A new proof that the reals are uncountable

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Abstract. We show that the reals are uncountable using Russell's Paradox, in a proof reminiscent of Gödel's proof of the Incompleteness Theorem. This simple proof could be offered as an alternative to people who find it difficult to accept Cantor's Diagonalization argument.

Let us assume, for the sake of contradiction, that the real numbers between 0 and 1 (inclusive) are countable. Since *"…the continuum of numbers, or real numbers system … is the totality of infinite decimals,"* [1] then a listing of the real numbers between 0 and 1 in base 3 (a ternary numeral system) would have all possible sequences of the digits 0, 1, and 2 after the radix point.[2]

Without loss of generality, we will let the following be true of this countable list:

- Each ternary expansion represents the "text" of a book.
- Every digit 2 is replaced by a space to separate "words" without digit 2. Multiple spaces are replaced with a single space.
- The ternary expansions encompass all possible "words" in all possible books. A "word" is a base 2 combination of the digits 0 and 1 for a number (*i.e*, the totality of words in base 2 for all possible books is given by this list).
- Each "word" of a book that is a finite number in base 2 refers to the position of a book (itself or another) on the countable list. Therefore, books are identified by their position on the countable list, as opposed to a book title.
- There is a book *among all possible books* that lists all the books that do not list themselves.

The last bullet is similar to Russell's Paradox.[3] Does this book that catalogues "all the books that do not include themselves" include itself? If it does, then it must not list itself in the book. If it doesn't, then it must include itself in the list. This is a contradiction and *undecideable*.

So, there is one book (a sequence of digits representing a real number) which is *undecidable* when the sequence is given meaning (*i.e.*, books as described above).[4] Therefore, no countable listing of all the real numbers can be complete, since one real number would represent something that is undecidable. Accordingly, the real numbers between 0 and 1 cannot be countable, since a contradiction would result.

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[1] Courant, Richard, and Herbert Robbins. 1996. *What Is Mathematics? : An Elementary Approach to Ideas and Methods*. Oxford: Oxford University Press. p.68.

[2] Note that each real number is not expressed uniquely on the list (*i.e.*, since 1 = 0.222… in base 3). However, the list can be converted to a countably unique list by going through the list and eliminating duplicate real numbers that end in 222…

[3] Weisstein, Eric W. "Russell's Antimony." From *MathWorld*—A Wolfram Web Resource. https://mathworld.wolfram.com/RussellsAntinomy.html (accessed Dec. 18, 2021).

[4] This is reminiscent of Kurt Gödel's 1931 mathematically rigorous proof that:

> … we could not set out a complete set of rules for arithmetic. Gödel showed that we could use any set of possible rules to create sentences similar to the sentence, "this sentence is false." If it's true, then it's false. But if it's false, then it's true. Any attempt to create rules would either allow sentences like, "this sentence is unprovable" to be proven. And so, we would have sentences that can be proved but are false. We would have just proven the sentence that says it can't be proven. Alternatively, we could strengthen our rules to exclude these sentences. But then, because we could no longer prove the sentence, the sentence, "this sentence is unprovable" would be true. And so, we would have true sentences that we can't prove in our system, making our system incomplete.

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