Motion in a gravitational field in second approximation

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Abstract. We calculate the equation of motion of a body in a gravitational field in the second-order approximation in the inverse of *c*. We express the result as a function of proper distance and proper time. Finally, we comment the results found by other authors.

1. Introduction

To find the equation of motion of a test particle in a gravitational field, we previously solve Einstein's gravitation equations, and then we follow one of the following two methods: find the Lagrangian of the particle from the line element and apply the Euler-Lagrange equation; or find the Christoffel symbols and apply the geodesic equation.

For a weak gravitational field, we develop the metric tensor components in power series of the inverse of the velocity of light in vacuum *c*. The zero-order approximation is Newtonian. In this investigation, we are interested in the following approximation, the one of order two in the inverse of *c*.

The authors who have dealt with this problem have expressed the equation of motion as a function of space-time coordinates, which do not have a defined physical meaning. Following Brans (1962), we express the equation of motion as a function of proper distance and proper time, which are directly measurable magnitudes.

2. Proper time

The coordinate time *t* is a value arbitrarily associated with an event, not a real-time measure. A physical clock measures proper time; therefore, it is a real measure of time.

In General Relativity, there are three proper times:

-*Proper time measured by a fixed observer in the field*, which is related to the time coordinated by

$$d\tau_f = \sqrt{g_{00}}\,dt$$

g_{00} is the component 0,0 of the metric tensor.

-*Synchronized proper time* elapsed between two events separated by an infinitesimal distance dx^α (Landau-Lifshitz, 1970, 349-353), (Rizzi-Ruggiero, 2004, 179-205) and is the difference between the proper times of synchronized clocks located where the events occur

$$d\tau = \sqrt{g_{00}}\left(dt - \frac{\gamma_\alpha}{c}dx^\alpha\right)$$

γ_α is defined by

$$\gamma_\alpha = -\frac{g_{0\alpha}}{g_{00}}$$

$g_{0\alpha}$ is the component $0,\alpha$ of the metric tensor (the Greek indices represent the spatial components from 1 to 3; the index 0 corresponds to the temporal component, and the

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Latin indices correspond to four-dimensional components from 0 to 3).

-*Proper time of the particle* is the time marked by a clock that moves with the particle in the gravitational field and that we then calculate.

3. Proper time of the particle

We define the spatiotemporal line element by

$$ds^2 = g_{00}\left(dx^0\right)^2 + 2g_{0\alpha}dx^\alpha dx^0 + g_{\alpha\beta}dx^\alpha dx^\beta = g_{00}\left(dx^0 - \gamma_\alpha dx^\alpha\right)^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta \tag{1}$$

$\gamma_{\alpha\beta}$ is the metric tensor of three-dimensional space, related to the four-dimensional metric tensor by

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}},$$

and $d\sigma^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta$ is the square of the proper distance between the two positions in which the two events occur whose spatiotemporal interval is *ds*, then

$$ds^2 = c^2 d\tau^2 - d\sigma^2.$$

We define the three-dimensional velocity concerning the synchronized proper time by

$$v'^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{\sqrt{g_{00}}\left(dt - \frac{\gamma_\alpha}{c}dx^\alpha\right)} \tag{2}$$

then

$$v' = \sqrt{\gamma_{\alpha\beta}v'^\alpha v'^\beta} = \frac{d\sigma}{\sqrt{g_{00}}\left(dt - \frac{\gamma_\alpha}{c}dx^\alpha\right)} \quad \Rightarrow \quad d\sigma^2 = g_{00}\left(dt - \frac{\gamma_\alpha}{c}dx^\alpha\right)^2 v'^2,$$

substituting in (1), we calculate the proper time of the particle

$$ds^2 = c^2 g_{00}\left(dt - \frac{\gamma_\alpha}{c}dx^\alpha\right)^2\left(1 - \frac{v'^2}{c^2}\right) \quad \Rightarrow \quad d\tau_p = \sqrt{g_{00}}\left(dt - \frac{\gamma_\alpha}{c}dx^\alpha\right)\sqrt{1 - \frac{v'^2}{c^2}}. \tag{3}$$

4. Proper time in second approximation

The components $g_{0\alpha}$ of the metric tensor are of third order with respect to the inverse of the velocity of light; therefore, we can neglect them in the second approximation

$$d\tau_p \approx \sqrt{g_{00}}\sqrt{1 - \frac{v'^2}{c^2}}\,dt. \tag{4}$$

As we will see later, the component 0,0 in the weak field approximation is

$$g_{00} = 1 + \frac{2\Phi}{c^2}$$

Φ is a function of position and time and that is not equal to the Newtonian potential ϕ. Then the particle's proper time in the second approach is

$$d\tau_p \approx \sqrt{1 - \frac{v'^2}{c^2} + \frac{2\Phi}{c^2}}\,dt. \tag{5}$$

In the same approximation, the synchronized proper time is

$$d\tau \approx d\tau_f \approx \sqrt{g_{00}}\,dt = \sqrt{1 + \frac{2\Phi}{c^2}}.$$

5. Three-dimensional velocity definitions

We can define three-dimensional velocity in several ways:

-*Coordinate velocity*

$$v^\alpha = \frac{dx^\alpha}{dt}.$$

-*Velocity with respect to synchronized proper time*

$$v'^{\alpha} = \frac{dx^{\alpha}}{d\tau}.$$

-*Velocity with respect the proper time of the field*

$$v''^{\alpha} = \frac{dx^{\alpha}}{d\tau_f}.$$

-*Velocity with respect the proper time of the particle*

$$w'^{\alpha} = \frac{dx^{\alpha}}{d\tau_p}.$$

-*Spatial components of tetravelocity*

$$u^k = \frac{dx^k}{d\tau_p} \quad \Rightarrow \quad u^k = \left(\frac{dx^0}{d\tau_p}, \frac{v'^{\alpha}}{\sqrt{1 - v'^2/c^2}} \right) \quad \Rightarrow \quad u^{\alpha} = \frac{v'^{\alpha}}{\sqrt{1 - v'^2/c^2}},$$

the fourth component of the tetravelocity can be put (Landau-Lifshitz, 1970, p. 251)

$$u^0 = \frac{c}{\sqrt{g_{00}}\sqrt{1 - v'^2/c^2}} + \frac{\gamma_{\alpha} v'^{\alpha}}{\sqrt{1 - v'^2/c^2}}.$$

-*Proper velocity*. The proper distance between two fixed points with respect to the observer is determined by physical methods. In the second-order approximation, the three-dimensional metric tensor is

$$\gamma_{\alpha\beta} \approx \delta_{\alpha\beta} \left(1 - \frac{2\Phi}{c^2} \right) \quad \Rightarrow \quad d\sigma^2 = \left(1 - \frac{2\Phi}{c^2} \right) \sum_{\alpha} \left(dx^{\alpha} \right)^2.$$

We define the components of the three-dimensional proper distance vector by

$$d\sigma^{\alpha} = \sqrt{1 - \frac{2\Phi}{c^2}}\, dx^{\alpha} \quad \Rightarrow \quad d\sigma^2 = \sum_{\alpha} \left(d\sigma^{\alpha} \right)^2.$$

With this result we define the proper velocity by *

$$w^{\alpha} = \frac{d\sigma^{\alpha}}{d\tau} \quad \Rightarrow \quad w^2 = \sum_{\alpha} \left(w^{\alpha} \right)^2.$$

In the second-order approximation, the relationships between the velocities defined above are

$$v'^{\alpha} = v''^{\alpha} = \frac{v^{\alpha}}{\sqrt{1 + 2\Phi/c^2}}; \qquad w^{\alpha} = \frac{\sqrt{1 - 2\Phi/c^2}}{\sqrt{1 + 2\Phi/c^2}} v^{\alpha};$$

$$w'^{\alpha} = u^{\alpha} = \frac{v^{\alpha}}{\sqrt{1 + 2\Phi/c^2}\sqrt{1 - v'^2/c^2}}$$

We find the square of the velocities with the metric tensor of three-dimensional space

$$v'^2 = \gamma_{\alpha\beta} v^{\alpha} v^{\beta}; \qquad w'^2 = \gamma_{\alpha\beta} w'^{\alpha} w'^{\beta}; \qquad w^2 = \sum_{\alpha} \left(w^{\alpha} \right)^2.$$

6. Lagrangian of a particle in a gravitational field

The Lagrangian L of a particle of mass m is

$$L = -mc \frac{ds}{dt},$$

for example, for (3) the Lagrangian is

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* As we have defined it, the components of the proper velocity are possible with the harmonic coordinates we are using. Otherwise, it would only be possible to define the module of the proper velocity $w = d\sigma/d\tau$.

$$L=-mc^2\sqrt{g_{00}}\left(1-\frac{\gamma_\alpha}{c}\frac{dx^\alpha}{dt}\right)\sqrt{1-\frac{v'^2}{c^2}}\,. \tag{6}$$

From the Lagrangian, we find the equation of motion by applying the Euler-Lagrange equation. However, we can apply various Euler-Lagrange equation depending on the functional dependence of the Lagrangian. For example, if $L=L\left(v^\alpha,x^\alpha,t\right)$

$$\frac{d}{dt}\left[\frac{\partial L\left(v^\alpha,x^\alpha,t\right)}{\partial v^\alpha}\right]-\frac{\partial L\left(v^\alpha,x^\alpha,t\right)}{\partial x^\alpha}=0. \tag{7}$$

If the Lagrangian were a function of other spatial and temporal coordinates, we would have to make the appropriate changes in (7).

We can also find the equations of motion by the equation of the geodesic

$$\frac{d^2x^k}{d\tau_p^2}+\Gamma^k_{ij}\frac{dx^i}{d\tau_p}\frac{dx^j}{d\tau_p}=0 \qquad \text{or} \qquad \frac{d}{d\tau_p}\left(g_{ik}\frac{dx^k}{dt}\right)=\frac{1}{2}\frac{\partial g_{pq}}{\partial x^i}\frac{dx^p}{d\tau_p}\frac{dx^q}{d\tau_p} \tag{8}$$

which is a method equivalent to (7), and therefore gives the same results.

7. Equation of motion in second approximation

The method we use to determine the equation of motion of a particle in a weak gravitational field consists of developing the metric tensor and the components of the Christoffel symbols in inverse powers of the velocity of light.

As demonstrated in (Weinberg, 1972, 209-233) and (Segura, 2013, 41-48), the terms that interest us in harmonic coordinates are

$$g_{00}\approx 1+\overset{(2)}{g}_{00}+\overset{(4)}{g}_{00}; \qquad g_{0\alpha}\approx\overset{(3)}{g}_{0\alpha}; \qquad g_{\alpha\beta}\approx-\delta_{\alpha\beta}\left(\alpha\neq\beta\right); \qquad g_{\alpha\alpha}=-1+\overset{(2)}{g}_{\alpha\alpha} \tag{9}$$

the number in parentheses means the dependence of the term on the inverse of the velocity of light. For example, if it is 2, the term depends on $1/c^2$.

We find the equation of motion by the geodesic equation (8), for which we previously found the components of the Christoffel symbols. Three gravitational potentials are defined: the scalar potentials ϕ and ψ and the vector potential **A** by the relations

$$\overset{(2)}{g}_{00}=\frac{2\phi}{c^2}; \qquad \overset{(4)}{g}_{00}=\frac{2\phi^2}{c^4}+\frac{2\psi}{c^4}; \qquad \overset{(2)}{g}_{\alpha\beta}=\delta_{\alpha\beta}\frac{2\phi}{c^2}; \qquad \overset{(3)}{g}_{0\alpha}=-\frac{4}{c}A^\alpha\,, \tag{10}$$

ϕ is the Newtonian potential, and ψ is the second scalar potential. The fourth-order component of g_{00} may seem unnecessary; however, when determining the equation of motion, the component 0,0 of the metric tensor must be multiplied by c^2, then the fourth-order components become second-order terms. We determined the three potentials by the field equations *.

The resulting equation of motion is

$$\frac{d\mathbf{v}}{dt}=-\nabla\phi-4\frac{\partial\mathbf{A}}{\partial t}+4\mathbf{v}\wedge\left(\nabla\wedge\mathbf{A}\right)-\nabla\left(\frac{2\phi^2}{c^2}+\frac{\psi}{c^2}\right)+3\frac{\mathbf{v}}{c^2}\frac{\partial\phi}{\partial t}+4\frac{\mathbf{v}}{c^2}\left(\mathbf{v}\cdot\nabla\right)\phi-\frac{v^2}{c^2}\nabla\phi, \tag{11}$$

v is the coordinate velocity. The first addend is of zero order and corresponds to the Newtonian solution, all the remaining addends are of second order and can have similar values, so it is impossible to neglect any of them.

The line element corresponding to the second-order approximation and that we find from (9) and (10) using Cartesian harmonic coordinates is

$$ds^2=\left(1+\frac{2\phi}{c^2}+\frac{2\phi^2}{c^4}+\frac{2\psi}{c^4}\right)c^2dt^2-\frac{8}{c}\mathbf{A}\cdot d\mathbf{r}cdt-\left(1-\frac{2\phi}{c^2}\right)\left[\left(dx\right)^2+\left(dy\right)^2+\left(dz\right)^2\right] \tag{12}$$

$d\mathbf{r}=\left(dx,dy,dz\right)$. Expression (12) will be different for other coordinates.

The proper time of the particle derived from (12) in the second-order approximation is

* The term $2\phi^2/c^4$ in $\overset{(4)}{g}_{00}$ may seem strange and arbitrary. However, this decomposition makes it easier to solve the field equation in the weak approximation (see below).

$$d\tau_p = \left[1 + \frac{2\phi}{c^2} - \frac{v^2}{c^2} + \frac{2\phi^2}{c^4} + \frac{2\psi}{c^4} - \frac{8}{c^2}\mathbf{A}\cdot\mathbf{v} + \frac{2\phi}{c^4}v^2\right]^{\frac{1}{2}} dt,$$

with the approximation valid for $x \ll 1$

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

we find

$$L = -mc^2\frac{d\tau_p}{dt} = -mc^2 - m\phi - m\frac{\psi}{c^2} + \frac{1}{2}mv^2 + \frac{1}{8}m\frac{v^4}{c^2} + 4m\mathbf{A}\cdot\mathbf{v} - \frac{3}{2}m\phi\frac{v^2}{c^2} - \frac{1}{2}m\frac{\phi^2}{c^2} \quad (13)$$

applying the Euler-Lagrange equation (7) (Segura, 2013, 48-60), we return to find the equation of motion (11).

8. Equation of motion with proper distance and time

Equation (11) is a function of the spatial and temporal coordinates, but these coordinates are not directly observable; for this reason, we put (11) as a function of proper distance and synchronized proper time, which are the magnitudes that an observer measures of motion of the particle subjected to gravitoelectromagnetic potentials (Brans, 1962).

The acceleration that has to appear in (11) is the proper acceleration

$$\frac{d^2\sigma^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d\sigma^\alpha}{d\tau}\right) = \frac{1}{\sqrt{1+\frac{2\Phi}{c^2}}}\frac{d}{dt}\left[\left(1-\frac{2\Phi}{c^2}\right)\frac{dx^\alpha}{dt}\right] =$$

$$= \left(1-\frac{\Phi}{c^2}\right)\left[\left(-\frac{2}{c^2}\frac{\partial\Phi}{\partial x^\beta}\frac{dx^\beta}{dt} - \frac{2}{c^2}\frac{\partial\Phi}{\partial t}\right)\frac{dx^\alpha}{dt} + \left(1-\frac{2\Phi}{c^2}\right)\frac{d^2x^\alpha}{dt^2}\right],$$

in vector notation

$$\frac{d^2\boldsymbol{\sigma}}{d\tau^2} = -\frac{2}{c^2}\mathbf{v}(\mathbf{v}\cdot\nabla)\Phi - \frac{2}{c^2}\mathbf{v}\frac{\partial\Phi}{\partial t} + \left(1-\frac{3\Phi}{c^2}\right)\frac{d\mathbf{v}}{dt} \quad (14)$$

in which there are no fourth-order terms of the inverse of ligth velocity.

The differential operator nabla has to be a function of proper distance, then

$$\nabla = \left(\frac{\partial}{\partial x^\alpha}\right)_t = \frac{\partial\sigma^\beta}{\partial x^\alpha}\left(\frac{\partial}{\partial\sigma^\beta}\right)_t = \left(1-\frac{\Phi}{c^2}\right)\nabla'$$

∇' is a function of proper coordinates σ^α. We note that in equation (11), we can substitute, within the second order approximation, the proper velocity **w** by the coordinate velocity **v** and *t* by the proper time τ. Then by (14) and taking into account that $\Phi = \phi + \phi^2/c^2 + \psi/c^2$ we find

$$\frac{d\boldsymbol{\sigma}}{d\tau} = -\nabla'\phi - 4\frac{\partial\mathbf{A}}{\partial\tau} + 4\mathbf{w}\wedge(\nabla'\wedge\mathbf{A}) - \frac{1}{c^2}\nabla'\psi + \frac{\mathbf{w}}{c^2}\frac{\partial\phi}{\partial\tau} + 2\frac{\mathbf{w}}{c^2}(\mathbf{w}\cdot\nabla')\phi - \frac{w^2}{c^2}\nabla'\phi, \quad (15)$$

wich is the equation of motion in a gravitational field in the second approximation (the first approximation is the Newtonian one) as a function of the proper time and distance,

We call inductive terms those that depend on the velocity of the source; that is, the second, third, and fourth terms of (15). We call the third term of (15) gravitomagnetic. We must pay attention to all of the addends of (15) because all of them can have values of the same order. With the equation of motion (15), we find the correct value of the Einstein planetary perturbation or precession of the pericenter (Segura, 2013, 107-110) *.

9. Weak Gravitational Field Equations

We find the gravitational potentials with Einstein's field equation

* In this reference, we find the precession of the pericenter of a planet using equation (11). We obtain the same result with (15).

$$R_{ik}-\frac{1}{2}g_{ik}R=-\chi T_{ik} \tag{16}$$

which we can solve in the linear approximation. However, the last term of (13) is nonlinear in potentials, so (16) must be solved by the weak field procedure. As demonstrated in (Segura, 2013, pp. 59-60), (16) in the weak field approximation is

$$\begin{aligned}
&\nabla^2 \overset{(2)}{g}_{00}+\nabla^2 \overset{(4)}{g}_{00}=\chi \overset{(-2)}{T}{}^{00}+\frac{1}{c^2}\frac{\partial^2 \overset{(2)}{g}_{00}}{\partial t^2}-\overset{(2)}{g}_{\alpha\beta}\frac{\partial^2 \overset{(2)}{g}_{00}}{\partial x^\alpha \partial x^\beta}+\left(\frac{\partial \overset{(2)}{g}_{00}}{\partial x^\alpha}\right)^2+\chi\left(\overset{(0)}{T}{}^{00}+2\overset{(2)}{g}_{00}\overset{(-2)}{T}{}^{00}+\overset{(0)}{T}{}^{\alpha\alpha}\right)\\
&\nabla^2 \overset{(3)}{g}_{0\alpha}=-2\chi \overset{(-1)}{T}{}^{0\alpha}\\
&\nabla^2 \overset{(2)}{g}_{\alpha\beta}=\chi\delta_{\alpha\beta}\overset{(-2)}{T}{}^{00}\ .
\end{aligned} \tag{17}$$

The non-linear term ϕ^2 in (13) deserves a comment, this term does not appear in linear theory, but it is a term that we have to consider in the second order approximation. However, it disappears when we calculate the proper acceleration (15). In other words, equation (15) is the same as the one obtained with the linear theory (which does not take into account the term ϕ^2); in addition, the sec ond scalar potential ψ found by (17) is identical to the one found by the linear theory. Therefore, we find the same equation (15) with the linear theory, which we understand as a circumstantial fact.

The equation of motion must be invariant in the recalibration of the potential; that is to say, that the origin of the potential is arbitrary, and only the differences of potential make physical sense and not the potential itself; this means that the equation of motion has to be linear with respect to potential, which is not the case in (11). However, as indicated, (11) is a non-physical equation because the coordinate acceleration is not directly measurable. However, (15) is invariant on the recalibration of the potential because it is a physical equation.

10. Others equations of motion in second approximation

Various equations of motion applicable to weak fields have been proposed, but not all are correct. Let us look at some of these proposals below.

-*Lagrangian in the linear approximation.* From the linearized gravitational field equations, we determine the line element and the corresponding Lagrangian (Segura, 2013, 41-48)

$$ds^2=\left(1+\frac{2\phi}{c^2}+\frac{2\psi}{c^4}\right)c^2dt^2-\frac{8}{c}\mathbf{A}\cdot d\mathbf{r}cdt-\left(1-\frac{2\phi}{c^2}\right)\left(dx^2+dy^2+dz^2\right)$$

$$L=-mc^2-m\phi-m\frac{\psi}{c^2}+\frac{1}{2}mv^2+\frac{1}{8}m\frac{v^4}{c^2}+4m\mathbf{A}\cdot\mathbf{v}-\frac{3}{2}m\phi\frac{v^2}{c^2},$$

all terms are linear with respect to potentials. Applying the Euler-Lagrange equation (7), we find the second-order equation of motion as a function of the coordinate velocity

$$\frac{d\mathbf{v}}{dt}=-\nabla\phi-4\frac{\partial\mathbf{A}}{\partial t}+4\mathbf{v}\wedge(\nabla\wedge\mathbf{A})-\frac{1}{c^2}\nabla\psi+3\frac{\mathbf{v}}{c^2}\frac{\partial\phi}{\partial t}+4\frac{\mathbf{v}}{c^2}(\mathbf{v}\cdot\nabla)\phi-\frac{v^2}{c^2}\nabla\phi$$

from where we deduce the proper acceleration and not considering ϕ^2 because it is nonlinear

$$\frac{d^2\boldsymbol{\sigma}}{d\tau^2}=-\nabla'\phi-4\frac{\partial\mathbf{A}}{\partial t}+4\mathbf{w}\wedge(\nabla'\wedge\mathbf{A})-\frac{1}{c^2}\nabla\psi+\frac{\mathbf{w}}{c^2}\frac{\partial\phi}{\partial t}+2\frac{\mathbf{w}}{c^2}(\mathbf{w}\cdot\nabla')\phi-\frac{w^2}{c^2}\nabla'\phi$$

which matches (15); however, it should be noted that the procedure followed is unsatisfactory because it neglects non-linear terms that produce second-order effects. However, as we have stated before when calculating the proper acceleration, the term that contains ϕ^2 disappears.

-*Lense-Thirring Lagrangian*. The line element and the corresponding Lagrangian is

$$\begin{aligned}
&ds^2=\left(1+\frac{2\phi}{c^2}\right)c^2dt^2-\frac{8}{c}\mathbf{A}\cdot d\mathbf{r}cdt-\left(1-\frac{2\phi}{c^2}\right)\left(dx^2+dy^2+dz^2\right)\\
&L=-mc^2-m\phi+4m\mathbf{A}\cdot\mathbf{v}+\frac{1}{2}mv^2+\frac{1}{8}m\frac{v^4}{c^2}+\frac{1}{2}m\frac{\phi^2}{c^2}-\frac{3}{2}m\frac{v^2}{c^2}\phi
\end{aligned} \tag{18}$$

these equations do not consider the fourth-order term of the component g_{00}, which, as we have said, gives rise to second-order terms in the equation of motion.

We can put the Lagrangian in the linear order as (Mashhoon, 2003)

$$L = -mc^2\sqrt{1 - v^2/c^2} - m\phi\frac{1 + v^2/c^2}{\sqrt{1 - v^2/c^2}} + 4m\mathbf{A}\cdot\mathbf{v}.$$

From (18) we find the equation of motion as a function of the coordinate velocity

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi - 4\frac{\partial\mathbf{A}}{\partial t} + 4\mathbf{v}\wedge(\nabla\wedge\mathbf{A}) - \frac{1}{c^2}\nabla\phi^2 + 3\frac{\mathbf{v}}{c^2}\frac{\partial\phi}{\partial t} + 4\frac{\mathbf{v}}{c^2}(\mathbf{v}\cdot\nabla)\phi - \frac{v^2}{c^2}\nabla\phi,$$

or proper acceleration

$$\frac{d^2\boldsymbol{\sigma}}{d\tau^2} = -\nabla'\phi - 4\frac{\partial\mathbf{A}}{\partial\tau} + 4\mathbf{w}\wedge(\nabla'\wedge\mathbf{A}) + \frac{1}{c^2}\nabla'\phi^2 + \frac{\mathbf{w}}{c^2}\frac{\partial\phi}{\partial\tau} + 2\frac{\mathbf{w}}{c^2}(\mathbf{w}\cdot\nabla')\phi - \frac{w^2}{c^2}\nabla'\phi,$$

which have small differences with (11) and (15).

It is interesting to note that we can derive the gravitomagnetic potential **A** from the Lorentz transformation. Indeed, let us consider a particle of mass M at rest at the origin of an inertial reference frame K'. For this frame, the line element in the weak field approximation is

$$ds^2 = \left(1 + \frac{2\phi'}{c^2}\right)c^2dt'^2 - \left(1 - \frac{2\phi'}{c^2}\right)\left(dx'^2 + dy'^2 + dz'^2\right) \quad (19)$$

ϕ' is the gravitational potential produced by the particle M measured in K'. Let us suppose a new inertial reference frame K with respect to which K' moves with a speed u in the positive direction of the x-axis. If we neglect u^2/c^2, that is, we assume that the particle that creates the field is non-relativistic, the Lorentz transformation equations are

$$x' \approx x - \frac{u}{c}ct, \qquad ct' \approx ct - \frac{u}{c}x$$

substituting in (19), defining the gravitomagnetic potential by

$$\mathbf{A} = \frac{\phi\mathbf{u}}{c^2}.$$

and taking into account that in the second-order approximation $\phi' \approx \phi$ *, we found (18) (Malekolkalmi-Farhoudi, 2006), (Strel'tsov, 1993) and (McDonald, 2017).

-*Einstein*. In the famous book *The Meaning of Relativity*, Einstein obtained from the linear theory of gravitation the equation of motion in the second order approximation, which in the notation we are using, is

$$\frac{d}{dt}\left[\left(1 - \frac{\phi}{c^2}\right)\mathbf{v}\right] = -\nabla\phi - 4\frac{\partial\mathbf{A}}{\partial t} + 4\mathbf{v}\wedge(\nabla\wedge\mathbf{A}) \quad (20)$$

or depending on the proper coordinates

$$\frac{d^2\boldsymbol{\sigma}}{d\tau^2} = -\nabla'\phi - 4\frac{\partial\mathbf{A}}{\partial\tau} + 4\mathbf{w}\wedge(\nabla'\wedge\mathbf{A}) - \frac{1}{c^2}\mathbf{w}(\mathbf{w}\cdot\nabla')\phi - \frac{\mathbf{w}}{c^2}\frac{\partial\phi}{\partial\tau} + \frac{3}{2c^2}\nabla'\phi^2$$

which differs from (15). The deduction of (20) needs to be corrected since it neglects components of the Christoffel symbols that must be considered, as well as the fourth-order components 0,0 of the metric tensor (Einstein, 1971, 119-124).

- *Davidson*. Davidson (1957) warned that equation (20) was wrong and instead derived equation

$$\frac{d}{dt}\left[\left(1 - \frac{3\phi}{c^2}\right)\mathbf{v}\right] = -\nabla\phi - 4\frac{\partial\mathbf{A}}{\partial t} + 4\mathbf{v}\wedge(\nabla\wedge\mathbf{A}) \quad (21)$$

neglecting terms containing v^2/c^2. From (21), we find the equation of motion as a function of the coordinate velocity and the proper velocity

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi - 4\frac{\partial\mathbf{A}}{\partial t} + 4\mathbf{v}\wedge(\nabla\wedge\mathbf{A}) - \frac{3}{2c^2}\nabla\phi^2$$

* Like electromagnetism, we can define a gravitational tetra potential $\phi^k = (\phi, c\mathbf{A})$ from which we deduce the gravitational potential transformation law.

$$\frac{d^2\boldsymbol{\sigma}}{d\tau^2} = -\nabla'\phi - 4\frac{\partial \mathbf{A}}{\partial t} + 4\mathbf{w} \wedge (\nabla \wedge \mathbf{A}) + \frac{1}{2c^2}\nabla'\phi^2 .$$

expressions that also differ of (11) and (15).

- *Pascual-Sánchez*. Applying the linearized equations and for the case of potentials independent of time, Pascual-Sánchez (2000) finds the equation of motion from the geodesic equation, obtaining

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi + 4\mathbf{v} \wedge (\nabla \wedge \mathbf{A}) \tag{22}$$

from here we deduce

$$\frac{d^2\boldsymbol{\sigma}}{d\tau^2} = -\nabla'\phi + \frac{2}{c^2}\nabla'\phi^2 + 4\mathbf{w} \wedge (\nabla' \wedge \mathbf{A}) - \frac{2}{c^2}\mathbf{w}(\mathbf{w}\cdot\nabla')\phi$$

which is very different of (15). Remember that the equations of motion dependent on ϕ^2 are not physically satisfactory.

- *Ruggiero-Tartaglia*. These authors also find (22) using the geodesic equation and the linearized theory. They assume that the fields are static and neglect terms containing v^2/c^2 . Like all the other authors, they calculate the coordinate acceleration, which, as we have said, is not a directly measurable quantity (Ruggiero-Tartaglia, 2002).

Even with the simplifications above, equation (22) does not coincide with (11) since it lacks the second scalar potential ψ which gives rise to second-order terms in the equation of motion.

-*Ciufolini-Wheeler*. These authors affirm that for the case of a weak field and slow motion, it is fulfilled (22), although they do not require the condition of constancy of the field (Ciufolini-Wheeler, 1995, 315-326).

-*Landau-Lifshitz*. These authors suppose that the gravitational field is constant, that is, it is possible to choose a reference system where the metric tensor does not depend on time. They define the spatial components of the force as

$$f^\alpha = c\sqrt{1 - v'^2/c^2}\,\frac{Dp^\alpha}{d\tau_p} = c\frac{Dp^\alpha}{d\tau},$$

D is the covariant derivative for the three-dimensional metric tensor, then from the geodesic equation, they find, with no other limitation than the independence of time of the metric tensor, that the equation of motion is

$$\mathbf{f} = \frac{mc^2}{\sqrt{1 - v'^2/c^2}}\left[-\nabla \ln\sqrt{g_{00}} + \sqrt{g_{00}}\,\frac{\mathbf{v}}{c} \wedge (\nabla \wedge \mathbf{g})\right] \tag{23}$$

$\mathbf{g} = (\gamma_\alpha)$, $\mathbf{v} = (v^\alpha)$ and $\mathbf{f} = (f_\alpha)$. Developing (23) in the second-order approximation, we find that it agrees with (11) in the time-independent field assumption (Landau-Lifshitz, 1970, 349-353).

-*Weinberg*. In the second-order approximation, Weinberg (1972, 209-233) finds (11) and (17). When integrating the first equation (17), he divides it into two differential equations according to their order.

* *Rizzi-Ruggiero*. These authors find an equation of motion formally similar to the Lorentz force (Rizzi-Ruggiero, 2004). They define the three-dimensional force similarly to Landau-Lifshitz, with the three-dimensional covariant derivative concerning the synchronized proper time

$$f_\alpha = \frac{Dp_\alpha}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}\frac{D}{d\tau}(mv'_\alpha)$$

$v'_\alpha = \gamma_{\alpha\beta}v'^\beta$ is the velocity with respect to the synchronized proper time, and *m* is the mass at rest. The equation of motion is

$$f_\alpha = \frac{m}{\sqrt{1 - v^2/c^2}}\left[-E_\alpha + \sqrt{g_{00}}\left(\frac{\mathbf{v}'}{c} \wedge \mathbf{B}\right)_\alpha\right] \tag{24}$$

with definitions

$$\mathbf{v}' = (v'^\alpha); \quad -\frac{\phi}{c^2} = \frac{1}{\sqrt{g_{00}}}; \quad A = (A_\alpha) = \left(-c^2\frac{g_{0\alpha}}{g_{00}}\right); \quad \tilde{\partial}_\alpha = \partial_\alpha - \frac{g_{o\alpha}}{g_{00}}\partial_0 ;$$

$$E_\alpha = \tilde{\partial}_\alpha \phi + \tilde{\partial}_0 A_\alpha ; \quad \mathbf{B} = \left(B^\alpha\right) = \tilde{\nabla} \wedge \mathbf{A}.$$

In second approximation and assuming time-independent metric, (24) coincides with (23), which, as we have already said, reduces to (11).

-*Pérez-Espinoza-Chubykalo*. Without claiming to be exhaustive, we add the recent research by (Pérez-Espinoza-Chubykalo, 2020), who, following the techniques exposed by Weinberg, found the equation of motion

$$\frac{d}{dt}\left[\left(1-\frac{3\phi}{c^2}+\frac{1}{2}\frac{v^2}{c^2}\right)\mathbf{v}\right]=\left(1-\frac{\phi}{c^2}+\frac{1}{2}\frac{v^2}{c^2}\right)\left[-\nabla\phi-\nabla\left(\frac{\phi^2}{c^2}+\frac{\psi}{c^2}\right)-4\frac{\partial A}{\partial t}+4\mathbf{v}\wedge\left(\nabla\wedge\mathbf{B}\right)-\frac{v^2}{c^2}\nabla\phi\right] \quad (25)$$

with some correction and using our notation. Expanding (25), we find (11) and (15).

11. Gravitational induction force

According to D'Alembert's principle, the sum of the external forces **F** acting on a body and the inertial force $\mathbf{F}_i$ is zero

$$\mathbf{F} + \mathbf{F}_i = 0$$

the force of inertia is a real force (and therefore produced by other bodies) and equal to $\mathbf{F}_i = -m\mathbf{a}$, **a** is the acceleration of the body with respect to an inertial reference frame, and *m* is the inertial mass.

We understand Mach's principle as the statement that inertial mass and the phenomenon of inertia are the results of the gravitational action of the Universe as a whole, which exerts a gravitational induction force on every accelerated body (Segura, 2021) and (Veto, 2013).

Let us suppose a body of gravitational mass *m*, acceleration **a**, and velocity **v** with respect to the inertial reference frame that is at rest with respect to the Universe. As the particle is accelerated with respect to the Universe as a whole, it will feel a gravitoelectromagnetic field (passive induction*), which produces the acceleration $d\mathbf{w}/d\tau$ on the particle (15). So if Mach's principle is correct and the inertia is the result of gravitational induction, it must be fulfilled

$$\frac{d\mathbf{w}}{d\tau} = -\mathbf{a}.$$

To verify that the last statement is correct, we must calculate the acceleration induced by the entire Universe on the accelerated body. Since we want to calculate the force of inertia in the classical approximation, we do not take into account in the calculation the terms that contain w^2/c^2, then the formula to determine the action induced by the Universe is

$$\frac{d\mathbf{w}}{d\tau} = -\nabla'\phi - 4\frac{\partial \mathbf{A}}{\partial t},$$

note that **A** depends on the particle's velocity, and by expanding $\nabla'\phi$ we also find inductive or velocity-dependent terms **.

Some research indicates that the induced gravitational force can explain the phenomenon of inertia (Martín-Rañada-Tiemblo, 2007), (Veto, 2013) and (Segura, 2019).

* We call active gravitoelectromagnetic induction the one produced by the movement of the source, while passive induction results from the motion of the body on which the force acts. For the problem we are considering, they are different since the Universe defines the inertial frame or privileged. In passive induction, the velocity that causes the inductive fields is the opposite of the velocity of the particle on which the force acts.

** The motion of the test particle does not alter the metric of the fixed observer in the Universe. Therefore, neither does it modify the proper time or distance.

The potential ϕ produced by the Universe is of the order of c^2, which seems that the gravitational field is very intense. Therefore the weak or linear gravitational theory would be inapplicable. If the cosmological principle is accepted, ϕ is the same everywhere in the Universe and can be recalibrated by adding an arbitrary constant, reducing the potential to zero. We add that the equation of motion (15) is invariant when we recalibrate the Newtonian potential, for this reason it cannot contain nonlinear terms like ϕ^2.

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