Scattering matrix based on time-evolution operators without needing renormalization: an alternative solution to infinity problem

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Abstract The current situation of research challenging the demanding tasks of renormalization implies that the present framework of quantum scattering theory does not offer good prospect, and therefore it is necessary to construct a new theory able to solve the infinity problem fundamentally in a general way. Our purpose is to construct an alternative mathematical formulation capable of ensuring the convergence of the scattering matrix without recourse to renormalization theory, thus preventing overlapping divergences of the scattering matrix in principle. We demonstrate that the infinity problem is due mainly to the mathematical representation of the scattering operator and present, as a solution to the problem, alternative mathematical representations of the scattering matrix in terms of the local and global time-evolution operators which replace the Dyson series and do not need the Feynman diagram. Importantly, the obtained results clarify that substantially, there does not exist the infinity problem of the scattering matrix. Ultimately, we draw the successful conclusion that it is possible to conceive of an alternative to renormalization theory and as a new proposal, our formalism can lay the foundation for formulating a consistent theory without infinity and renormalization.

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1 Introduction

In our view, one of the key questions of quantum field theory is whether renormalization theory is able to reach the ultimate goal to resolve the divergence problem of scattering matrix in a general way. Such an opinion seems paradoxical and challenging but the present situation of research showing Odyssei in renormalization naturally makes it burgeon.

The Feynman diagrams as the graphical representations of Dyson's formula for the scattering matrix (S-matrix) which the Wick theorem produces had promoted quantum scattering theory to a physically elaborated theory [1]. However, as a rule, we encounter formidable divergence problems on calculating the scattering matrix based on the Feynman diagram. Renormalization acknowledged as an astounding mathematical trick enables one to overcome some overlapping divergences in the Feynman diagrams.

It is possible to understand this situation by considering the historical milestones of the development of renormalization. Freeman Dysons research which had begot the Dyson series had contributed significantly to the development of quantum scattering theory [2,3]. As a result, the foundations of the future research on the scattering problem had been laid out, but at the same time, the problem of overlapping divergences arose. The conventional method for solving the problem of overlapping divergence of the scattering matrix is renormalization, which provides an approach to calibrating fundamental physical quantities so that computational results of the scattering matrix could coincide with experiments.

The researches on renormalization have spawned a diversity of approaches. John Ward's approach to renormalization featured in the advantage to be considerably simple [4] and the Yang-Mills method for finding a finite renormalized amplitude contributed to the early development of renormalization theory. An innovative attempt to resolve the problem of overlapping divergences was made by Salam [5], which attracted a lots of attention because the connection with significant work by Steven Weinberg [6] could provide a mathematically coherent version of renormalization. Another foundation for renormalization was presented by Stueckelberg and Green [7], which was distinguished from the previous ones. The mathematically important questions on the overlapping divergence had been proposed in detail by Bogoliubov and Parasiuk [8]. Klaus Hepp's review of an important theorem of Bogoliubov and Parasiuk's work was significant for the research in that direction [9]. On the other hand, Wolfhart Zimmermann discovered an explicit formula for Bogoliubov's iterative method [10], which was finalized as Bogoliubov, Parasiuk, Hepp and Zimmermann's (BPHZ) method. The proposition of Gelfand and Yaglom and Cameron's comment were an important success in making the Feynman history integral rigorous [11,12].

The recent researches show the ramifications of renormalization technique which covers a wide range of the studies concerning the exact renormalization group equation [13, 14], renormalized perturbation theory [15], renormalization theory on the perturbative Feynman graph expansion [16], scattering amplitudes and renormalization group [17,18,19, 20,21,22,23,24,25], and on the other side, connections between these different techniques of renormalization[26]. Recently, the development of the BPHZ renormalization which shows the tendency to develop into a configuration space formulation is remarkable because it does not introduce an auxiliary mass term, instead including massless fields [27].

Renormalization theory which has its long and challenging history from perturbative quantum electrodynamics developed by Dyson, Gell-Mann, Low and others to Wilson's formulation and Polchinski's functional equation has enjoyed so much successes in addressing overlapping divergences of the scattering matrix [28]. At present, renormalization maintains its dominant status in resolving the infinity problem of the scattering matrix. Nevertheless, many researchers cast doubt on the Dyson series and the Feynman diagram because they seem imperfect and in a sense, factitious. In fact, in this formulation, invariable physical quantities such as mass and electric charge are treated as variable ones, whose physics is not clear yet. The survey of the researches shows that the heart of renormalization theory still has not been finished and the centre of research continues to shift [29,30]. The facts, in a sense, imply that it is necessary to reconsider the significance of renormalization theory, since it is not natural that most of terms of the perturbation expansion are given as divergent integrals which must be renormalized [28,31,32,33,34].

It still remains undetermined whether renormalization theory can develop as a complete theory capable of addressing a diversity of renormalization in a general way and in addition giving physically reasonable interpretations. The present situation of the research of renormalization leads to the practical view that renormalization theory would not develop favorably in the future and exactly, it is desirable to find out a certain general method available to all the case of calculations of the scattering matrix without dealing with the infinity problem. In this regard, it is remarkable that there are critical opinions on renormalization and attempt to construct a new formulations of scattering theory without renormalization [28,33].

For this reason, we aim to construct a mathematical formulation of quantum scattering theory free of infinity. Our formulation is based on two new time-evolution operators called the local time-evolution operator and the global time-evolution operator which replace the Dyson series. These operators are characterized by being independent of the overlapping divergence and the Feynman diagram, thus not needing renormalization.

2 Infinity problem of scattering matrix and analysis

To help to understand the motivation of our work, let us examine the Dyson series from the mathematical aspect. The Schrödinger equation in the interaction picture is written as

$$i\hbar\frac{\partial}{\partial t}\Phi(t) = H_{int}(t)\Phi(t). \tag{1}$$

For convenience, afterward H_{int} is denoted briefly H. For given initial state $\Phi(t_0)$, the unique solution of the above equation is represented as

$$\Phi(t) = S(t, t_0)\Phi(t_0).$$

Intuitively, the functions $\Phi(t_0)$ and $\Phi(t)$ correspond to the states of an incoming free particle Φ_{in} and an outgoing free particle Φ_{out}, respectively.

The real number

$$W(t, t_0) = |\langle\Phi_{out}(t)|S(t, t_0)\Phi_{in}(0)\rangle|^2 .$$

is the transition probability from an incoming free state to an outgoing free state. Here, the complex number $\langle\Phi_{out}(t)|S(t, t_0)\Phi_{in}(0)\rangle$ is called the S-matrix element. As a rule, solving the Schrödinger equation (1) by use of the method of successive approximation, one can obtain a representation of the scattering operator. Integrating both sides of equation (1) with respect to time yields

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle + (-i/\hbar)\int_{t_0}^{t}\hat{H}(t')\Phi(t')dt'. \tag{2}$$

The zeroth approximation becomes

$$|\Phi^{(0)}(t)\rangle = |\Phi(t_0)\rangle, \tag{3}$$

which is the state function for an initial state. It is necessary to improve approximation, regarding this as the zeroth approximation.

By means of the method of successive approximation, starting with the zeroth approximation, one can obtain the first approximation as

$$|\Phi^{(1)}(t)\rangle \approx |\Phi(t_0)\rangle + (-\mathrm{i}/\hbar)\int_{t_0}^{t} \hat{H}(t')dt'|\Phi(t_0)\rangle \tag{4}$$

and then by successive substitution of the first approximation into the integrand can determine the second approximation. The iteration done in this way improves the approximation of the solution. The sufficient iteration may lead to an exact solution.

The nth order approximation is written as

$$\begin{aligned}|\Phi^{(n)}(t)\rangle \approx \Bigg[&1 + \frac{-\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t_1)dt_1 \\ &+\left(\frac{-\mathrm{i}}{\hbar}\right)^2\int_{t_0}^{t}\hat{H}(t_1)dt_1\int_{t_0}^{t_1}\hat{H}(t_2)dt_2 + \cdots + \left(\frac{-\mathrm{i}}{\hbar}\right)^n \\ &\int_{t_0}^{t}\hat{H}(t_1)dt_1\int_{t_0}^{t_1}\hat{H}(t_2)dt_2\cdots\int_{t_0}^{t_{n-1}}\hat{H}(t_n)dt_n\Bigg]|\Phi(t_0)\rangle.\end{aligned} \tag{5}$$

The complete solution is represented with the aid of an infinite series as

$$\begin{aligned}|\Phi(t)\rangle = \sum_{n=0}^{\infty}\left(\frac{-\mathrm{i}}{\hbar}\right)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n \\ \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)|\Phi(t_0)\rangle,\end{aligned} \tag{6}$$

where using the Lagrangian density of the Hamiltonian $\mathcal{H}$, we write $\hat{H} = \int \mathrm{d}x\ \mathcal{H}(x)$. As an example, in the case of the interaction between electron-positron field and electromagnetic field, the Lagrangian density of interaction is represented as

$$\mathcal{H} = -j_\mu(x)A^\mu(x) = -\frac{e}{2}\left[\bar{\psi}\gamma_\mu\psi\right]A^\mu = eN\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu.$$

Taking into consideration the domain of integration and the permutability of upper limits of integration variables, equation (6) is modified as an overlapping integral with like limits of integration:

$$\begin{aligned}|\Phi(t)\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mathrm{i}}{\hbar}\right)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_n \\ \left[\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)\right]|\Phi(t_0)\rangle.\end{aligned} \tag{7}$$

Considering that the Hamilton operators $\{\hat{H}(t_k)|k = 1, 2, \cdots, n\}$ may be noncommutative, i.e., $\left[\hat{H}(t_i), \hat{H}(t_j)\right]_- \neq 0$, one writes the final result with the help of the time ordering operator $\mathcal{T}$ as

$$\begin{aligned}|\Phi(t)\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mathrm{i}}{\hbar}\right)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_n \\ \mathcal{T}\left[\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)\right]|\Phi(t_0)\rangle.\end{aligned} \tag{8}$$

What should be stressed here is that while the reduction of equation (6) to equation (7) presupposes the commutativity of the Hamiltonians $\{\hat{H}(t_k)|k = 1, 2, \cdots\}$, the final result (8) assumes noncommutativity. This is not consistent from the logical aspect.

It is important to check whether the time-ordering operator is necessary. In fact, $\{\hat{H}(t_k)|k = 1, 2, \cdots\}$ should be considered to be commutative. This is because $\{\hat{H}(t_k)|k = 1, 2, \cdots\}$ have one and the same structure in relation to differential operators with respect to coordinates and $\{t_k|k = 1, 2, \cdots\}$ should be qualified as parameters involved in multiplication operator. In the end, it follows that $\{\hat{H}(t_k)|k = 1, 2, \cdots, n\}$ are commutative and thus the time-ordering operator is not needed.

Equation (8) can be written in a simple form as

$$|\Phi(t)\rangle = S(t, t_0)|\Phi(t_0)\rangle. \tag{9}$$

Here, the operator called the Dyson series:

$$\begin{aligned}S(t, t_0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mathrm{i}}{\hbar}\right)\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_n \\ \mathcal{T}[\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)]\end{aligned} \tag{10}$$

becomes the scattering operator. With the help of this series, it is possible to calculate elements of the scattering matrix based on the perturbation theory.

It is usual to calculate higher-order approximation of the scattering matrix to consider concrete effects. We then encounter such difficulties that higher-order terms of the $S-$matrix diverge. If terms of higher-order approximation become infinite, the perturbation theory obviously is not of significance. In this connection, renormalization theory is being investigated as the best way to solve the problem of the overlapping divergence of scattering matrix. This method is characterized by the manner that based on analyzing the cause of divergence, a series of fundamental quantities of physics such as mass and charge and the like is redefined, and then infinity is removed by applying a definite procedure.

Renormalization theory claims the following principles. Since measured physical quantities really are finite, infinity which occurs on calculating scattering matrices should be necessarily eliminated by applying a proper method, even if it may be factitious. After renormalization, within the limits of the approximation of one-particle state, i.e., a free state of particle, all physical quantities should approximate those of a free particle. The long history of the investigation of renormalization method showed examples of successful solution of a series of problems.

However, this method cannot be assessed as providing the final solution to the problem of overlapping divergence for reasons given by physics and mathematics. It can be reviewed from the mathematical and physical point of view whether or not renormalization theory possesses generality.

First, let us consider the problem from the physical point of view. If one should renormalize a certain series of physical quantities universally defined, it implies that a physical formula in which these quantities are contained is not valid, since every measured physical quantity must be always finite. In this sense, renormalization should be regarded as a formal method for modifying an inexact formula so as to provide a proper result by changing invariable physical quantities entering the formula. From the physical aspect, it should be considered that it is immoderate to make a particular physical process correspond to every higher-order approximation of the scattering operator by using the Feynman diagram. If an arbitrary higher-order approximation corresponds to a definite real physical process, then we must give answer to the question about how to explain immense space and time necessary for the internal process. From the mathematical point of view, we in principle may imagine infinite-order approximation of the scattering operator. In this case, we should consider that a higher-order physical process is required to proceed instantaneously without the constraints of space-time. However, it is impossible to think of such a physical process. Therefore, the Feynman diagram cannot be assessed to be legitimate in the context of the Dyson series. On that account, renormalization theory may give an ad hoc solution to some problems but cannot arrive at the final solution. A particular higher-order scattering process that the Feynman diagram represents can be involved in a given scattering. In this case, the whole scattering outcomes to a superposition of possible higher-order scattering processes. In this sense, the Feynman diagram is of significance. However, it is not realistic to consider that all higher-order scattering processes that the Feynman rule dictates are possible. In fact, the mathematical concept of the higher-order of the Dyson series and the physical picture of the Feynman diagram are different matters.

Next, let us consider the problem from the mathematical point of view. In our view, in order to solve the infinity problem, it is necessary to correct the mathematical formulations causing the infinity problem rather than to resort to renormalization which treats invariable physical quantities as adjustable parameters. To understand the infinity problem, let us start from equation (2). By the mean value theorem of integral calculus:

$$\int_a^b f(x)g(x)dx = f(c)(b-a) \qquad (c \in [a,b]),$$

it is possible to take

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle + \frac{-\mathrm{i}}{\hbar}(t-t_0)\hat{H}(\bar{t})|\Phi(\bar{t})\rangle, \tag{11}$$

where $\bar{t} \in [0,t]$.

For $\hat{H}(\bar{t})|\Phi(\bar{t})$, we can imagine an eigenvalue equation with parameter $\bar{t}$:

$$\hat{H}(\bar{t})|\Phi(\bar{t})\rangle = E(\bar{t})|\Phi(\bar{t})\rangle. \tag{12}$$

Then, we have

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle + \frac{-\mathrm{i}}{\hbar}(t-t_0)E(\bar{t})|\Phi(\bar{t})\rangle. \tag{13}$$

Since the Schrödinger equation presupposes finiteness of solution, equation (12) is finite and thus equation (13) is finite as well.

In this case, in a formal manner, the scattering operator should be taken as

$$S(t,t_0) = 1 + \frac{-\mathrm{i}}{\hbar}(t-t_0)E(\bar{t})\frac{|\Phi(\bar{t})\rangle}{|\Phi(t_0)\rangle}. \tag{14}$$

Of course, since $|\Phi(\bar{t})\rangle$ is unknown, the solution should be considered to be formal, but equation (14) is enough to verify that the scattering operator should be finite. In fact, in terms of the definition of the Schrödinger equation, $|\Phi(\bar{t})\rangle, |\Phi(t_0)\rangle$ and $E(\bar{t})$ should be finite and non-zero valued. Considering in this way, we draw the conclusion there is no infinity problem.

It is important to review renormalization theory from the aspect of the convergence of successive approximation. Purely from the mathematical point of view, we cannot ensure that solution (8) is always convergent. Of course, it is possible to find solutions of simultaneous linear equation with the help of a similar method of successive approximation to solution (8). The formula of the method of successive approximation for obtaining the solution of a simultaneous linear equation $f(\mathbf{X}) = 0$ in general can be written in the form

$$\mathbf{X}^{(n+1)} = A\mathbf{X}^{(n)}, \tag{15}$$

where A denotes the iterating operation and n, the order of successive approximation. For the solution of this equation by means of the method of successive approximation to converge, it is necessary to choose a proper initial point $\mathbf{X}^{(\mathbf{0})}$ and to apply a proper method of iteration. If not so, the solution of the equation may diverge. Mathematicians do not alter A itself in the above equation for the purpose of obtaining a convergent solution. In other words, they do not modify the relation or equation. On the contrary, according to renormalization theory, physicists vary invariable physical quantities entering the equation, e.g., mass and charge etc., thus changing the formula itself describing physics. As it is, from the mathematical point of view, we cannot guarantee that choosing an arbitrary initial wave function always leads to an exact convergent solution for a problem of scattering on applying the method of successive approximation. Therefore, it is important to make the exact choice of initial wave function and the reasonable determination of iterative operation for the successive approximation so as to ensure the convergence of solution.

Let us consider that it is possible to imagine another approach to the successive approximation. From equation (2), we can take the equation for iterative operation as

$$|\Phi^{(n)}(t)\rangle \approx |\Phi(t_0)\rangle + \frac{-\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')\Phi^{(n-1)}(t')dt',$$

where the superscripts n and $n-1$ denote the orders of approximation, respectively. By the mean value theorem of integral calculus:

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx,$$

where $c \in [a, b]$, we have

$$|\Phi^{(n)}(t)\rangle = |\Phi(t_0)\rangle + \frac{-\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'|\Phi^{(n-1)}(\bar{t})\rangle, \tag{16}$$

where $\bar{t} \in [t_0, t]$. Then, the zeroth approximation can be taken as

$$\Phi^0(\bar{t}) = \Phi(t_0). \tag{17}$$

From equations (16) and (17), it follows that

$$|\Phi(t)\rangle \approx \sum_{n=0}^{\infty}\left(\frac{-\mathrm{i}}{\hbar}\right)^n \left[\int_{t_0}^{t} \hat{H}(t')dt'\right]^n |\Phi(t_0)\rangle. \tag{18}$$

Again, by the mean value theorem of integral calculus, we obtain

$$|\Phi(t)\rangle \approx \sum_{n=0}^{\infty}\left(\frac{-\mathrm{i}}{\hbar}\right)^n \left[(t-t_0)\hat{H}(\bar{t})\right]^n |\Phi(t_0)\rangle. \tag{19}$$

Assuming eigenvalue equation

$$\hat{H}(\bar{t})|\Phi(t_0)\rangle = E|\Phi(t_0)\rangle, \tag{20}$$

equation (19) is recast as

$$|\Phi(t)\rangle = \sum_{n=0}^{\infty}\left(\frac{-\mathrm{i}}{\hbar}\right)^n [(t-t_0)E]^n |\Phi(t_0)\rangle. \tag{21}$$

From this, we can see that for the series to converge, it is necessary that

$$\frac{E(t-t_0)}{\hbar} < 1. \tag{22}$$

If the above condition is not satisfied, this method of successive approximation is not significant. Therefore, this method is not of generality.

Also, it is easy to understand the problem of convergence of the Dyson series by taking into consideration expression

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \mathcal{T}[\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)]. \tag{23}$$

Purely from the mathematical point of view, it is self-evident that equation (23) is equivalent to

$$\prod_{k=1}^{n}\mathcal{T}\int_{t_0}^{t}\hat{H}(t_k)dt_k = \left[\int_{t_0}^{t}\hat{H}(t')dt'\right]^n, \tag{24}$$

since $t_1, t_2, \cdots, t_n$ are independent integration variables and thus the members of integration $\int_{t_0}^{t}\hat{H}(t_k)dt_k$ are identical. Straightforwardly, we can see that the following holds:

$$\int_{t_0}^{t}\hat{H}(t_1)dt_1\int_{t_0}^{t}\hat{H}(t_2)dt_2 = \int_{t_0}^{t}\hat{H}(t_2)dt_2\int_{t_0}^{t}\hat{H}(t_1)dt_1$$

and furthermore,

$$\int_{t_0}^{t}\hat{H}(t_1)dt_1\int_{t_0}^{t}\hat{H}(t_2)dt_2 = \left[\int_{t_0}^{t}\hat{H}(t_1)dt_1\right]^2 = \left[\int_{t_0}^{t}\hat{H}(t_2)dt_2\right]^2.$$

Therefore, the time ordering operator and Wick's theorem are needless. Naturally, the Feynman diagram which is framed topologically in terms of the Dyson series is unnecessary for the mathematical reason as well.

This conclusion is serious but provides the key to solving the infinity problem of scattering matrix. This procedure which is mathematically legitimate is sufficient to keep us from imaging higher-order scattering processes which the Feynman diagram dictates and overlapping divergence thereof. According to equation (24), Nth higher-order approximation is described by N successive applications of $\int_{t_0}^{t}\hat{H}(t_k)dt_k$ to the wave function.

Thus, the time-evolution operator in a series, equation (10) can be represented without using the time ordering operator and the Feynman diagram as

$$S(t, t_0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mathrm{i}}{\hbar}\right)^n\left[\int_{t_0}^{t}\hat{H}(t')dt'\right]^n. \tag{25}$$

Using the power series expansion $\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, formula (25) is recast as

$$S(t, t_0) = \exp\left[\frac{-\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right]. \tag{26}$$

Equation (10) is different from the Dyson series (**??**). In the end, the time ordering operator makes the difference. It is obvious that ignoring the time ordering operator in the Dyson series yields equation (26). So far as the Hamiltonian $\bar{H}$ is correct, equation (26) ensures convergence.

The infinity problem of scattering matrix, in a sense, is an instance illustrative of the imperfection of the adopted mathematical language for quantum field theory. Purely from the point of view of mathematics, such a mathematical theory that one separates a finite quantity from a given infinity cannot be legitimate. We can understand the points in question of renormalization theory by considering it based merely on mathematical logic. As the starting point, we should presuppose the fact that the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\Phi}{\partial t} = \hat{H}\Phi \tag{27}$$

gives an exact finite solution in its domain of definition.

Let us remember that the solution in terms of scattering operator is represented as

$$\Phi(t) = S(t, t_0)\Phi(t_0). \tag{28}$$

Obviously, equations. (27) and (28) should be considered to be identical from the mathematical viewpoint. If the solution to equation (28) becomes infinite, it gives nothing but the conclusion that just the scattering operator $S(t, t_0)$ is incorrect. In this case, we must examine the exactitude of the interaction Hamiltonian defined.

There is neither need for nor possibility of necessarily conceiving of every mathematical approximation carrying physical meaning. If renormalization is needed on firm physical ground, it indicates that the Schrödinger equation (27) too holds only on condition that renormalization is performed and the renormalization is necessary for all problems to which the fundamental equation of quantum mechanics is applied. Evidently, it is not logic because there are many cases when it is possible to obtain exact results without renormalization.

3 Local time-evolution operator and analysis of convergence

It is not reasonable to introduce the successive approximation in the form of equation (8). This is because in order that it is possible to represent

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')\Phi(t')dt' \tag{29}$$

as

$$|\Phi(t_1)\rangle \approx |\Phi(t_0)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1} \hat{H}(t')dt'|\Phi(t_0)\rangle, \tag{30}$$

a close proximity of t_0 in general should comprise the limits of integration. Only if so, equation (30) in general is justified from the mathematical aspect. Actually, it is easy to understand that only for sufficiently short integration interval, it is possible to guarantee convergence of the above solution. The violation of this condition may lead to the divergence of solution. In this connection, it is necessary to improve the successive approximation considered above so as to guarantee convergence. What is best is to prevent the infinity problem.

Let us consider that it is possible to avoid the infinity problem by applying the successive approximation properly. If one first partitions the time interval properly and then makes phased evolution of solutions in every time interval rather than applies the successive approximation using the whole time interval, the desired convergence is possible.

Using equation (16), we take as the first time-evolution

$$\begin{aligned}|\Phi(t_1)\rangle &= |\Phi(t_0)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1} \hat{H}(t')dt'|\Phi(t_0)\rangle \\ &= \left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1} \hat{H}(t')dt'\right]|\Phi(t_0)\rangle,\end{aligned} \tag{31}$$

where $t_1 \in [t_0, t]$ and as the second time-evolution

$$\begin{aligned}|\Phi(t_2)\rangle &= |\Phi(t_1)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2} \hat{H}(t')dt'|\Phi(t_1)\rangle \\ &= \left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2} \hat{H}(t')dt'\right]|\Phi(t_1)\rangle \\ &= \prod_{k=1}^{2}\left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'\right]|\Phi(t_0)\rangle,\end{aligned} \tag{32}$$

where $t_2 \in [t_1, t]$. Let us partition the interval, $[t_0, t]$ into n equal elementary subintervals defined by a set of points $\{t_0, t_1, t_2, \cdots t_n\}$ such that $t_0 = 0 < t_1 < t_2, \cdots < t_n = t$. Then, the state function in general is written as

$$|\Phi(t_n)\rangle = \prod_{k=1}^{n}\left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'\right]|\Phi(t_0)\rangle. \tag{33}$$

With reference to the above equation, the time-evolved state function is represented concisely as

$$|\Phi(t)\rangle = \lim_{n\to\infty}\prod_{k=1}^{n}\left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'\right]|\Phi(t_0)\rangle. \tag{34}$$

From this, we should take the time-evolution operator as

$$S(t,t_0) = \lim_{n\to\infty} \prod_{k=1}^{n} \left[1 - \frac{\mathrm{i}}{\hbar} \int_{t_{k-1}}^{t_k} \hat{H}(t')dt' \right]. \tag{35}$$

We shall refer to this as the local time-evolution operator in the sense that it is represented by integrals in small time intervals. According to the theorem of mean value of integral calculus, we have

$$\int_{t_{k-1}}^{t_k} \hat{H}(t')dt' = \hat{H}(\bar{t}_k)(t_k - t_{k-1}) = \hat{H}(\bar{t}_k)\triangle t, \tag{36}$$

where $\bar{t}_k$ is between $[t_{k-1}, t_k]$. By equation (35), the nth order approximation of the time-evolution operator can be represented as

$$S^{(n)}(t,t_0) = \prod_{k=1}^{n} \left[1 - \frac{\mathrm{i}}{\hbar} \int_{t_{k-1}}^{t_k} \hat{H}(t')dt' \right]. \tag{37}$$

Let us consider the convergence of the scattering matrix represented in terms of the local time-evolution operator. Of course, for a finite partition of the time interval, there is no infinity problem, since it gives the scattering matrix in a finite series. Let us consider the case of infinite partition. We begin with the fact that there exists E_k which satisfies

$$\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'|\Phi(t_0)\rangle = \triangle t\hat{H}(\bar{t}_k)|\Phi(t_0)\rangle = \triangle tE_k|\Phi(t_0)\rangle,$$

namely,

$$\hat{H}(\bar{t}_k)|\Phi(t_0)\rangle = E_k|\Phi(t_0)\rangle,$$

where $\triangle t = \dfrac{t-t_0}{n}$. To clarify this, it is necessary to take into consideration that for an element of the scatering matrix, there holds

$$\langle\Phi(t_k)|\hat{H}(\bar{t}_k)|\Phi(t_0)\rangle = \langle\hat{H}(\bar{t}_k)\Phi(t_k)|\Phi(t_0)\rangle = E_k\langle\Phi(t_k)|\Phi(t_0)\rangle,$$

where we took into consideration that $\hat{H}(\bar{t}_k)$ is Hermitian. Therefore, we should consider that in the sense of the scattering matrix, there holds

$$\hat{H}(\bar{t}_k)|\Phi(t_0)\rangle = E_k|\Phi(t_0)\rangle.$$

In doing so, equation (37) is represented as

$$S^{(n)}(t,t_0) = \prod_{k=1}^{n} \left[1 - \frac{\mathrm{i}}{\hbar} E_k\triangle t \right]. \tag{38}$$

Evidently, of $\{E_k\}$, there exists E_{max} that satisfies

$$\left\| \left[1 - \frac{\mathrm{i}}{\hbar} E_{max}\triangle t \right]^n \right\| \geq \left\| \prod_{k=1}^{n} \left[1 - \frac{\mathrm{i}}{\hbar} E_k\triangle t \right] \right\|. \tag{39}$$

From equation (39), we should examine the convergence of

$$S_m = \left[1 - \frac{\mathrm{i}}{\hbar} E_{max}\frac{t-t_0}{n} \right]^n. \tag{40}$$

Setting $A = -\dfrac{\mathrm{i}}{\hbar}E_{max}(t-t_0)$, equation (40) is reduced to

$$S_m = \left[1 + \frac{A}{n} \right]^n. \tag{41}$$

In virtue of the binomial theorem, equation (41) is expanded into a power series:

$$S_m = \sum_{k=0}^{n} C_n^k \left(\frac{A}{n} \right)^k, \tag{42}$$

where C_n^k are the binomial coefficients. Verifying that S_m is absolutely convergent, namely,

$$S_{abs} = \sum_{k=0}^{n} \left| C_n^k \left(\frac{A}{n} \right)^k \right| \tag{43}$$

is convergent is sufficient for checking that S_m is convergent. After the manner of equation (41), equation (43) again is recast as

$$S_{abs} = \left[1 + \frac{|A|}{n} \right]^n.$$

Setting $x = \dfrac{n}{|A|}$, we immediately obtain

$$S_{abs} = \left[1 + \frac{1}{x} \right]^{x|A|}.$$

Since

$$\lim_{x\to\infty} \left[1 + \frac{1}{x} \right]^{x|A|} = e^{|A|},$$

S_m is absolutely convergent.

Thus, it has been demonstrated that in the case of the infinite partition of a time interval, the local time-evolution operator ensures the convergence of the scattering matrix. This shows that it is possible to avoid the divergence of the scattering matrix by using time evolution based on the partition of a given time interval.

In conclusion, introducing equation (35) instead of the Dyson series, i.e., equation (10) enables us to prevent the infinity problem of the scattering matrix.

4 Global time-evolution operator and analysis of convergence

The Heisenberg picture is an important mathematical formulation for investigating the time evolution of quantum states together with the Schrödinger formulation. In particular, the Heisenberg picture plays a key role in the case of the investigation of the scattering problem. The scattering matrix is closely related to the Heisenberg picture.

It is important to review whether this formulation is possessed of mathematical generality. To begin with, let us consider how the Heisenberg picture is derived. The state function for a microscopic system in the Schödinger picture is determined by

$$\mathrm{i}\hbar\frac{\partial\Phi_S}{\partial t} = \hat{H}\Phi_S. \tag{44}$$

The formal solution to this equation is considered to be

$$\Phi_S\,(t) = e^{-\frac{i}{\hbar}\hat{H}t}\Phi_H, \tag{45}$$

where Φ_H as a time-independent function is defined as the wave function in the Heisenberg picture [35,36]. Here, subscript H refers to the Heisenberg picture. Operator $\hat{S} = e^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}$ is considered to be the unitary operator making transformation between the two representations. By equation (45), the matrix elements of a certain operator $\hat{F}$ are represented as

$$\begin{aligned} F_{mn}(t) &= \left\langle \Phi_{Sm}\,(t)\middle|\hat{F}_S\middle|\Phi_{Sn}\,(t)\right\rangle \\ &= \left\langle \Phi_{Hm}\,(t)\middle|e^{\frac{i}{\hbar}\hat{H}t}\hat{F}_S e^{-\frac{i}{\hbar}\hat{H}t}\middle|\Phi_{Hn}\,(t)\right\rangle. \end{aligned} \tag{46}$$

The last expression in equation (46) can be interpreted as matrix elements of the operator with respect to the wave function in the Heisenberg picture.

Naturally, the expression in equation (46):

$$\hat{F}_H(t) = e^{\frac{\mathrm{i}}{\hbar}\hat{H}t}\hat{F}_S e^{-\frac{\mathrm{i}}{\hbar}\hat{H}t} \tag{47}$$

can be considered the operator in the Heisenberg picture, which is time-dependent unlike the Schrödinger picture.

From equation (47), it follows that the equation of motion in the Heisenberg picture is

$$\mathrm{i}\hbar\frac{\partial\hat{F}_H(t)}{\partial t} = \left[\hat{F}_H(t), \hat{H}\right]. \tag{48}$$

In order to examine whether the Heisenberg picture is correct, it is necessary to obtain the formal solution of the Schrödinger equation in a rigorous way.

It is always possible that the wave function is set as

$$\Phi_S(\mathbf{q},t) = \hat{f}(\mathbf{q},t)\varphi(\mathbf{q}), \tag{49}$$

where $\hat{f}(\mathbf{q},t)$ is an operator dependent on time and position. Thus, the Schrödinger equation (44) is represented as

$$\mathrm{i}\hbar\frac{\partial\left[\hat{f}(\mathbf{q},t)\varphi(\mathbf{q})\right]}{\partial t} = \hat{H}\left[\hat{f}(\mathbf{q},t)\varphi(\mathbf{q})\right]. \tag{50}$$

Since operator $i\hbar\frac{\partial}{\partial t}$ is applied only to $\hat{f}(\mathbf{q},t)$, we have

$$\mathrm{i}\hbar\frac{\partial\hat{f}(\mathbf{q},t)}{\partial t}\varphi(\mathbf{q}) = \hat{H}\hat{f}(\mathbf{q},t)\varphi(\mathbf{q}). \tag{51}$$

For equation (51) to hold, the time-dependent part of the equation except for $\varphi(\mathbf{q})$ should be an operator equation. Therefore, we can imagine the following equation for the operator $\hat{f}(\mathbf{q},t)$:

$$\mathrm{i}\hbar\frac{\partial\hat{f}(\mathbf{q},t)}{\partial t} = \hat{H}\hat{f}(\mathbf{q},t). \tag{52}$$

Here, operator $\hat{f}(\mathbf{q},t)$ is assumed to be able to be treated algebraically, though it is formal to do so.

Thus, we have

$$\frac{1}{\hat{f}(\mathbf{q},t)}\frac{\partial\hat{f}(\mathbf{q},t)}{\partial t} = -\frac{\mathrm{i}}{\hbar}\hat{H}(\mathbf{q},t).$$

By integrating both sides of this equation with respect to time, we get

$$\int_{t_0}^{t}\frac{\partial}{\partial t'}\ln\hat{f}(\mathbf{q},t')dt' = -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'.$$

Consequently, we obtain operator $\hat{f}$ as

$$\hat{f}(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]. \tag{53}$$

where $\hat{f}(\mathbf{q},t)$ is treated as an algebraic function, while operating on $\varphi(\mathbf{q})$. Evidently, equation (53) becomes the improved time-evolution operator which supersedes that of Heisenberg. Accordingly, we arrive at the conclusion that the wave function should be represented formally as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\varphi(\mathbf{q}). \tag{54}$$

For initial condition $t = t_0$, equation (54) reduces to

$$\Phi_S(\mathbf{q},t_0) = \varphi(\mathbf{q}) = \Phi_0(\mathbf{q}) = \Phi_H(\mathbf{q}). \tag{55}$$

Then, the wave function is written as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\Phi_0(\mathbf{q}) \tag{56}$$

or

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\mathit{\Phi}_H(\mathbf{q}). \tag{57}$$

Hence, it follows that the time-evolution operator is

$$S(t,t_0) = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right). \tag{58}$$

This operator is represented by an expansion in the Taylor series as

$$S(t,t_0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right)^n. \tag{59}$$

We shall refer to $S(t,t_0)$ as the global time-evolution operator, since it gives an analytical representation of time evolution from an initial time to a final time, i.e., with respect to the whole interval of time, $[t_0,t]$. Evidently, our formal solution, i.e., equation (57) is substantially distinguished from equation (45)

Interestingly, the global time-evolution operator coincides with equation (25) as an expression of the Dyson series in the case of neglecting the time ordering operator. This fact tells us that if equation (59) is right, it means that the time ordering operator and the Feynman diagram are meaningless and unnecessary.

For the scattering problem, the global time-evolution operator becomes the scattering matrix. Thus, we have

$$\Phi(\mathbf{q},t) = S(t,t_0)\Phi(\mathbf{q},t_0). \tag{60}$$

The substitution of equation 57 into the Schrödinger equation yields

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Phi_S(\mathbf{q},t) = \hat{H}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right]\varphi(\mathbf{q}) = \hat{H}(\mathbf{q},t)\Phi_S(\mathbf{q},t).$$

Therefore, we have

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Phi_S(\mathbf{q},t) = \hat{H}(\mathbf{q},t)\Phi_S(\mathbf{q},t).$$

This confirms that the representation of state function in terms of the global time-evolution operator satisfies the Schrödinger equation.

It should be emphasized that the time-evolution operator is taken as $\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]$ instead of $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)$.

If $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})$ does not satisfy the Schrödinger equation, this is sufficient to confirm that the Heisenberg picture is not perfect.

Substituting $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})$ into the Schrödinger equation proves that this formal solution does not satisfy the equation.

In fact, in view of the time dependence of $\hat{H}$, we get

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})\right] = \left[\hat{H} + t\frac{\partial\hat{H}}{\partial t}\right]\Phi_S(\mathbf{q},t) \neq \hat{H}\Phi_S(\mathbf{q},t). \tag{61}$$

Obviously, the formal solution of the Schrödinger equation $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})$ which is really supposed is not mathematically correct, since it does not satisfy the wave equation itself.

Therefore, it is necessary to adopt the improved representation of time-evolution operator in place of the Heisenberg picture as

$$\hat{F}_H(t) = \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right]\hat{F}_S\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right]. \tag{62}$$

Let us consider that the scattering matrix in terms of the global time-evolution operator is always convergent. From the solution of the Schrödinger equation in terms of the global time-evolution operator:

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right]\Phi_0(\mathbf{q}),$$

we can see that the time-evolution operator, $\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right]$ becomes the scattering operator. Thus, we get the following representation.

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right]\Phi_0(\mathbf{q}) = S(t,t_0)\Phi_0(\mathbf{q}). \tag{63}$$

The global time-evolution operator can be expanded in a Taylor series. The Taylor series expansion of the exponential operator is given as

$$S(t,t_0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{-\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q},t')dt'\right]^n. \tag{64}$$

The expansion of the opertor in a Taylor series makes sense only if it is convergent. Evidently, the number of terms of a series means the order of approximation. It is obvious that a finite order approximation converges.

Now, let us consider that the infinite series of the global time-evolution operator ensures convergence.

First, according to the mean value theorem of integral calculus, we get

$$\int_{t_0}^{t} \hat{H}(\mathbf{q}, t')dt' = \hat{H}(\mathbf{q}, \bar{t})(t - t_0), \tag{65}$$

where $\bar{t}$ becomes a parameter. equation (65) helps us eliminate the integral symbol from the time-evolution operator to get the following representation:

$$\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q}, t')dt'\right] = \exp\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)\hat{H}(\mathbf{q}, \bar{t})\right]. \tag{66}$$

Next, we expand the time-evolution operator as the following Taylor series:

$$S(\mathbf{q}, t) = \sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}(t - t_0)\hat{H}(\mathbf{q}, \bar{t})\right)^n. \tag{67}$$

The solution of the time-dependent Schrödinger equation is represented with the help of the time-evolution operator as

$$\Phi_S(\mathbf{q}, t) = \sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}(t - t_0)\hat{H}(\mathbf{q}, \bar{t})\right)^n \Phi_0(\mathbf{q}). \tag{68}$$

It is possible to verify that in any case, interaction does not give rise to infinity. We represent the Hamilton operator $\hat{H}(\mathbf{q}, \bar{t})$ in equation (68) as the sum of a free part, $\hat{H}_0$ and an interaction part, $\hat{H}_I$, namely,

$$\hat{H}(\mathbf{q}, \bar{t}) = \hat{H}_0 + \hat{H}_I(\mathbf{q}, \bar{t}). \tag{69}$$

Taking into consideration that for operators $\hat{H}_0$ and $\hat{H}(\mathbf{q}, \bar{t})$ the eigen equations hold, namely,

$$\hat{H}_0\Phi_0(\mathbf{q}) = E_0\Phi_0(\mathbf{q}) \tag{70}$$

and

$$\hat{H}(\mathbf{q}, \bar{t})\Phi(\mathbf{q}) = E\Phi(\mathbf{q}), \tag{71}$$

we consider the following integrals:

$$\langle\Phi(\mathbf{q})|\hat{H}_0\Phi_0(\mathbf{q})\rangle = E_0\langle\Phi(\mathbf{q})|\Phi_0(\mathbf{q})\rangle \tag{72}$$

and

$$\langle\Phi_0(\mathbf{q})|\hat{H}(\mathbf{q}, \bar{t})\Phi(\mathbf{q})\rangle = E\langle\Phi_0(\mathbf{q})|\Phi(\mathbf{q})\rangle. \tag{73}$$

Since $\hat{H}(\mathbf{q}, \bar{t})$ is Hermitian, equation (73) becomes

$$\langle\hat{H}(\mathbf{q}, \bar{t})\Phi_0(\mathbf{q})|\Phi(\mathbf{q})\rangle = E\langle\Phi_0(\mathbf{q})|\Phi(\mathbf{q})\rangle. \tag{74}$$

The complex conjugate of equation (74) is

$$\langle\Phi(\mathbf{q})|\hat{H}(\mathbf{q}, \bar{t})\Phi_0(\mathbf{q})\rangle = E\langle\Phi(\mathbf{q})|\Phi_0(\mathbf{q})\rangle. \tag{75}$$

Then, the subtraction of equation (72) from equation (75) yields

$$\langle|\Phi(\mathbf{q})|\left[\hat{H}(\mathbf{q}, \bar{t}) - \hat{H}_0(\mathbf{q})\right]\Phi_0(\mathbf{q})\rangle = (E - E_0)\langle\Phi(\mathbf{q})|\Phi_0(\mathbf{q})\rangle.$$

Taking equation (69) into consideration, we have

$$\langle\Phi(\mathbf{q})|\hat{H}_I(\mathbf{q}, \bar{t})\Phi_0(\mathbf{q})\rangle = \langle\Phi(\mathbf{q})|(E - E_0)\Phi_0(\mathbf{q})\rangle.$$

Hence, we obtain

$$\hat{H}_I(\mathbf{q}, \bar{t})\Phi_0(\mathbf{q}) = (E - E_0)\Phi_0(\mathbf{q}). \tag{76}$$

In the end, we reach the conclusion that since $(E - E_0)$ is finite, $\hat{H}_I(\mathbf{q}\bar{t})\Phi_0(\mathbf{q})$ is finite as well. Thus, it is obvious that $\hat{H}_I(\mathbf{q}, \bar{t})$ is impossible to carry infinity. Even if $\hat{H}_I(\mathbf{q}, \bar{t})$ is represented in the language of quantum field theory, it should give a finite result.

From equations (68) and (76), the solution of the time-dependent Schrödinger equation is represented as

$$\begin{aligned}\Phi_S(\mathbf{q}, t) &= \sum_n \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)\right]^n \left[\hat{H}(\mathbf{q}, \bar{t})\right]^n \Phi_0(\mathbf{q}) \\ &= \sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^n \left[(t - t_0)E\right]^n \Phi_0(\mathbf{q}).\end{aligned} \tag{77}$$

Hence, we can examine the convergence based on the relation:

$$S(t, t_0) = \sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^n \left[(t - t_0)E\right]^n. \tag{78}$$

The following consideration leads to the conclusion that the scattering operator, equation (78) is always convergent. The nth and $n + 1$th terms in series $\sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^n [(t - t_0)E]^n$ read as

$$a(n + 1) = \frac{1}{(n + 1)!}\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)E\right]^{n+1}, \tag{79}$$

$$a(n) = \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)E\right]^n. \tag{80}$$

The ratio between them is

$$\frac{a(n + 1)}{a(n)} = \frac{1}{(n + 1)}\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)E\right]. \tag{81}$$

Since its limit is zero, namely,

$$\lim_{n\to\infty} \frac{a(n + 1)}{a(n)} = 0, \tag{82}$$

according to the convergence condition of infinite series, it follows that series S converges to a finite value. Thus, the use of the global time-evolution operator keeps the scattering matrix from divergence. Hence, it is concluded that for quantum field theory, there is no physical world causing the infinity problem and the problem is nothing but an inevitable result due to the imperfection of the conventional formulation.

5 Conclusion

Our purpose is to present an alternative mathematical formulation capable of solving the divergence problem of the scattering matrix without recourse to renormalization theory. Should it be true, it undoubtedly would be a major advance in the final solution to the infinity problem and the development of quantum field theory.

We demonstrated that the infinity problem is due mainly to the mathematical representation of the scattering operator. Our work has shown that the representation of scattering matrix by means of the method of successive approximation based on the Dyson series and the Feynman diagram is not perfect from the mathematical and physical views.

We have presented an alternative mathematical representation of scattering matrix in terms of the local and global time-evolution operators independent of the Dyson series and the Feynman diagram. Using a mathematically rigorous method, the local time-evolution operator and the global time-evolution operator have been derived, which can supersede the Dyson series and the Heisenberg picture, and importantly, does not need the Feynman diagram. These operators are distinguished from the scattering matrix in terms of the Dyson series by being irrelevant to infinity. The global time-evolution operator which provides the exact solution of the wave equation supersedes the Heisenberg picture.

Within the confines of the present formulation, there does not exist the divergence problem of the scattering matrix. The fact that despite the long study of the infinity problem, we have not yet found a final solution to renormalization needs innovative perspectives which are based on consideration from a new angle. Complying with such a requirement, we have explored the local and global time-evolution operators which always guarantee convergence of the scattering matrix. Our formulation is simple and mathematically uncomplicated, but gives clear explanations of the divergence problems of scattering matrix. Our work has revealed that the Dyson series is not unique, and it is possible to find out more consistent scattering operator than the Dyson series.

Since there is no paper associated with the research direction like ours, we believe that our work will contribute significantly to developing the mathematical formalism of quantum scattering theory in a new direction.

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