - \frac{\nu}{3}; 1 - \frac{2\nu}{3}; \left(\frac{r}{r_0} \right)^3 \right] + B_0 \left(\frac{r}{r_0} \right)^{\frac{3}{2}+\nu} {}_2F_1 \left[\frac{1}{2} + \frac{\nu}{3}, \frac{1}{2} + \frac{\nu}{3}; 1 + \frac{2\nu}{3}; \left(\frac{r}{r_0} \right)^3 \right], \tag{55}
$$

where $\nu = \sqrt{\left(\frac{3}{2}\right)^2 + m_\phi^2}$. From the properties of hypergeometric functions (Gauss's theorem), we find that

$$
\phi_0(r_0) = \left[A_0 \frac{\Gamma\left(1 - \frac{2\nu}{3}\right)}{\Gamma\left(\frac{1}{2} - \frac{\nu}{3}\right)\Gamma\left(\frac{1}{2} - \frac{\nu}{3}\right)} + B_0 \frac{\Gamma\left(1 + \frac{2\nu}{3}\right)}{\Gamma\left(\frac{1}{2} + \frac{\nu}{3}\right)\Gamma\left(\frac{1}{2} + \frac{\nu}{3}\right)} \right] \Gamma(0) = \infty, \tag{56}
$$

unless we specially tune the integration constants. It is also possible to get a finite dilaton at the horizon for $m_\phi = 0$, when $\nu = 3/2$. In that case

$$
\phi_0(r) = A_0 + B_0 \ln\left(1 - r^3/r_0^3\right), \tag{57}
$$

which is constant at the horizon for $B_0 = 0$.

The same result is obtained for

$$
\phi_1(r) = A_1 \left(\frac{r}{r_0} \right)^{\frac{3}{2}-\nu} {}_2F_1 \left[\frac{1}{2} - \frac{\nu}{3}, \frac{1}{2} - \frac{\nu}{3}; 1 - \frac{2\nu}{3}; \left(\frac{r}{r_0} \right)^3 \right] + B_1 \left(\frac{r}{r_0} \right)^{\frac{3}{2}+\nu} {}_2F_1 \left[\frac{1}{2} + \frac{\nu}{3}, \frac{1}{2} + \frac{\nu}{3}; 1 + \frac{2\nu}{3}; \left(\frac{r}{r_0} \right)^3 \right]. \tag{58}
$$

However, because ϕ_1 is treated perturbatively compared to ϕ_0, this is actually inconsistent with the expansion. We cannot have a divergent small perturbation at the horizon, unless we specially tune A_1 and B_1 to

$$
B_1 = -A_1 \frac{\Gamma\left(1 - \frac{2\nu}{3}\right)\Gamma\left(\frac{1}{2} + \frac{\nu}{3}\right)^2}{\Gamma\left(1 + \frac{2\nu}{3}\right)\Gamma\left(\frac{1}{2} - \frac{\nu}{3}\right)^2}. \tag{59}
$$

Although we cannot solve exactly for ϕ_2, we still see that because $e^{-2\eta\phi_0} \to 0$ at the horizon, at least the horizon behaviour of ϕ_2 is the same as that of ϕ_1 and we must again tune the integration constants to avoid perturbation expansion inconsistencies.

Let us now look at Eq. (49). It is now easy to see that in the presence of a diverging ϕ at the horizon, which is necessary to have the possibility of an insulator at strong disorder, the kinetic term has

$$
e^{-2\eta\phi_0} \left[1 - 2\eta\varepsilon\phi_1 + 2\varepsilon^2\eta\left(\eta\phi_1^2 - \phi_2\right) + \ldots \right], \tag{60}
$$

which goes to zero at the horizon at all orders of ε. Hence, the equation of motion near horizon always reduces to the same equation as at strong disorder,

$$\partial_a V = -\partial_a Z F^2. \tag{61}$$

at all order in ε. Now, again, because we are working at weak disorder, we must expand the equation out in ε and solve it order-by order. Thus, we get that all $a_i = 0$ [most likely, unless we again pick some strange V and Z and play the ε expansion of the vector field A_μ against the expansion for the axion]. A possible way out would be to have a potential with flat directions (like a moduli space), but that's probably a bit silly.

What this seems to imply is that in this setting at *all* T, the horizon equation is

$$\partial_a V = -\partial_a Z F^2, \tag{62}$$

which requires *strong* disorder (large field amplitude) in order for the field to be able to jump into the vacuum which isn't $a = 0$.

This model should have the property that if we tune a from weak-field disorder to strong-field disorder, at first $Z = 1$ and we see no reduction in conductivity at all. Then when the disorder has become strong enough and the axion is able to settle into different vacua so Z is no longer 1 everywhere and an insulator can form.

---

[1] D. Anninos, T. Anous, F. Denef, and L. Peeters, (2013), arXiv:1309.0146 [hep-th].
[2] E. Witten, Annals Phys. **128**, 363 (1980).
[3] M. Cvetic and H. H. Soleng, Phys.Rev. **D51**, 5768 (1995), arXiv:hep-th/9411170 [hep-th].