The Erdös-Borwein constant

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abstract

The Erdös-Borwein constant is the sum of the reciprocals of the Mersenne numbers .It is named after Paul Erdös and Peter Borwein.

keywords: Erdös-Borwein constant, series

1. Introduction

The Erdös-Borwein constant is a sum over all Mersenne reciprocals, namely

$$E = \sum_{n=1}^{\infty} \frac{1}{2^n - 1} = \sum_{n=1}^{\infty} \frac{1}{1 + 2 + 2^2 + \ldots + 2^{n-1}} = 1.6066951524 \ldots \tag{1}$$

Although E is known to be irrational (Erdös, 1948 ; Borwein, 1992).

In this note we give some formulas related to (1).

2. Formulas for E

Entry 1.

$$E = \sum_{n=1}^{\infty} \frac{1}{2^{n^2}} \left(\frac{2^n + 1}{2^n - 1} \right) \tag{2}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{n k}} \tag{3}$$

$$E = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n (2^n - 1)} \tag{4}$$

$$E = \sum_{n=1}^{\infty} \frac{\sigma_0(n)}{2^n} \tag{5}$$

where $\sigma_0(n) = d(n)$ is the divisor function .

Entry 2.

$$E = 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\left(2^{n+1} - 1\right)^k} \tag{6}$$

$$E = 1 + \sum_{n=1}^{\infty} n\, 2^{-n^2} \left(1 - 2^{-2 n-1}\right) + \sum_{n=1}^{\infty} \frac{2^{-n^2-2 n}}{2^{n+1} - 1} \tag{7}$$

$$E = 1 + \sum_{n=1}^{\infty} n\, 2^{-n^2} \left(1 - 2^{-2 n-1}\right) + 2 \cdot \sum_{n=1}^{\infty} 2^{-n^2} \left(1 - 2^{-2 n-1}\right) \sum_{k=1}^{n} \frac{1}{2^k - 1} \tag{8}$$

$$E = 3 \cdot \sum_{n=1}^{\infty} 2^{-n^2} - \sum_{n=1}^{\infty} 2^{-n^2-2n} \sum_{k=1}^{n} \frac{2^k}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} \tag{9}$$

$$E = \frac{3}{2} + \sum_{n=3}^{\infty} 2^{-n} \cdot \sum_{k=2}^{n-1} 2^{-(n-k)(k-1)} \tag{10}$$

$$E = 1 + \sum_{n=1}^{\infty} 2^{-n-1} \cdot \sum_{k=1}^{n} 2^{-(n-k)k} \tag{11}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} 2^{-nk} + 2 \cdot \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 2^{-2nk} \tag{12}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 2^{-2nk} (2^n + 1) \tag{13}$$

$$E = 2 \sum_{n=1}^{\infty} \frac{1}{2^{n+1} - 1} + \sum_{n=1}^{\infty} \frac{1}{(2^n - 1)\left(2^{n+1} - 1\right)} \tag{14}$$

$$E = 2^m \sum_{n=1}^{\infty} \frac{1}{2^{n+m} - 2^m + 1} + \sum_{n=1}^{\infty} \frac{1}{(2^n - 1)(2^{n+m} - 2^m + 1)} \quad , \; m = 1,\, 2,\, 3,\, \ldots \tag{15}$$

$$E = 3 \sum_{n=1}^{\infty} \frac{1}{2^{2n} - 1} + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\left(2^{2k} - 1\right)^{n+1}} \tag{16}$$

$$E = 1 + 3 \sum_{n=1}^{\infty} \frac{1}{2^{2n+1} - 1} + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\left(2^{2k+1} - 1\right)^{n+1}} \tag{17}$$

$$E = 3 \sum_{n=1}^{\infty} \frac{n\, 2^{2n}}{\left(2^{2n} - 1\right)\left(2^{2n+2} - 1\right)} + 6 \sum_{n=1}^{\infty} \frac{2^{2n}}{(2^n + 1)\left(2^{2n+2} - 1\right)} \tag{18}$$

$$E = \sum_{n=1}^{\infty} \frac{1}{2^n + 1} + 2 \sum_{n=1}^{\infty} \frac{1}{2^{2n} - 1} \tag{19}$$

$$E = \sum_{n=1}^{\infty} \frac{1}{2^n + 1} + \sum_{n=1}^{\infty} \frac{2^{n+1}}{(2^n - 1)\left(2^{n+1} - 1\right)} \sum_{k=1}^{n} \frac{1}{2^k + 1} \tag{20}$$

$$E = 1 + \frac{1}{6} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{-n} \left(3 - 2^{-n+1}\right)(1 - (-2)^{-n})}{(1 - 2^{-n})\left(1 - 2^{-n-1}\right)} \tag{21}$$

$$E = \sum_{n=1}^{\infty} \frac{1}{(2^n - 1)^2} + \sum_{n=1}^{\infty} \frac{1}{2^{n+1} - 1} \sum_{k=1}^{n} \frac{2^k}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} \tag{22}$$

$$E = 1 + 4 \sum_{n=1}^{\infty} \frac{1}{\left(2^{n+1} - 1\right)^2} + \sum_{n=1}^{\infty} \frac{1}{2^n (2^n - 1)\left(2^{n+1} - 1\right)^2} \tag{23}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{2^{-n(k+m)}}{k\,(k+1)} - \sum_{n=1}^{\infty} \ln(1 - 2^{-n}) \tag{24}$$

$$E = \sum_{n=1}^{\infty} \frac{n}{(n+1)\left(2^{n+1} - 1\right)} - \sum_{n=1}^{\infty} \ln(1 - 2^{-n}) \tag{25}$$

$$E = \sum_{n=1}^{\infty} \frac{n\, 2^n}{(2^n - 1)\left(2^{n+1} - 1\right)} \tag{26}$$

$$E = \sum_{n=1}^{\infty} \frac{1}{n\,(2^n - 1)} + \sum_{n=1}^{\infty} \frac{n}{(n+1)\left(2^{n+1} - 1\right)} \tag{27}$$

$$E = \sum_{n=1}^{m-1} \frac{1}{2^n - 1} + \frac{1}{m} \sum_{n=m}^{\infty} \frac{n}{2^n - 1} - \frac{1}{m} \sum_{n=1}^{\infty} \frac{n}{2^{n+m} - 1} \quad , \; m = 1,\, 2,\, 3,\, \ldots \tag{28}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{m} \frac{2^{2^{k-1} n}}{2^{2^k n} - 1} + \sum_{n=1}^{\infty} \frac{1}{2^{2^m n} - 1} \quad , \; m = 1,\, 2,\, 3,\, \ldots \tag{29}$$

$$E = \sum_{n=1}^{\infty} \frac{n}{2^n - 1} - \sum_{n=1}^{\infty} \frac{n\,(n+1)\,2^n}{\left(2^{n+1} - 1\right)\left(2^{n+2} - 1\right)} \tag{30}$$

$$E = \ln 2 + \sum_{n=1}^{\infty} \frac{n}{(n+1)\left(2^{n+1} - 1\right)} + \sum_{n=1}^{\infty} \frac{1}{n\,2^n\,(2^n - 1)} \tag{31}$$

$$E = 2 \ln 2 + \sum_{n=1}^{\infty} \frac{n}{(n+2)\left(2^{n+2} - 1\right)} + \sum_{n=1}^{\infty} \frac{1}{(n+1)\,2^n \left(2^{n+1} - 1\right)} \tag{32}$$

$$E = 7 \sum_{n=1}^{\infty} \frac{1}{2^{3n} - 1} + 7 \sum_{n=1}^{\infty} \frac{1}{2^{n+1} - 1} \sum_{k=1}^{n} \frac{2^k \left(3 \cdot 2^k + 1\right)}{\left(2^{2k} + 2^k + 1\right)\left(2^{2k+2} + 2^{k+1} + 1\right)} \tag{33}$$

$$E = \sum_{n=1}^{\infty} n\, 2^{-n} \prod_{k=n+1}^{\infty} \left(1 - 2^{-k}\right) \tag{34}$$

$$E = \frac{5}{4} + \frac{\ln 2}{2} + \sum_{n=3}^{\infty} 2^{-n} \sum_{k=2}^{n-1} \left(2^{-(n-k)(k-1)} - \frac{1}{2\,k} \right) \tag{35}$$

$$E = \sum_{n=1}^{\infty} 2^{-n^2} + 2 \sum_{n=1}^{\infty} \frac{2^{-(n+1)} - 2^{-(n+1)^2}}{1 - 2^{-(n+1)}} \tag{36}$$

$$E = 2 - \sum_{n=1}^{\infty} 2^{-n^2} + 2 \sum_{n=1}^{\infty} \frac{2^{-(n+1)^2}}{1 - 2^{-(n+1)}} \tag{37}$$

$$E = 2 \sum_{n=1}^{\infty} \frac{1}{2^{2n} - 1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n^2}} \left(\frac{2^{2n} + 1}{2^{2n} - 1} \right) \tag{38}$$

$$E = 2 \sum_{n=1}^{\infty} \frac{1}{2^{2n-1} - 1} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n^2}} \left(\frac{2^{2n} + 1}{2^{2n} - 1} \right) \tag{39}$$

$$E = 2 \sum_{n=1}^{\infty} \frac{1}{2^{2n} - 1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n - 1} \tag{40}$$

$$E = 2 \sum_{n=1}^{\infty} \frac{1}{2^{2n-1} - 1} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n - 1} \tag{41}$$

$$E = 2 - \sum_{n=1}^{\infty} \frac{1}{2^n + 1} + 2 \sum_{n=1}^{\infty} \frac{1}{2^{3n} - 2^n} \tag{42}$$

$$E = \frac{1}{2} \sum_{n=1}^{\infty} \left(\coth\left(\frac{n \ln 2}{2} \right) - 1 \right) \tag{43}$$

$$E = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, 2^{-k}}{(2^n - 1)^k} = 1 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{k-1}\, 2^{-k}}{(2^n - 1)^k} = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}\, 2^{-k}}{\left(2^{n-k+1} - 1\right)^k} \tag{44}$$

$$E = \int_0^\infty e^x \left(\sum_{n=1}^{\infty} e^{-2^n x} \right) dx \tag{45}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{n k} - 1} - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{n (k+1)} - 1} \tag{46}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^{n k} + 1} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{n (k+1)} - 1} \tag{47}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ln\left(1 + 2^{-n k}\right) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k + 1) \left(2^{n (k+1)} - 1\right)} \tag{48}$$

$$E = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \ln\left(1 - 2^{-n k}\right) - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(k + 1) \left(2^{n (k+1)} - 1\right)} \tag{49}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2^{k-1}}{2^{n 2^{k-1}} + 1} \tag{50}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{2^{2^k (n-k) + 2^{k-1}} - 1} \tag{51}$$

$$E = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2^{-n} \cdot 3^{-k} \left(2 + 2^{-n 3^{-k}}\right)}{1 + 2^{-n 3^{-k}} + 2^{-2 n 3^{-k}}} \tag{52}$$

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{d(k) \, \mu(n)}{2^{n k} - 1} \tag{53}$$

Remark: $d(k)$ is the divisor function and $\mu(n)$ is the Moebius function.

$$E = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} d(k) \, \phi(n) \left(\frac{\left(1 - 2^{-k}\right)^2}{2^{n k} - 1} \right) \tag{54}$$

Remark: $d(k)$ is the divisor function and $\phi(n)$ is the Euler totient function.

$$E = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{2^n} - 1} + \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{2^{n+1}-1} \frac{1}{2^k - 1} = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{2^n} - 1} + \sum_{n=1}^{\infty} \sum_{k=1}^{2^n - 1} \frac{1}{2^{k+2^n} - 1} \tag{55}$$

$$E = 2 \sum_{n=1}^{\infty} \frac{1}{2^{4 n-2} - 1} + \sum_{n=1}^{\infty} \frac{1}{2^n - (-1)^n} \tag{56}$$

$$E = 2 \sum_{n=1}^{\infty} \frac{1}{2^{4 n} - 1} + \sum_{n=1}^{\infty} \frac{1}{2^n + (-1)^n} \tag{57}$$

$$E = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{(n-k+1)(k+1)} \binom{(n - k + 1) (k + 1)}{m} (-2)^{-m} \tag{58}$$

$$E = \gamma + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2 \left(2^{n-k} - 1\right) + k} \tag{59}$$

Remark: γ is the Euler-Mascheroni constant.

Entry 3.

$$E = \frac{3}{2} + \int_0^\infty \frac{1}{e^{2 \pi x} - 1} \left(\frac{4 \sin(x \ln 2)}{5 - 4 \cos(x \ln 2)} \right) dx \tag{60}$$

$$E = \frac{1}{4} + \frac{\gamma - \ln(\ln 2)}{\ln 2} + \int_0^\infty \left(\cot\left(\frac{x \ln 2}{2} \right) - \frac{2}{x \ln 2} \right) \frac{1}{e^{2\pi x} - 1} \, dx \tag{61}$$

Remark: γ is the Euler-Mascheroni constant.

$$E = \sum_{n=1}^{\infty} \frac{\zeta(2^n)}{2^n} + \int_1^\infty (x - [x]) \left(\sum_{n=1}^{\infty} x^{-1-2^n} \right) dx \tag{62}$$

Remark: $\zeta(x)$ is the Riemann zeta function.

Entry 4. for $N = 1, 2, 3, \ldots$, we have

$$E = \frac{1}{4} + \frac{\gamma - \ln(\ln 2)}{\ln 2} - \sum_{n=0}^{N-1} \frac{(\ln 2)^{2n+1} B_{2n+2}^2}{(2n+2)(2n+2)!} + O\left((\ln 2)^{2N}\right) \tag{63}$$

where

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n \right) \tag{64}$$

$$B_n = \left\{ 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, \ldots \right\} \tag{65}$$

Remark: γ is the Euler-Mascheroni constant, and B_n are the Bernoulli numbers.

Entry 5. for $m = 1, 2, 3, \ldots$, we have

$$E = a_m + \sum_{n=1}^{\infty} \frac{1}{2^{mn}(2^n - 1)} \tag{66}$$

where

$$a_{m+1} = a_m + \frac{1}{2^{m+1} - 1}, \quad a_1 = 1, \quad m = 1, 2, 3, \ldots \tag{67}$$

$$a_m = \left\{ 1, \frac{4}{3}, \frac{31}{21}, \frac{54}{35}, \ldots \right\}, \quad a_m \longrightarrow E \tag{68}$$

Examples:

$$E = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n (2^n - 1)} = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{1}{2^{2n}(2^n - 1)} \tag{69}$$

$$E = \frac{31}{21} + \sum_{n=1}^{\infty} \frac{1}{2^{3n}(2^n - 1)} = \frac{54}{35} + \sum_{n=1}^{\infty} \frac{1}{2^{4n}(2^n - 1)} \tag{70}$$

Endnote

Entry 6. for $m = 0, 1, 2, 3, \ldots$, we have

$$\sum_{n=1}^{m+1} \frac{1}{2^n - 1} - \frac{\ln\left(1 - 2^{-m-2}\right)}{\ln 2} < E < \sum_{n=1}^{m+1} \frac{1}{2^n - 1} - \frac{\ln\left(1 - 2^{-m-1}\right)}{\ln 2} \tag{71}$$

References

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