Collatz Conjecture

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The problem

Conjecture: The following operation is applied on an arbitrary positive integer n

$$f(n) = \begin{cases} \frac{n}{2}, & if\ n \cong 0\ mod2 \\ 3n+1, & if\ n \cong 1\ mod2 \end{cases}$$

The Collatz conjecture states: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

Abstract

We consider n to have only odd values, and even values are written in the form; $n.2^b$. We create a predefined function $r_b(n)$.Define, $g(n) = r_b(n) + r_{b-1}(n)$ and prove $g(n) = f(n)$.

$g(n)$ being an identical function to Collatz transformations, we use the properties of said function to probe if some number n can explode to infinity.

We study n_x in detail, establish pattern for n_x modulo 3. We use our understanding to probe if some number n, can loop to itself with more than one transformation.

Format of the solution: The solution does not adhere to the conventional framework of paragraphed proof writing, every piece of maths that is important (to conjecture) is tabular.

- The solution template is inspired from Leslie Lamport; how to write a 21st century proof
- The Solution is framed in a structured template with every argument followed its proof.
- All the subsections are tabulated to study, IF-THEN clause: for main case and sub cases.
- Tabulation should help the reader understand the larger picture in context to some specific case.

Current understanding: The heuristic and probabilistic arguments that support the conjecture are well known. The conjecture has been proven valid for numbers upto 2^{68} but hasnt been proven yet for all numbers. There has been a lot of interesting work done in this problem by notable mathematicians. Few of the notable efforts have been by; Terras showing almost all values n eventually iterated to a value less than n, Krasikov and Lagarias showed that for any large number x, there were at least x0.84 initial values n between 1 and x whose Collatz iteration reached 1. Terrence tao showed Almost all Collatz orbits attain almost bounded values.

The conjecture has been studied using Benford's law, Markovs chains, binary systems among other approaches. Variants of the Collatz function have been studied, John Conway invented a computer language called fractran in which every program was a variant of the Collatz function, it turned out to be Turing complete.

There has been some interesting commentary by reputed names, regarding the problem; Paul Erdos said about the Collatz conjecture: "Mathematics may not be ready for such problems." Jeffery Lagarias stated in 2010 that the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present-day mathematics. Richard K guy stated "Don't try to solve these problems! " Some call it the most dangerous problem in mathematics. All this commentary makes us more interested in looking into the problem.

For verbal explanation refer: https://www.youtube.com/watch?v=ZXK56OdwdrE

Definition 0.1 Transformation: Application of 3n+1 followed by application of n/2(one or more times) till we get odd number is termed as transformation. Application of 3n+1 always results the form of $n'.2^b$ and we just need to divide $n'.2^b$ by 2, b number of times, to get n' which may go through transformation once again.

Notation

{ } : square brackets are used to represent sets. All the sets in the analysis are open ray sets, that is having a certain starting point and can be extended to infinity.

≡ : Equivalence is used for operations under the defined transformations in the problem, that is 3n+1 & n/2. Example; $5 \equiv 1$. One may consider ≡ as applying transformation on odd element and dividing it by max power of 2 with result being an integer.

n is defined to be only odd and we may apply 3n+1 upon it. Any even entity shall be represented as $even = n_{odd}.2^b$

≅ : is used to describe congruence modulo some number.

Definition 0.2

$$\mathrm{n}_{\mathrm{x(before\ transformation;\ applying\ 3n+1)}}$$

$$\equiv \mathrm{n}_{\mathrm{s(after\ transformation;\ applying\ 3n+1\ and\ dividing\ it\ by\ max\ power\ of\ 2)}}$$

$$n_x \,\&\, n_s \text{ are always odd}$$

The co-application of 3n+1 and n/2 shall be considered as a single step

$$3n_x + 1 = n_s.2^b \mid n_x \,\&\, n_s = 2k+1 \,\&\, k, b \in \mathbb{Z}^+$$

D0.2	$3n_x + 1 = n_s.2^b \text{ is same as } n_x \equiv n_s$

Take the Universal set of all positive integers {U}

$$\{U\} = \{1,2,3,4,5 \ldots .\}$$

On all even elements, apply map (n/2 till we get odd) on {U}, we get:

$$\frac{n}{2} \to \{U\}, we\ get \{U'\} = \{1,3,5,7,9 \ldots .\}$$

We begin our study considering set $\{U'\}$ with only positive odd integers

Rooster Notation: $\{U'\} = \{1,3,5,7,9 \ldots.\}$ Set Builder Notation: $\{U'\} = \{2k-1\} |\ k\ \in \mathbb{Z}^{+}$

We define $\{\mathrm{r_y}\}$& $\{\mathrm{r_b}\}$, formulate expansion for $\{\mathrm{r_b}\}$ and establish the relationship between $\mathrm{r_b}$& $\mathrm{n_s}$

Definition 1: $\{\mathbf{r_y}\}$ is a set of sets contains elements corresponding to values of {U'} based upon parity of y with the given definition;

D1	Condition	$\mathrm{r_y} = \frac{\mathrm{r_{y-1}} \pm 1}{2} \vert$ $\{r_0\} = \{\mathrm{U'}\} \Rightarrow \mathrm{r_0} = \mathrm{n_x}\ and\ r, \mathrm{y} \in \{\mathbb{Z}^{+}\} \cup \{0\}$
D1.1	$\mathrm{y} \cong 1 \bmod 2$ (y=odd)	$\mathrm{r_y} = \frac{\mathrm{r_{y-1}} + 1}{2}$
D1.2	$\mathrm{y} \cong 0 \bmod 2$ (y=even)	$\mathrm{r_y} = \frac{\mathrm{r_{y-1}} - 1}{2}$

$\mathrm{r_{y-1}} \pm 1$ implies, we add or subtract 1 to the value of r for any given subset (y-1)

$\mathrm{r_{y-1}}$ is mapped to $\mathrm{r_y}$ if and only if value of r in $\mathrm{r_{y-1}}$ is odd. The mapping continues till r is even.

For value of r being even, we define said set as $\mathrm{r_b}$.

Example: Say, $\mathrm{n_x} = 13$, $\mathrm{r_0} = 13_0$(by definition)

- For $\mathrm{r_y} = \mathrm{r_1}$: because y is odd, $\mathrm{r_y} = \frac{\mathrm{r_{y-1}}+1}{2}$ implies $\mathrm{r_1} = \frac{\mathrm{r_0}+1}{2} = 7$, so $\mathrm{r_1} = 7_1$
  Since value of r in $\mathrm{r_1}$ is odd, we extent the set further;
- For $\mathrm{r_y} = \mathrm{r_2}$: because y is even, $\mathrm{r_y} = \frac{\mathrm{r_{y-1}}-1}{2}$ implies $\mathrm{r_2} = \frac{\mathrm{r_1}-1}{2} = 3$, so $\mathrm{r_2} = 3_2$
  Since value of r in $\mathrm{r_2}$ is odd, we extend the set further.
- For $\mathrm{r_y} = \mathrm{r_3}$: because y is odd, $\mathrm{r_y} = \frac{\mathrm{r_{y-1}}+1}{2}$ implies $\mathrm{r_3} = \frac{\mathrm{r_2}+1}{2} = 2$, so $\mathrm{r_2} = 2_3$
  Since value of r in $\mathrm{r_3}$ is even, we cannot extend the set further. Thus, b=3 and $\mathrm{r_b} = 2_3$

Definition 2: $\{\mathbf{r_b}\}$

$$\mathrm{r_b} = \mathrm{r_y} |\ r\ in\ \mathrm{r_y} = 2k, k \in \mathbb{Z}^{+}$$

Since, $\mathrm{r_b}$ is same as $\mathrm{r_y}$ with the only condition is that value of r in $\mathrm{r_y}$ is even. So, $\mathrm{r_b}$ carries the same defination as $\mathrm{r_y}$

D2	Condition	$\mathrm{r_b} = \frac{\mathrm{r_{b-1}} \pm 1}{2} \vert\ \mathrm{b} \in \mathbb{Z}^{+}$
D2.1	$\mathrm{b} \cong 1 \bmod 2$ (b=odd)	$\mathrm{r_b} = \frac{\mathrm{r_{b-1}} + 1}{2}$
D2.2	$\mathrm{b} \cong 0 \bmod 2$ (b=even)	$\mathrm{r_b} = \frac{\mathrm{r_{b-1}} - 1}{2}$

If one applies relevant map on $\mathrm{r_b}$ where value of r is even, result is a rational solution which is not a positive integer or zero, thus is invalid.

Remark: For condition r=0, we use the classification of zero being even described by Penner 1999, p. 34: Lemma B.2.2

Define: $\{\boldsymbol{R}_{\mathrm{b}}\} = \{\{r_1\} \cup \{r_2\} \cup \{r_3\} \cup \{r_4\} \cup \{r_5\} \cup \{r_6\} \cup \ldots\}$

Lemma 1.0: There does not exist n_x that is a subset of {U'}, and does not have an associated representation in $\{r_b\}$. In other words, all elements of {U'} are a subset of r_b such that $b = 1 \rightarrow \infty$.

$$\forall\, n_x \in \{U'\}\ \exists\ \{r_b(n_x)\} \in R_b \text{for } b = 1 \rightarrow \infty | 3n_x + 1 = n_s . 2^b \ \&\ n_x\, , n_s = 2k - 1\ \&\ k, b \in \mathbb{Z}^+$$

$$\Rightarrow q\{U'\} = \sum_{b=1}^{b\rightarrow\infty} q\{r_b\} \,|\, \text{for } b \cong 1 \bmod 2, r_b = \frac{r_{b-1}+1}{2}\ \&b \cong 0 \bmod 2, r_b = \frac{r_{b-1}-1}{2}\ \&b \in \mathbb{Z}^+$$

Proof: let number of elements in any given set be represented by q{x}

L1		$\forall\, n_x \in \{U'\}\ \exists\ \{r_b(n_x)\} \in R_b \text{for } b = 1 \rightarrow \infty$
L1.1		$q\{U'\} = \text{total numberofelements in universal set } \{U'\}$ $q\{r_y\} = \text{total numberofelements in set } \{r_y\}$
Proof:		By definition
L1.2	Base Case	$q\{r_1\} = \frac{1}{2^1} q\{U'\}$
Proof:		By definition 2 $\{\text{even}r_y\} = \{r_b\}\ \&\{\text{odd}r_y\} = \{r_{y+1}\}$ $q(r_{y+1}) = q(\text{odd}r_y) = q(\text{odd}r_{y+1}) + q(\text{even}r_{y+1})$ Quantity of odd numbers are equal to quantity of even numbers $q(\text{odd}r_y) = q(\text{even}r_y)$ $q(\text{even}r_y) = \frac{1}{2} q(r_{y+1})$ $q\{r_{b=1}\} = \frac{1}{2^1} q\{r_{b=0}\} = \frac{1}{2^1} q\{U'\}$
L1.3		$q\{r_{b=2}\} = \frac{1}{2^2} q\{U'\}$
Proof:		$q\{r_{b=1}\} = q\{r_{y=2}\} + q\{r_{b=2}\} = 2q\{r_{b=2}\}$ $q\{r_{b=2}\} = \frac{1}{2} q\{r_{b=1}\} = \frac{1}{2^2} q\{U'\}$
L1.4	Mathematical Induction Assumed case	$q\{r_{b=x}\} = \frac{1}{2^{b=x}} q\{U'\} \,|\, x \in \mathbb{Z}^+$
Proof:		Assumed for induction
L1.5		$q\{r_{b=(x+1)}\} = \frac{1}{2^{b=(x+1)}} q\{U'\} \,|\, x \in \mathbb{Z}^+$

Proof:		$q\{r_{b=x}\} = q\{r_{y=(x+1)}\} + q\{r_{b=(x+1)}\} = 2q\{r_{b=(x+1)}\}$ $q\{r_{b=(x+1)}\} = \frac{1}{2} q\{r_{b=x}\} = \frac{1}{2^{b=(x+1)}} q\{U'\}$
L1.6		$q\{U'\} = \sum_{b=1}^{b\to\infty} q\{r_b\}$
Proof:		$\sum_{b=1}^{b\to\infty} q\{r_b\} = q\{r_{b=1}\} + q\{r_{b=2}\} + q\{r_{b=3}\} + q\{r_{b=4}\} + q\{r_{b=5}\} \ldots$ Using L1.6 $\sum_{b=1}^{b\to\infty} q\{r_b\} = q\{U'\}\left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} \ldots\right) = q\{U'\}(1)$ $= q\{U'\}$
L1.0		$\forall\, n_x \in \{U'\} \exists \{r_b(n_x)\} \in R_b \text{for } b = 1 \to \infty$
Proof:		By L1.6

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Theorem 1.0: for all values of n_x, the r_b has well defined values that depend upon the parity of b

$$\Leftrightarrow\ b = \text{even}, r_b = \frac{3n_x - 2^b + 1}{3.2^b} \wedge \Leftrightarrow\ b = \text{odd}, r_b = \frac{3n_x + 2^b + 1}{3.2^b}\ |3n_x + 1 = n_s.2^b\ \&\ n_x, n_s = 2k - 1\ \&\ a, k, b \in \mathbb{Z}^+$$

Proof:

T1.0	Condition	$\Leftrightarrow\ b = \text{even}, r_b = \frac{3n_x - 2^b + 1}{3.2^b} \wedge \Leftrightarrow\ b = \text{odd}, r_b = \frac{3n_x + 2^b + 1}{3.2^b}\ \|$ $3n_x + 1 = n_s.2^b\ \&\ n_x, n_s = 2k - 1\ \&\ k, b \in \mathbb{Z}^+$
T1.1	IF	$r_b = \frac{r_{b-1} \pm 1}{2}$
Proof:		By definition D2
T1.2.1	If b=even Base case b=2	$r_2 = \frac{3n_x - 2^2 + 1}{3.2^2}$
Proof:		$r_2 = \frac{r_{2-1} - 1}{2^1} = \frac{\frac{n_x + 1}{2^1} - 1}{2^1} = \frac{n_x - \frac{3}{3}}{2^2} = \frac{3n_x - 3}{3.2^2} = \frac{3n_x - 2^2 + 1}{3.2^2}$
T1.2.2	$b = 2a\| \in \mathbb{Z}^+$	$r_{2a} = \frac{3n_x - 2^{2a} + 1}{3.2^{2a}}$

Proof:		Assumed for induction
T1.2.3	$b = 2a + 2\|$ $a \in \mathbb{Z}^+$	$$\mathrm{r}_{2a+2} = \frac{3\mathrm{n}_x - 2^{2a+2} + 1}{3.2^{2a+2}}$$
Proof:		Using T1.2.2 $$\mathrm{r}_{2a+2} = \frac{\mathrm{r}_{2a+2-1} - 1}{2^1} \Rightarrow \mathrm{r}_{2a+2} = \frac{\frac{\frac{3\mathrm{n}_x - 2^{2a} + 1}{3.2^{2a}} + 1}{2^1} - 1}{2^1}$$ $$\mathrm{r}_{2a+2} = \frac{3\mathrm{n}_x - 2^{2a+2} + 1}{3.2^{2a+2}} \text{(by algebra)}$$
T1.2.4	Then,	$$\mathrm{r}_b = \frac{3\mathrm{n}_x - 2^b + 1}{3.2^b}$$
Proof:		Using mathematical induction in T1.2.2 & T1.2.3 and substituting 2a with b
T1.3.1	If, b=odd Base case b=1	$$\mathrm{r}_1 = \frac{3\mathrm{n}_x + 2^1 + 1}{3.2^1}$$
Proof:		$$\mathrm{r}_1 = \frac{\mathrm{n}_x + 1}{2^1} = \frac{\mathrm{n}_x + \frac{3}{3}}{2^1} = \frac{\mathrm{n}_x + \frac{2^1 + 1}{3}}{2^1} = \frac{3.\mathrm{n}_x + 2^1 + 1}{3.2^1}$$
T1.3.2	$b = 2a + 1\|$ $a \in \mathbb{Z}^+$	$$\mathrm{r}_{2a+1} = \frac{\mathrm{r}_{2a} + 1}{2^1}$$
Proof:		Using definition D2.1
T1.3.3	Then,	$$\mathrm{r}_{2a+1} = \frac{3\mathrm{n}_x + 2^{2a+1} + 1}{3.2^{2a+1}}$$
Proof:		Using T1.2.2 $$\mathrm{r}_{2a+1} = \frac{\mathrm{r}_{2a} + 1}{2^1} \Rightarrow \mathrm{r}_{2a+1} = \frac{\frac{3\mathrm{n}_x - 2^{2a} + 1}{3.2^{2a}} + 1}{2^1}$$ $$\mathrm{r}_{2a+1} = \frac{3\mathrm{n}_x + 2^{2a+1} + 1}{3.2^{2a+1}} \text{(by algebra)}$$
T1.0	THEN	$$if\ \mathrm{b} = \text{even}, \mathrm{r}_b = \frac{3\mathrm{n}_x - 2^b + 1}{3.2^b} \wedge if\ \mathrm{b} = \text{odd}, \mathrm{r}_b = \frac{3\mathrm{n}_x + 2^b + 1}{3.2^b}$$
Proof:		By T1.2.4 & T1.3.3

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Upon calculating based on Theorem 1, for values in r_b, we get;

$$r_1 = \frac{\mathrm{n}_x + 1}{2^1}, r_2 = \frac{\mathrm{n}_x - 1}{2^2}, r_3 = \frac{\mathrm{n}_x + 3}{2^3}, r_4 = \frac{\mathrm{n}_x - 5}{2^4}, r_5 = \frac{\mathrm{n}_x + 11}{2^5}, r_6 = \frac{\mathrm{n}_x - 21}{2^6} \ldots$$

Theorem 2.0:

$$\forall(\mathrm{r_b}+\mathrm{r_{b-1}})=\mathrm{n_s} \mid \mathrm{r_b} \,\&\, \mathrm{r_{b-1}} \in \mathrm{n_x} \,\&\, 3n_x+1=n_s.2^b \,\&\, n_x, n_s=2k-1 \,\&\, k,b \in \mathbb{Z}^+$$

We establish the operation "$\mathrm{r_b}+\mathrm{r_{b-1}}$" is identical to application of 3n+1 (on odd) followed by n/2 (on even) till we get odd

Proof:

T2.0		$\forall(\mathrm{r_b}+\mathrm{r_{b-1}})=\mathrm{n_s} \mid$ $\mathrm{r_b} \,\&\, \mathrm{r_{b-1}} \in \mathrm{n_x} \,\&\, 3n_x+1=n_s.2^b \,\&\, n_x, n_s=2k-1 \,\&\, k,b \in \mathbb{Z}^+$
T2.1	IF	$\forall(\mathrm{r_b}+\mathrm{r_{b-1}})=\mathrm{n_s} \Longrightarrow \forall(\mathrm{r_{beven}}+\mathrm{r_{b-1}})=\mathrm{n_s} \wedge \forall(\mathrm{r_{bodd}}+\mathrm{r_{b-1}})=\mathrm{n_s}$
Proof:		Since, parity of b seems to play a role, we put in the effort to study each case separately.
T2.2.1	If. Case 1: b=even=2j $\mid j \in \mathbb{Z}^+$	$\mathrm{r_{2j}}=\dfrac{3\mathrm{n_x}-2^{2j}+1}{3.2^{2j}} \,\&\, r_{2k-1}=\dfrac{3n_x+2^{2j-1}+1}{3.2^{2j-1}}$
Proof:		Using Theorem 1
T2.2.2		$\mathrm{r_{2j}}+\mathrm{r_{2j-1}}=\dfrac{(3n_x+1)}{2^{2j}}$
Proof:		Using Algebra
T2.2.3	Then	$\forall(\mathrm{r_{beven}}+\mathrm{r_{b-1}})=\mathrm{n_s}$
Proof:		Substitute 2j with beven & 2j-1 with b-1 in T2.2.2 and equate with D0.2 $\mathrm{r_{beven}}+\mathrm{r_{b-1}}=\mathrm{n_s}=\dfrac{(3n_x+1)}{2^b}$
T2.3.1	If, Case 2: b=odd=2j+1$\mid j \in \mathbb{Z}^+$	$r_b=\dfrac{3n_x+2^{\mathrm{2j+1}}+1}{3.2^{\mathrm{2j+1}}} \,\&\, r_{\mathrm{2j+1-1}}=\dfrac{3n_x-2^{\mathrm{2j+1-1}}+1}{3.2^{\mathrm{2j+1-1}}}$
Proof:		Using Theorem 1
T2.3.2		$r_{\mathrm{2j+1}}+r_{\mathrm{2j+1-1}}=\dfrac{(3n_x+1)}{2^{\mathrm{2j+1}}}$
Proof:		Using Algebra
T2.3.3	Then,	$\forall(\mathrm{r_{bodd}}+\mathrm{r_{b-1}})=\mathrm{n_s}$
Proof:		Substitute 2j+1 with bodd & 2j+1-1 with b-1 in T2.3.2 and equate with D0.2 $\mathrm{r_{bodd}}+\mathrm{r_{b-1}}=\mathrm{n_s}=\dfrac{(3n_x+1)}{2^b}$
T2.0	THEN,	$\forall(\mathrm{r_b}+\mathrm{r_{b-1}})=\mathrm{n_s}$
Proof:		Using T2.2.3 & T2.3.3

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Let $g(n_x) = r_b(n_x) + r_{b-1}(n_x)$

Then, $(r_b + r_{b-1}) = n_s \Longrightarrow g(n_x) = f(n_x)$

Thus, we create an identical function to the collatz transformations

Theorem 2 can also be re-written in an interesting form: sum of two continued fractions (using definition r_b and r_{b-1}) of for all the possible positive integer values of b;

$$(r_b + r_{b-1}) = n_s \Rightarrow \lim_{b=1\to\infty} \left\{ \underbrace{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{n_x+1}{2}-1}{2}+1}{2}-1}{2}+1}{2}\cdots}_{b\ times} \right. + \left\{ \underbrace{\cfrac{\cfrac{\cfrac{\cfrac{n_x+1}{2}-1}{2}+1}{2}-1}{2}\cdots}_{b-1\ times} \right. = \frac{3n_x+1}{2^b}$$

The continued fraction expression is pretty simple to prove. One may reach the same conclusion without going through Theorem1

Now, we explore if there exists some element n_x, which under defined collatz transformations becomes infinity.

$$n_x \equiv n_x |\, n_s = \infty$$

Corollary 1.0: We identify the condition when any given element after undergoing transformation will definitely increase.

$$\text{if } b = 1, \forall n_s > \forall n_x \wedge \text{if } b > 1, \forall n_s < \forall n_x |\, 3n_x + 1 = n_s . 2^b \ \& \ n_x , n_s = 2k - 1 \ \& \ k, b \in \mathbb{Z}^+$$

$$\text{increase/decrease: condition for any transformation} = \begin{cases} \text{for } b = 1, & \forall n_s > \forall n_x \\ \text{for } b > 1, & \forall n_s < \forall n_x \end{cases} |n_s > 1$$

Proof:

C1.0	Condition	$\text{if } b = 1, \forall n_s > \forall n_x \wedge \text{if } b > 1, \forall n_s < \forall n_x | 3n_x + 1 = n_s . 2^b \ \& \ n_x , n_s = 2k - 1 \ \& \ k, b \in \mathbb{Z}^+$
C1.1	IF	$r_b + r_{b-1} = n_s$
Proof:		By Theorem 2
C1.2.1	If Case 1: $b = 1$	$r_1 + r_0 = n_s$
Proof:		By definition D1: $r_0 = n_x$
C1.2.2	Then	$n_s > n_x$
Proof:		$r_1 + r_0 = \frac{n_x + 1}{2} + n_x > n_x \Longrightarrow n_s > n_x$
C1.3.1	If Case 2: $b = 2$	$n_s = r_2 + r_1$
Proof:		By Theorem 2

C1.3.2		$n_s = \frac{3n_x + 1}{4}$
Proof:		$n_s = \frac{n_x - 1}{2^2} + \frac{n_x + 1}{2} = \frac{3n_x + 1}{4}$
C1.3.2.1	If $n_x = 1$ Then	$n_s = n_x$
Proof:		$3n_x + 1 = n_s.2^2 \ if \ n_x = 1 \ then \ n_s = 1$
C1.3.2.2	If $n_x > 1$ Then	$n_x > n_s$
Proof:		$3n_x + 1 = n_s.2^2 \ \& \ n_x = 1 + n' \Longrightarrow n_s = \frac{3 + 1 + 3n'}{4} = 1 + \frac{3n'}{4}$ $n' = 2k' \ \& \ k' \in \mathbb{Z}^+$
C1.4.1	If Case 3: $b \geq 3$	$3n_x + 1 = n_s.2^{\geq 3}$
Proof:		By definition D0.2: because $b \geq 3$
C1.4.2	Then	$n_x > n_s$
Proof:		if $n_s > n_x$, then $n_s = n_x + j \mid j \in \mathbb{Z}^+$ $3n_x + 1 = n_s.2^{\geq 3} \Longrightarrow 3n_x + 1 = (n_x + j).2^{\geq 3}$ $1 - j.2^{\geq 3} = n_x.(2^{\geq 3} - 3)$ for $j \geq 1$, left hand side is negative, implying n_x is negative, implying $n_x \notin \mathbb{Z}^+$. This is false.
C1.5		$n_s < n_x$ with $b > 2$
Proof:		By C1.3.2.2 & C1.4.2
C1.0	THEN	$\text{if } b = 1, \forall n_s > \forall n_x \wedge \text{if} b > 1, \forall n_s < \forall n_x$
Proof:		By C1.2.2 & C1.5

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We consider applying transformation on some number multiple times such that it will definitely increase in all the applied transformations. Thus, the sub condition as per Corollary 1; n_s is always greater than n_x during all of these multiple transformations needs to be probed.

Corollary 2.0:

$$r_1(s) = \frac{3}{2} r_1(x) \mid r_1(x) \text{ is } r_b \text{ for } n_x, r_1(s) \text{ is } r_b \text{ for } n_s \ \& \ b = 1, 3n_x + 1 = n_s.2^b \ \& \ n_x, n_s = 2k - 1 \ \& \ k \in \mathbb{Z}^+$$

$r_1(s)$ is the value of r_b for n_s. Similarly, $r_1(x)$ is the value of r_b for n_x When we repeatedly apply transformation: we always label the element that we apply transformation upon as n_x, the transformed element is always labelled as n_s

Example: Say, $n_x = 9$ then $n_s = 7$, now apply transformation on 7, so 7 becomes n_x
$n_x = 7,\ \mathrm{r_b}(7) = 4_1$ then $n_s = 11\ \mathrm{r_b}(11) = 6_1$, again continue applying transformation upon 11, so 11 becomes n_x
$n_x = 11, \mathrm{r_b}(11) = 6_1$ then $n_s = 17,\ \mathrm{r_b}(11) = 4_2$... and so on.

Proof:

C2.0	Condition	$\mathrm{r_1(s)} = \frac{3}{2}\mathrm{r_1(x)} \mid \mathrm{r_1(x)}$ is $\mathrm{r_b}$ for $\mathrm{n_x}$, $\mathrm{r_1(s)}$ is $\mathrm{r_b}$ for $\mathrm{n_s}$ & $\mathrm{b} = 1$, $3\mathrm{n_x} + 1 = \mathrm{n_s}.2^{\mathrm{b}}$ & $\mathrm{n_x}$, $\mathrm{n_s} = 2\mathrm{k} - 1$ & $\mathrm{k} \in \mathbb{Z}^+$
C2.1	IF	$\mathrm{r_b}$ for $\mathrm{n_x} = \mathrm{r_b(x)}$ & $\mathrm{r_b}$ for $\mathrm{n_s} = \mathrm{r_b(s)} \mid 3\mathrm{n_x} + 1 = \mathrm{n_s}.2^{\mathrm{b}}$
Proof:		By definition
C2.2		$\mathrm{n_x} = 2\mathrm{r_1(x)} - 1$ & $\mathrm{n_s} = 2\mathrm{r_1(s)} - 1$
Proof:		*By algebra on definition of* r_1 $r_1(x) = \frac{n_x + 1}{2}$ & $r_1(s) = \frac{n_s + 1}{2}$
C2.3		$\mathrm{r_1(x)} = \mathrm{n_s} - \mathrm{n_x}$
Proof:		$\mathrm{r_1} + \mathrm{r_0} = \mathrm{n_s} \Rightarrow \mathrm{r_1(x)} + \mathrm{n_x} = \mathrm{n_s}$
C2.4		$\mathrm{r_{1(x)}} = (2\mathrm{r_{1(s)}} - 1) - (2\mathrm{r_{1(x)}} - 1)$
Proof:		Using substitution of n_s & $\mathrm{n_x}$ from C2.2 in C2.3
C2.0	THEN	$\mathrm{r_1(s)} = \frac{3}{2}\mathrm{r_1(x)}$
Proof:		Using algebra on C2.4

■

Corollary 1 implies for n greater than 1; b greater than 1 is the only condition for increase during transformations. Corollary 2 implies for n greater than 1, an element can grow finite number of times, as any number ($3r_1$(x)) that is divided by 2 will eventually result; an odd number. Thus, after some finite number of transformations, the element n will definitely decrease because b happens to be greater than 1. We do not conclude that n reaches a value less than itself, we only conclude that for all there does not exist n that can grow continuously infinite number of times.

Thus, the transformational process, n continuously grows and transforms to infinity; that is described by the following equation

$$n_{u1} \equiv n_{u2} \equiv n_{u3} \equiv n_{u4} \equiv n_{u5} \dots \equiv n_{u\infty} \mid n_{u1} < n_{u2} < n_{u3} < n_{u4} < n_{u5} < \cdots < n_{u\infty} \text{ where } n_{u\infty} = \infty \;\&\; r_b = r_1 \;\forall\; n_{u1}, n_{u2}, n_{u3}, n_{u4}, n_{u5} \dots$$

is false and invalid. One concludes that *continuous* increase to infinity is not possible.

Notation:

<≠> is used to describe relationship between 2 elements; one element may be greater than or smaller than the other element, but both the elements are not equal.

Note: It would seem improper to use "<≠> " notation describing any series. However, It is okay to use such notation in the context of our analysis; we don't know if when they are larger or smaller to adjacent element, all we know is none of the elements in the series can be equal to any other element. We consider every element during the transformational process to be not equal to any other element, as that would imply, the elements loops, thus n cannot transform to infinity.

Consider the transformational process described as:

$$n_{u1} \equiv n_{u2} \equiv n_{u3} \equiv n_{u4} \equiv n_{u5} \dots \equiv n_{u\infty} \mid n_{u1} <\neq> n_{u2} <\neq> n_{u3} <\neq> n_{u4} <\neq> n_{u5} \dots$$

The transformation from n_{u1} *to* $n_{u\infty}$ with discontinuous growth may be described by the above equation. So, it is still possible for some number to grow to infinity at a relatively slower rate. Hence, the question of discontinuous growth to infinity remains valid and thus open.

Proposition 1.0:

$$n_x \not\equiv \infty \mid 3n_x + 1 = n_s.2^b \;\&\; n_x, n_s = 2k - 1 \;\&\; k, b \in \mathbb{Z}^+$$

We prove proposition by contradiction.

Proof:

P1.0	Condition	$n \not\equiv \infty \mid 3n_x + 1 = n_s.2^b \;\&\; n_x, n_s = 2k + 1 \;\&\; k, b \in \mathbb{Z}^+$
P1.1	IF	$n_s \equiv \infty \mid n_x, n_s = 2k + 1 \;\&\; k, b \in \mathbb{Z}^+$
Proof:		Assumed to establish contradiction
P1.2		$3r_b \pm 1 = n_s$
Proof:		By applying definition 2 on Theorem 2 $r_b + r_{b-1} = 3r_b \pm 1$
P1.3		$r_b \notin \{U\}$
Proof:		By P1.1 & P1.2 $3r_b \pm 1 = \infty \Rightarrow r_b = \frac{\infty \mp 1}{3}$
P1.4		$\forall\; r_b \in \{U\}$
Proof:		By Definition 2

		$\forall\, r_b \in \mathbb{Z}^+ \& \mathbb{Z}^+ \in \{U\} \Longrightarrow r_b \in \{U\}$
P1.0	THEN	$n \not\equiv \infty$
Proof:		By contradiction in P1.3 & P1.4

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Thus, no number can transform to infinity.

Corollary 3.0

$$n_s \not\cong 0 \bmod 3 \mid 3n_x + 1 = n_s 2^b, n_x, n_s = 2k - 1 \& k, b \in \mathbb{Z}^+$$

Despite the argument being trivial in nature, we will still prove it as it is instrumental in our study of the conjecture.

Proof:

C3.0		$n_s \not\cong 0 \bmod 3 \mid 3n_x + 1 = n_s 2^b, n_x, n_s = 2k - 1 \& k, b \in \mathbb{Z}^+$
C3.1	IF	$3n_x + 1 = n_s 2^b, n_x, n_s = 2k - 1 \& k, b \in \mathbb{Z}^+$
Proof:		By definition
C3.2		$n_s \cong 0 \bmod 3 \Longrightarrow n_s = 3j \mid j \in \mathbb{Z}^+$
Proof:		Assumed to establish contradiction
C3.3		$n_x \notin \mathbb{Z}^+$
Proof:		$n_x = \frac{3j2^b - 1}{3} = j2^b - \frac{1}{3} \Longrightarrow n_x \notin \mathbb{Z}^+$
C3.0	Then	$n_s \not\cong 0 \bmod 3$
Proof:		By contradiction in C3.1 & C3.3

■

Corollary 4.0

$$\text{if } n_1 \cong 1 \bmod 3 \;\&\; n_2 \cong 1 \bmod 3, \text{then } n_3 \cong 1 \bmod 3 \;\wedge$$
$$\text{if } n_1 \cong 2 \bmod 3 \;\&\; n_2 \cong 2 \bmod 3, \text{then } n_3 \cong 1 \bmod 3 \;\wedge$$
$$\text{if } n_1 \cong 1 \bmod 3 \;\&\; n_2 \cong 2 \bmod 3, \text{then } n_3 \cong 2 \bmod 3 \mid n_1.n_2 = n_3 \;\&\; n_1, n_2, n_3, k_1, k_2, k_3, k \in \mathbb{Z}$$

The above arguments may be written in multiplicative format;

$$2 \bmod 3 * 2 \bmod 3 \cong 1 \bmod 3 \wedge 1 \bmod 3 * 1 \bmod 3 \cong 1 \bmod 3 \wedge 1 \bmod 3 * 2 \bmod 3 \cong 2 \bmod 3$$

Proof:

C4		$\Leftrightarrow n_1 \cong 1 \bmod 3 \;\&\; n_2 \cong 1 \bmod 3, n_3 \cong 1 \bmod 3 \;\wedge$ $\Leftrightarrow n_1 \cong 2 \bmod 3 \;\&\; n_2 \cong 2 \bmod 3, n_3 \cong 1 \bmod 3 \;\wedge$ $\Leftrightarrow n_1 \cong 1 \bmod 3 \;\&\; n_2 \cong 2 \bmod 3, n_3 \cong 2 \bmod 3$

C4.1.1	If	$n_1 \cong 1\ mod3 = 3k_1 + 1\ \&\ n_2 \cong 1\ mod3 = 3k_2 + 1, n_3 \cong 1\ mod3 = 3k_3 + 1$
Proof:		By definition
C4.1.2	Then	$n_1 * n_2 = n_3 \cong 1\ mod3$
Proof:		$n_1 * n_2 = (3k_1 + 1)(3k_2 + 1) = 3.3k_1k_2 + 3k_2 + 3k_1 + 1 = 3(3k_1k_2 + k_2 + 1k_1) + 1 = 3k + 2 \cong 1\ mod3$
C4.2.1	If	$n_1 \cong 2\ mod3\ \&\ n_2 \cong 2\ mod3, n_3 \cong 1\ mod3$
Proof:		By definition
C4.2.2	Then	$n_1 * n_2 = n_3 \cong 1\ mod3$
Proof:		$n_1 * n_2 = (3k_1 + 2)(3k_2 + 2) = 3.3k_1k_2 + 2.3k_2 + 2.3k_1 + 4 = 3(3k_1k_2 + 2k_2 + 2k_1 + 1) + 1 = 3k + 1 \cong 1\ mod3$
C4.3.1	If	$n_1 \cong 1\ mod3 = 3k_1 + 1, n_2 \cong 1\ mod3 = 3k_2 + 2, n_3 \cong 2\ mod3 = 3k_3 + 2$
Proof:		By definition
C4.3.2	Then	$n_1 * n_2 = n_3 \cong 2\ mod3$
Proof:		$n_1 * n_2 = (3k_1 + 1)(3k_2 + 2) = 3.3k_1k_2 + 3k_2 + 2.3k_1 + 2 = 3(3k_1k_2 + k_2 + 2k_1) + 2 = 3k + 2 \cong 2\ mod3$
C4.0	THEN	$\Leftrightarrow\ n_1 \cong 1\ mod3\ \&\ n_2 \cong 1\ mod3, n_3 \cong 1\ mod3\ \wedge$ $\Leftrightarrow\ n_1 \cong 2\ mod3\ \&\ n_2 \cong 2\ mod3, n_3 \cong 1\ mod3\ \wedge$ $\Leftrightarrow\ n_1 \cong 1\ mod3\ \&\ n_2 \cong 2\ mod3, n_3 \cong 2\ mod3$
Proof:		By C4.1.2, C4.2.2, C4.3.2

■

Theorem 3.0: There is a well-defined relationship between $n_s\ modulo$ 3 & parity of b in 2^b.One can determine $n_s\ modulo$ 3 by the parity of b and vice versa. The relationship is independent of n_x.

$$\text{if } n_s \cong 2\ mod3, \text{then } b = odd\ \wedge\ \text{if } n_s \cong 1\ mod3, \text{then } b = even, |\ 3n_x + 1 = n_s.2^b\ \&\ n_x, n_s = 2k - 1\ \ \&\ j, k, b \in \mathbb{Z}^+$$

Proof:

T3.0		$\Leftrightarrow n_s \cong 2\ mod3, b = odd\ \wedge\ n_s \cong 1\ mod3, b = even, |\ 3n_x + 1 = n_s.2^b\ \&\ n_x, n_s = 2k + 1\ \&\ k, b \in \mathbb{Z}^+$
T3.1	IF	T3.0 ⇒ $\dashv b = odd \Rightarrow b = even, n_s \cong 2\ mod3$ is false $\dashv b = even \Rightarrow b = odd, n_s \cong 1\ mod3$ is false
Proof:		$If\ P, then\ Q\ \Rightarrow \dashv Q, \dashv P$ *Modus Tollens*
T3.2.1	If. Case 1: b=even &	$3n_x + 1 = n_s * 2^{beven}|$

	$n_s = 3j + 2$	$\mathrm{n_s} \cong 2\ \mathrm{mod3} = 3\mathrm{j} + 2$
Proof:		By definition of $\mathrm{n_s}$ in terms of 3n+1
T3.2.2		$3\mathrm{n_x} + 1 = (3\mathrm{j} + 2).\,2^{\mathrm{beven}}$
Proof:		By definition of $\mathrm{n_s}$ in terms of $\mathrm{n_s}\ \mathrm{mod3}$
T3.2.3		$3\mathrm{n_x} + 1 = 3\mathrm{j}.\,2^{\mathrm{beven}} + 2^{\mathrm{bodd}}$
Proof:		Since $2.\,2^{\mathrm{bodd}} = 2^{\mathrm{beven}}$
T3.2.4	Then	$\dashv \mathrm{b} = \mathrm{odd} \Rightarrow \mathrm{b} = \mathrm{even}\,, \dashv \mathrm{n_s} \cong 2\ \mathrm{mod3}$ Thus, $\mathrm{b} = \mathrm{odd}, \mathrm{n_s} \cong 2\ \mathrm{mod3}$
Proof:		Using Corollary 4.0 & T3.2.3⇒ $3\mathrm{n_x} + 1 = 3\mathrm{j}.\,2^{\mathrm{beven}} + 2^{\mathrm{bodd}} \Rightarrow$ $1\ \mathrm{mod3} \cong 0\ \mathrm{mod3} * 1\ \mathrm{mod3} + 2\ \mathrm{mod3}$ $1\ \mathrm{mod3} \cong 2\ \mathrm{mod3}$ is $\Leftarrow\!\Rightarrow$, thus false
T3.3.1	If. Case 2: b=odd & $n_s = 3j + 1$	$3\mathrm{n_x} + 1 = \mathrm{n_s} * 2^{\mathrm{bodd}}\vert$ $\mathrm{n_s} \cong 1\ \mathrm{mod3} = 3\mathrm{j} + 1$
Proof:		By definition of $\mathrm{n_s}$ in terms of 3n+1
T3.3.2		$3\mathrm{n_x} + 1 = (3\mathrm{j} + 1).\,2^{\mathrm{bodd}}$
Proof:		By definition of $\mathrm{n_s}$ in terms of $\mathrm{n_s}\ \mathrm{mod3}$
T3.3.3		$3\mathrm{n_x} + 1 = 3\mathrm{j}.\,2^{\mathrm{bodd}} + 2^{\mathrm{bodd}}$
Proof:		Since $1.\,2^{\mathrm{bodd}} = 2^{\mathrm{bodd}}$
T3.3.4	Then	$\dashv \mathrm{b} = \mathrm{odd} \Rightarrow \mathrm{b} = \mathrm{even}\,, \dashv \mathrm{n_s} \cong 1\ \mathrm{mod3}$ Thus, $\mathrm{b} = \mathrm{even}, \mathrm{n_s} \cong 1\ \mathrm{mod3}$
Proof:		Using Corollary 4.0 & T3.3.3⇒ $3\mathrm{n_x} + 1 = 3\mathrm{j}.\,2^{\mathrm{bodd}} + 2^{\mathrm{bodd}} \Rightarrow$ $1\ \mathrm{mod3} \cong 0\ \mathrm{mod3} * 2\ \mathrm{mod3} + 2\ \mathrm{mod3}$ $1\ \mathrm{mod3} \cong 2\ \mathrm{mod3}$ is $\Leftarrow\!\Rightarrow$, thus false
T3.0	THEN	$\dashv \mathrm{b} = \mathrm{odd}\,, \dashv \mathrm{n_s} \cong 2\ \mathrm{mod3} \wedge \dashv \mathrm{b} = \mathrm{even}\,, \dashv \mathrm{n_s} \cong 1\ \mathrm{mod3}$ $\mathrm{b} = \mathrm{even}, \mathrm{n_s} \cong 1\ \mathrm{mod3} \wedge \mathrm{b} = \mathrm{odd}, \mathrm{n_s} \cong 2\ \mathrm{mod3}$
Proof		By T3.2.4 & T3.3.4

■

Definition 3.0 $\{n_x\}$: The set n_x, is a set that contains all the possible values of n_x that would satisfy definition 1.0;

$$n_x \equiv n_s:\ 3n_x + 1 = n_s.2^b \Longrightarrow n_x = \frac{n_s.2^b - 1}{3} \text{ for all valid values of b}$$

The set $\{n_x\}$ is infinitely large with values represented by;

$$\frac{n_s.2^\beta - 1}{3}, \frac{n_s.2^{\beta+2} - 1}{3}, \frac{n_s.2^{\beta+4} - 1}{3}, \frac{n_s.2^{\beta+6} - 1}{3}, \frac{n_s.2^{\beta+8} - 1}{3} \dots \frac{n_s.2^{\beta+2z} - 1}{3} \mid$$

$$\text{if } n_s \cong 2 \text{ mod}3, \beta = 1 \wedge \text{if } n_s \cong 1 \text{ mod}3, \beta = 2 \ \&\ z \in \mathbb{Z}^+$$

All the above give the same result n_s upon application of n/2.

Note: We have only even numbers and not odd numbers being added to the exponent of 2^β in the above set representation because Theorem 3 dictates; the parity of b has to be the same, if we happen to add odd number to the exponent of 2^β, then the parity of b changes, thus we would not get any valid solution for n_x.

Theorem 4.0: All elements of $\{n_x\}$ can be expressed in the form of its adjacent element.

$$n_{x1+(z+1)} = 4n_{x1+z} + 1 \mid$$

$$\{n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3}, n_{x1+4}, n_{x1+5}, n_{6x1+} \dots n_{x1+z}, n_{x1+(z+1)}\} \in \{n_x\} \ \&$$

$$3n_x + 1 = n_s.2^b \ \&\ n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3} \dots, n_x, n_s = 2k - 1 \ \&\ k, b \in \mathbb{Z}^+$$

Note: The notation n_{x1+1} instead of n_{x2}, would seem a bit strange.

There is a method to the madness;

$n_x \equiv n_s \Longrightarrow n_x = \frac{n_s.2^b - 1}{3}$ with b =1 for $n_s \cong 2$ mod3 or b=2 for $n_s \cong 1$ mod3

n_{x1}: We refer this as base case. It is the first/smallest solution such that

$$n_{x1} = \frac{n_s.2^1 - 1}{3} \mid n_s \cong 2 \text{ mod}3 \vee n_{x1} = \frac{n_s.2^2 - 1}{3} \mid n_s \cong 1 \text{ mod}3$$

Since, we write exponential of 2 in the form $2^{\beta+2z}(z \in \mathbb{Z}^+)$

$n_{x1+1}, n_{x1+2}, n_{x1+3}$ ... represent:

$$n_{x1+1} = \frac{n_s.2^{1+2} - 1}{3} \mid n_s \cong 2 \text{ mod}3 \vee n_{x1+1} = \frac{n_s.2^{2+2} - 1}{3} \mid n_s \cong 1 \text{ mod}3$$

$$n_{x1+2} = \frac{n_s.2^{1+4} - 1}{3} \mid n_s \cong 2 \text{ mod}3 \vee n_{x1+2} = \frac{n_s.2^{2+4} - 1}{3} \mid n_s \cong 1 \text{ mod}3$$

$$n_{x1+3} = \frac{n_s.2^{1+6} - 1}{3} \mid n_s \cong 2 \text{ mod}3 \vee n_{x1+3} = \frac{n_s.2^{2+6} - 1}{3} \mid n_s \cong 1 \text{ mod}3$$

The notation makes sense as the z in the expression2^{b+2z} is referred to as n_{x1+z}. it creates a simple direct link to the additional component of exponent (2z) in expression; 2^{b+2z}. The notation also helps identifying parity of n_x modulo3 which will be evident as we study further.

Proof:

T4.0		$$n_{x1+(z+1)} = 4n_{x1+z} + 1$$
T4.1	IF	$\{n_s.2^{\beta}, n_s.2^{\beta+2}, n_s.2^{\beta+4}, n_s.2^{\beta+6}, n_s.2^{\beta+8}, n_s.2^{\beta+10}, \dots\} \equiv n_s$
Proof:		The said set transforms to n_s upon application of n/2. Parity of b has been maintained same, complying with theorem 3.
T4.2.1		$$\{n_x\} = \begin{bmatrix} \frac{n_s.2^{\beta}-1}{3}, & \frac{n_s.2^{\beta+2}-1}{3}, & \frac{n_s.2^{\beta+4}-1}{3}, & \\ \frac{n_s.2^{\beta+6}-1}{3}, & \frac{n_s.2^{\beta+8}-1}{3}, & \frac{n_s.2^{\beta+10}-1}{3}, & \dots \end{bmatrix}$$
Proof:		By definition
T4.2.2		Let $[n_x] =$ $[n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3}, n_{x1+4}, n_{x1+5}, n_{x1+6}, n_{x1+7} \dots n_{x1+8} \dots]$
Proof:		By definition
T4.3	Base case	$$n_{x1+1} = 4n_{x1} + 1$$
Proof:		By substitution of $n_s.2^{\beta+2}$ $$n_{x1} = \frac{n_s.2^{\beta}-1}{3} \Rightarrow n_s.2^{\beta+2} = 2^2(3n_{x1}+1)$$ $$n_{x1+1} = \frac{n_s.2^{\beta+2}-1}{3} = \frac{2^2(3n_{x1}+1)-1}{3}$$ By algebra, we get; $$n_{x1+1} = 4n_{x1} + 1$$
T4.4	Mathematical Induction	$$n_{x1+z} = 4n_{x1+(z-1)} + 1$$ $$\frac{n_s.2^{\beta+2.(z-1)}-1}{3}, n_{x1+(z+1)} = \frac{n_s.2^{\beta+2z}-1}{3}$$
Proof:		Assumed for induction
T4.0	THEN,	$$n_{x1+(z+1)} = 4n_{x1+z} + 1$$
Proof:		$$n_{x1+(z+1)} = \frac{n_s.2^{\beta+2(z+1)}-1}{3}$$ $$n_{x1+(z+1)} = \frac{4.n_s.2^{\beta+2(z)}-1}{3}$$ Using $3n_{x1+z} + 1 = n_s.2^{\beta+2z}$ $$n_{x1+(z+1)} = \frac{4(3n_{x1+z}+1)-1}{3}$$

■

Corollary 5.0: Congruence modulo 3 is well ordered irrespective of the first solution, 0 mod3 is followed by 1 mod3 is followed by 2 mod3 is followed by 0 mod3 is followed by 1 mod3 and so on...

$$(\text{if } n_{x1} \cong 0 \text{ mod}3 \text{ then } n_{x1+1} \cong 1 \text{ mod}3, n_{x1+2} \cong 2 \text{ mod}3, n_{x1+3} \cong 0 \text{ mod}3, \dots) \wedge$$
$$(\text{if } n_{x1} \cong 1 \text{ mod}3, \text{then } n_{x1+1} \cong 2 \text{ mod}3, n_{x1+2} \cong 0 \text{ mod}3, \dots) \wedge$$
$$(\text{if } n_{x1} \cong 2 \text{ mod}3, n_{x1+1} \cong 0 \text{ mod}3, n_{x1+2} \cong 1 \text{ mod}3, \dots)|$$
$$\{n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3}, n_{x1+4}, n_{x1+5}, n_{x1+6} \dots\} \in \{n_x\} \ \&$$
$$3n_x + 1 = n_s.2^b \ \& \ n_x, n_s = 2k - 1 \ \& \ k, b \in \mathbb{Z}^+$$

Proof:

C5.0		$(\text{if } n_{x1} \cong 0 \text{ mod}3, \ n_{x1+1} \cong 1 \text{ mod}3, n_{x1+2} \cong 2 \text{ mod}3, n_{x1+3} \cong 0 \text{ mod}3, \dots)$ $\wedge (\text{if } n_{x1} \cong 1 \text{ mod}3, n_{x1+1} \cong 2 \text{ mod}3, n_{x1+2} \cong 0 \text{ mod}3, \dots)$ $\wedge (\text{if } n_{x1} \cong 2 \text{ mod}3, n_{x1+1} \cong 0 \text{ mod}3, n_{x1+2} \cong 1 \text{ mod}3, \dots)$
C5.1	IF	$\{n_{x1}, n_{x1+1}, n_{x1+2}, n_{x1+3}, n_{x1+4}, n_{x1+5}, n_{x1+6} \dots\} \in \{n_x\}$
Proof:		By definition
C5.2.1	Case1: If	$n_{x1} \cong 0 \text{ mod}3 \Rightarrow n_{x1} = 3m \mid m \in \mathbb{Z}^+$
Proof:		By definition
C5.2.2		$n_{x1+1} \cong 1 \text{ mod}3$
Proof:		By Theorem 4 $n_{x1+1} = 4n_{x1} + 1 = 4.3m + 1 \cong 1 \text{ mod}3 \mid m \in \mathbb{Z}^+$
C5.2.3		$n_{x1+2} \cong 2 \text{ mod}3$
Proof:		By Theorem 4 $n_{x1+2} = 4n_{x1+1} + 1 = 4.(4.3m + 1) + 1 = 16.3m + 3 + 2 \cong 2 \text{ mod}3$
C5.2.4		$n_{x1+3} \cong 0 \text{ mod}3$
Proof:		By Theorem 4 $n_{x1+3} = 4n_{x1+2} + 1 = 4.(16.3m + 3 + 2) + 1 = 64.3m + 12 + 8 + 1 \cong 0 \text{ mod}3$
C5.2.5		$n_{x1+4} \cong 1 \text{ mod}3$
Proof:		By C5.2.2
C5.2.6	Then	$n_{x1+5} \cong 2 \text{ mod}3, \ n_{x1+6} \cong 1 \text{ mod}3, \ n_{x1+7} \cong 0 \text{ mod}3, \ n_{x1+8} \cong 1 \text{ mod}3, \ n_{x1+9} \cong 2 \text{ mod}3, n_{x1+10} \cong 0 \text{ mod}3 \dots$
Proof:		By C5.2.3, C5.2.4, C5.2.2, C5.2.3, C5.2.4
C5.2.7		$n_{x1+1} \cong 1 \text{ mod}3, n_{x1+2} \cong 2 \text{ mod}3, n_{x1+3} \cong 0 \text{ mod}3,$ $n_{x1+4} \cong 1 \text{ mod}3, n_{x1+5} \cong 2 \text{ mod}3, n_{x1+6} \cong 1 \text{ mod}3, n_{x1+7} \cong 0 \text{ mod}3, \dots$
Proof:		By C5.2.2, C5.2.3, C5.2.4
C5.3.1	Case 2: If	Let $n_{x1} \cong 1 \text{ mod}3 \Rightarrow n_{x1} = 3m + 1 \mid m \in \mathbb{Z}^+$

Proof:		By definition
C5.3.2	Then	$n_{x1+1} \cong 2 \bmod 3, n_{x1+2} \cong 0 \bmod 3, n_{x1+3} \cong 1 \bmod 3, n_{x1+4} \cong 2 \bmod 3, \ldots$
Proof:		By C5.2.3, C5.2.4, C5.2.2
C5.4.1	Case3: If	Let $n_{x1} \cong 2 \bmod 3 \Rightarrow n_{x1} = 3m + 2 \mid m \in \mathbb{Z}^+$
Proof:		By definition
C5.4.2	Then,	$n_{x1+1} \cong 0 \bmod 3, n_{x1+2} \cong 1 \bmod 3, n_{x1+3} \cong 2 \bmod 3, n_{x1+4} \cong 0 \bmod 3, \ldots$
Proof:		By C5.2.4, C5.2.2, C5.2.3
C5.0	THEN	$(\text{if } n_{x1} \cong 0 \bmod 3, n_{x1+1} \cong 1 \bmod 3, n_{x1+2} \cong 2 \bmod 3 \ldots) \wedge (\text{if } n_{x1} \cong 1 \bmod 3, n_{x1+1} \cong 2 \bmod 3, n_{x1+2} \cong 0 \bmod 3 \ \ldots) \wedge (\text{if } n_{x1} \cong 2 \bmod 3, n_{x1+1} \cong 0 \bmod 3, n_{x1+2} \cong 1 \bmod 3, \ldots)$
Proof:		By C5.2.7, C5.3.2, C5.4.2

■

Classify elements of {U’} by using sets n_s modulo9 and n_s modulo27 definition.

Grouped by n_s mod9		
n_s mod 9	**n_s mod 27**	**n_x mod 9**
1 mod 9	19 mod 27	7 mod9
1 mod 9	10 mod 27	4 mod9
1 mod 9	1 mod 27	1 mod9
2 mod9	11 mod27	7 mod9
2 mod9	2 mod27	1 mod9
2 mod9	20 mod27	4 mod9
4 mod9	13 mod27	8 mod9
4 mod9	4 mod27	5 mod9
4 mod9	22 mod27	2 mod9
5 mod9	23 mod27	0 mod3
5 mod9	14 mod27	0 mod3
5 mod9	5 mod27	0 mod3
7 mod9	25 mod 27	0 mod3

grouped by n_s mod27		
n_s mod 27	**n_s mod 9**	**n_x mod 9**
1 mod 27	1 mod 9	1 mod 9
2 mod 27	2 mod9	1 mod9
4 mod 27	4 mod9	5 mod9
5 mod 27	5 mod9	0 mod3
7 mod 27	7 mod9	0 mod3
8 mod 27	8 mod9	5 mod9
10 mod 27	1 mod 9	4 mod9
11 mod 27	2 mod9	7 mod9
13 mod 27	4 mod9	8 mod9
14 mod 27	5 mod9	0 mod3
16 mod 27	7 mod9	0 mod3
17 mod 27	8 mod9	2 mod9
19 mod 27	1 mod 9	7 mod9

grouped by n_x mod9		
n_s mod 9	**n_s mod 27**	**n_x mod 9**
5 mod9	23 mod27	0 mod3
5 mod9	14 mod27	0 mod3
5 mod9	5 mod27	0 mod3
7 mod9	25 mod 27	0 mod3
7 mod9	16 mod 27	0 mod3
7 mod9	7 mod 27	0 mod3
2 mod9	2 mod27	1 mod9
1 mod 9	1 mod 27	1 mod9
4 mod9	22 mod27	2 mod9
8 mod9	17 mod27	2 mod9
2 mod9	20 mod27	4 mod9
1 mod 9	10 mod 27	4 mod9
4 mod9	4 mod27	5 mod9

7 mod9	16 mod 27	0 mod3
7 mod9	7 mod 27	0 mod3
8 mod9	17 mod27	2 mod9
8 mod9	8 mod27	5 mod9
8 mod9	26 mod27	8 mod9

Dist1

20 mod 27	2 mod9	4 mod9
22 mod 27	4 mod9	2 mod9
23 mod 27	5 mod9	0 mod3
25 mod 27	7 mod9	0 mod3
26 mod 27	8 mod9	8 mod9

Dist 2

8 mod9	8 mod27	5 mod9
2 mod9	11 mod27	7 mod9
1 mod 9	19 mod 27	7 mod9
4 mod9	13 mod27	8 mod9
8 mod9	26 mod27	8 mod9

Dist3

Table 2.0: $\mathrm{n_x}\ mod\ 9\ for\ \mathrm{n_s}\ mod9\ \&\ \mathrm{n_s}\ mod27$

Theorem 5.0

All elements in $\mathrm{n_x}$ are well ordered and for all n_s and $\mathrm{n_x}$ modulo 9 is well distributed.

$$values\ of\ \mathrm{n_x}\ mod9\ \forall\ \mathrm{n_s}\ mod9\ is\ well\ distributed\ |$$
$$3\mathrm{n_x} + 1 = \mathrm{n_s}.2^{\mathrm{b}}\ \&\ \mathrm{n_s} \neq \mathrm{n_x}\ \&\ \mathrm{n_x}, \mathrm{n_s} = 2\mathrm{k} - 1\ \ \&\ \mathrm{j}, \mathrm{j}', \mathrm{k}, \mathrm{b} \in \mathbb{Z}^+,$$
$$\mathrm{q}(\mathrm{n_s} \cong 1\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 2\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 4\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 5\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 7\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 8\ mod9)\ for\ \{U'\}, \mathrm{n_s} \ncong 0\ \mathrm{mod}3$$

Proof:

T5.		All elements in $\mathrm{n_x}$ are well ordered and for all n_s, $\mathrm{n_x}$ modulo 9 is well distributed.
T5.1	IF	$\mathrm{q}(\mathrm{n_s} \cong 1\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 2\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 4\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 5\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 7\ mod9) = \mathrm{q}(\mathrm{n_s} \cong 8\ mod9)\ for\ \{U'\}$
Proof:		Based upon the fact that there are always and exactly 3 sets of odd elements modulo 9, between 9(m) and 9(m+1) depending if m is odd or even. If m is odd, then odd elements that lie in between 9m & 9(m+1) are congruent to 2 mod9, 4 mod9 and 8 mod9 If m is even, then odd elements that lie in between 9m & 9(m+1) are congruent to 1 mod9, 5 mod9 and 7 mod9
T5.2.1		Table 2.0
Proof:		Substitute 1 mod9 with 1+9j, 2 mod9 with 2+9j, 4 mod9 with 4+9j, 5 mod9 with 5+9j, 7 mod9 with 7+9j, 8 mod9 with 8+9j; Find $\mathrm{n_{x1}}$ with $\beta = 1$ or 2 as per n_s modulo 3 following theorem 3 | $1 + 9\mathrm{k}, 2 + 9\mathrm{j}, 4 + 9\mathrm{j}, 5 + 9\mathrm{j}, 7 + 9\mathrm{j}, 8 + 9\mathrm{j} = 2\mathrm{k} - 1,\ \mathrm{k}, j \in \mathbb{Z}^+$

T5.2.2		$q(\mathrm{n_x} \cong 1mod9) = q(\mathrm{n_x} \cong 2mod9) = q(\mathrm{n_x} \cong 4mod9) = q(\mathrm{n_x} \cong 5mod9) = q(\mathrm{n_x} \cong 7mod9) = q(\mathrm{n_x} \cong 8mod9) = \frac{1}{9}q(\{U'\} - \{1\}) = \frac{1}{3}q(\mathrm{n}_x \cong 0\ mod3) = \frac{1}{3}q(\mathrm{n}_x \cong 1\ mod3) = \frac{1}{3}q(\mathrm{n}_x \cong 2\ mod3)$
Proof:		by table 2.0
T5.2.3		$values\ of\ \mathrm{n_x}\ mod9\ \forall\ \mathrm{n_s}\ mod9\ is\ well\ distributed$
Proof:		By T5.2.2
T5.0	THEN,	All elements in $\mathrm{n_x}$ are well ordered and for all n_s *and* $\mathrm{n_x}$ modulo 9 is well distributed.
Proof:		By C5 and T5.2.3

■

Theorem 5 is based upon modular analysis implying the cyclic nature of transformation from $\{\tau_t\}\ from\ \{\tau_{t+1}\}$. Thus, one may extend the understanding to whole universal set {U'} and all the reverse transformations elements can go through.

Notation:

$\equiv\equiv$ double equivalence implies more than 1 transformation.$\mathrm{n_s} \equiv\equiv \mathrm{n_s}$ implies that some number $\mathrm{n_s}$ transforms to $\mathrm{n_s}$ with more than 1 transformation; $\mathrm{n_x} \neq \mathrm{n_s}$

Proposition 2: some number loops to itself

Proposition 2.a: the loop happens with single transformation such that $n_s \equiv n_s$

Proposition 2.b: the loop happens with more than one transformation such that $n_s \equiv\equiv n_s$ such that $n_x \neq n_s$

Proposition 2.a $\boldsymbol{n_s} \equiv \boldsymbol{n_s}$: Case for single transformation loop has trivial solution $1 \equiv 1$ with no other possible solution. $\mathrm{n_x} = \mathrm{n_s}$

$$3\mathrm{n_x} + 1 = \mathrm{n_s}2^{\mathrm{b}} \Longrightarrow 3\mathrm{n_s} + 1 = \mathrm{n_s}2^{\mathrm{b}} \Longrightarrow \mathrm{n_s}(2^{\mathrm{b}} - 3) = 1 \Longrightarrow (2^{\mathrm{b}} - 3) = \frac{1}{\mathrm{n_s}}$$

For any value of $\mathrm{n_s} > 1$, right hand side gives a rational solution and on the right-hand side, no value of b could dish out rational solution. Thus, no other value of $\mathrm{n_s}$ satisfies the condition $n_s \equiv n_s$

$$n_s \equiv n_s |\ n_x = n_s = 1$$

Definition 4.0:

$\{\tau_0\}$ is an arbitrarily defined ordered set that is similar to {U'} such that $\{U'\} = \{1\} \cup \{\tau_0\}$

All elements of $\{\tau_0\}$ are zero reverse transformations away from {U'}. The element "1" is excluded as $n_s \equiv n_s|\, n_x = n_s = 1$

τ_t is a set that contains all elements that are t reverse transformations away from its associated element in {U'}

$$\{\tau_t\} = \{\tau_t^{1m3}\} \cup \{\tau_t^{2m3}\} \cup \{\tau_t^{0m3}\}$$

$\{\tau_t^{1m3}\}$ is a set that contains all the elements that congruent to 1 mod3 and are t reverse transformations away from its associated element in {U'}

$\{\tau_t^{2m3}\}$ is a set that contains all the elements that congruent to 2 mod3 and are t reverse transformations away from its associated element in {U'}

$\{\tau_t^{0m3}\}$ is a set that contains all the elements that congruent to 0 mod3 and are t reverse transformations away from its associated element in {U'}

Let q be an element counting function such that $q\{\tau_t^{1m3}\}$ represents total number of elements that are 1 mod3 and are t reverse transformations away from {U'}. Similarly, $q\{\tau_t^{2m3}\}$ represents total number of elements that are 2 mod3 that are t reverse transformations away from {U'} and $q\{\tau_t^{0m3}\}$ represents total number of elements that are 0 mod3 that are t reverse transformations away from {U'}.

$q\{\tau_t^{\dashv 0m3}\}$ refers to inverse of $q\{\tau_t^{0m3}\}$ that is elements that are not 0 mod3.

$$q\{\tau_t^{\dashv 0m3}\} = q\{\tau_t^{1m3}\} \cup q\{\tau_t^{2m3}\}$$

All elements that are no congruent to 0 mod3, will have a representation in $\{\tau_{t+1}\}$

$$q\{\tau_{t+1}\} = q\{\tau_t^{\dashv 0m3}\} = q\{\tau_t^{1m3}\} \cup q\{\tau_t^{2m3}\}$$

$$q\{\tau_0\} = q(\{U'\} - [1])$$

The notation $q(\{U'\} - [1])$ implies; all elements of {U'} except the element "1".

Proposition 2.b $n_s \equiv\equiv n_s$: We explore the possibility for any number to loop with more than one transformation such that $n_x \neq n_s$.

Methodology for checking validity of proposition 2.b

Loop $n_s \equiv\equiv n_s$ implies that when the number of transformations $t \to \infty$ one should have a valid value for n_s and all the possible interim values of n_x such that no value of n_x for any given n_s can be congruent 0 mod3, such that $n_x \neq n_s$

$$n_x \ncong 0\ mod3$$

$$\Longrightarrow \forall\, n_s: (n_x \cong 1\ mod3 \vee n_x \cong 2\ mod3)$$

Corollary 3; n_s cannot be congruent to 0 mod3 even as t → ∞. Using elimination of $n_s \cong 0\ mod3$ as the set τ_0 is expanded to τ_1 expanded to τ_2 expanded to τ_3… expanded to $\tau_{t\to\infty}$, one can test if loop is possible. If there is some element that loops with more than one transformation then said process of elimination should leave us with some definite value with n_s and n_x not being congruent to 0 mod3

For the set $\{\tau_0\}$, $t = 0$

$$\{\tau_0\} = \{\tau_0^{1m3}\} \cup \{\tau_0^{2m3}\} \cup \{\tau_0^{0m3}\}$$

Using Corollary 5; One third of all elements would be eliminated as they are congruent to 0 mod3. For deriving $\{\tau_1\}\ from\ \{\tau_0\}$, continue with reverse transformation for rest non-eliminated elements;

$$\{\tau_1\} = \{\tau_0^{\dashv 0m3}\} = \{\tau_0^{1m3}\} \cup \{\tau_0^{2m3}\}$$

For the set $\{\tau_1\}$, $t = 1$

$$\{\tau_1\} = \{\tau_0^{\dashv 0m3}\} = \{\tau_0^{1m3}\} \cup \{\tau_0^{2m3}\} = \{\tau_1^{1m3}\} \cup \{\tau_1^{2m3}\} \cup \{\tau_1^{0m3}\}$$

Using Corollary 5; One third of all elements would be eliminated as they are congruent to 0 mod3. For deriving $\{\tau_2\}\ from\ \{\tau_1\}$, continue with reverse transformation for rest non-eliminated elements; And so on…

Note: For deriving $\{\tau_t\}\ from\ \{\tau_{t-1}\}$, we do not consider the infinite values of n_x for any given n_s, we only consider the base solution of n_x that is n_{x1} as other solutions like n_{x1+1}, n_{x1+2}… would automatically be considered as it already exists in {U'} .

Including all the possible values of $\{n_x\}$, gives us infinite solutions for every element in {U'} and going back just one more step would break our analysis because of the infinities popping up everywhere. We keep the relationship between $\{\tau_t\}\ and\ \{\tau_{t-1}\}$ as bijective and invertible to able to keep track of number of elements in every set by avoiding the abyss of infinities. Also, no element is kept out of our study as all the possible solutions of $\{n_x\}$ are already a part of {U'}

Example:

$$\{\tau_0\} = \{U'\}\ = \{1\}\ \cup \{3,5,7,9,11,13,15\ldots.\}$$

{U'}	$\{\tau_0\}$	$\{\tau_1\}$	$\{\tau_2\}$	$\{\tau_3\}$	$\{\tau_4\}$
1	Does not exist $n_x = n_s$				
3	3	Does not exist $n_s \cong 0\ mod3$			
5	5	3	Does not exist $n_s \cong 0\ mod3$		
7	7	9	Does not exist $n_s \cong 0\ mod3$		
9	9	Does not exist $n_s \cong 0\ mod3$			
11	11	7	9	Does not exist $n_s \cong 0\ mod3$	
13	13	17	11	7	9
15	15	Does not exist $n_s \cong 0\ mod3$			
17	17	11	7	9	Does not exist

					$n_s \cong 0\ mod3$
19	19	25	33	Does not exist $n_s \cong 0\ mod3$	
21	21	Does not exist $n_s \cong 0\ mod3$			
23	23	15	Does not exist $n_s \cong 0\ mod3$		
25	25	33	Does not exist $n_s \cong 0\ mod3$		
27	27	Does not exist $n_s \cong 0\ mod3$			
29	29	19	25	33	Does not exist $n_s \cong 0\ mod3$

Table 1.0: deriving $\{\tau_t\}\ from\ \{\tau_{t-1}\}$

Consider; $n_s = 11, \{n_x\} = \{7,29,117,469,1877...\}$

$$For\ \ n_s = 11, \qquad \{n_{x1} = 7, n_{x1+1} = 29, n_{x1+2} = 117\ n_{x1+3} = 469, n_{x1+4} = 1877...\}$$

At T=1, only consider $n_s = 7$, other values like 29. 117, 469 etc may be ignored as they are going to be evaluated in their respective rows row.

$$\forall n_s \not\equiv\not\equiv n_s |\ 3n_x + 1 = n_s.2^b\ \&\ n_s \neq n_x\ \&\ n_x\ , n_s = 2k - 1\ \ \&\ j, j^{'}, k, b \in \mathbb{Z}^{+}\&$$

$$\dots n_s \equiv n_{(t-1)x} \equiv\dots\equiv n_{2x} \equiv n_{1x} \equiv n_x \equiv n_s \equiv n_{(t-1)x} \equiv\dots\equiv n_{2x} \equiv n_{1x} \equiv n_x \equiv n_s \equiv n_{(t-1)x} \dots$$

Proof:

P2.b.0		$\text{if } n_s \geq 3, \forall n_s \not\equiv\not\equiv n_s$
P2.b.1	IF	$\dots n_s \equiv n_{(t-1)x} \equiv\dots\equiv n_{2x} \equiv n_{1x} \equiv n_x \equiv n_s \equiv n_{(t-1)x} \equiv\dots\equiv n_{2x} \equiv n_{1x} \equiv n_x \equiv n_s \equiv n_{(t-1)x} \dots$
Proof:		By definition: if a loop exists then one may continue transformation (forward or backward) infinite times
P2.b.2.1		$\nexists n_{1x}, \text{if } n_x \cong 0 \text{ mod}3$
Proof:		$n_x \cong 0 \text{ mod}3 = 3j$ $n_{1x} = \frac{n_x 2^b - 1}{3} = \frac{3j2^b - 1}{3} = \frac{3j' - 1}{3} \mid j' = j2^b$ $n_{1x} \cong \frac{2 \text{ mod}3}{3} \notin \{U'\}$
P2.b.2.2	If	$n_x \cong 0 \text{ mod}3, \text{then } n_s \not\equiv\not\equiv n_s$
Proof:		$n_s \equiv n_{(t-1)x}$ $\text{if } n_x \cong 0 \text{ mod}3 \Rightarrow\ \nexists n_{1x} \Rightarrow\ \nexists n_{2x} \Rightarrow\ \nexists n_{3x} \dots \Rightarrow \nexists n_{(t-1)x}$ Due to contradiction the condition is false.

P2.b.3.1		$$q\{\tau_{t+1}\} = q\{\tau_t^{\neg 0m3}\}$$
Proof:		Using P2.b.2.2
P2.b.3.2		$$q\{\tau_t\} = q\{\tau_t^{0m3}\} + q\{\tau_t^{1m3}\} + q\{\tau_t^{2m3}\}$$
Proof:		According to Theorem 5 for all numbers upon applying reverse transformation there is ordered distribution of elements modulo 3. Addition being commutative, the order of elements modulo 3 does not matter.
P2.b.3.3		$$q\{\tau_{t+1}\} = \frac{2^1}{3^1} q\{\tau_t^{\neg 0m3}\}$$
Proof:		$q\{\tau_{t+1}\}$ is the set of elements that are one more reverse transformation from $q\{\tau_t\}$. According to P2.2.2, $q\{\tau_t^{0m3}\}$ will not have any representation in $q\{\tau_{t+1}\}$ as relevant n_{1x} does not exist. Eliminating such elements, we have; $$q\{\tau_{t+1}\} = q\{\tau_t^{\neg 0m3}\} = q\{\tau_t^{1m3}\} + q\{\tau_t^{2m3}\} = q\{\tau_t\} - q\{\tau_t^{0m3}\}$$ Using Theorem 5 describing one third of elements being 0 mod3 $$q\{\tau_{t+1}\} = q\{\tau_t^{\neg 0m3}\} = q\{\tau_t\} - \frac{1}{3} q\{\tau_t\} = \frac{2}{3} q\{\tau_t\}$$
P2.b.4.1	At t=0	$$q\{\tau_0^{0m3}\} = \frac{2^0}{3^1} q(\{U'\} - [1]) \cong 0 \text{ mod3}$$ $$q\{\tau_0^{\neg 0m3}\} = \frac{2^1}{3^1} q(\{U'\} - [1]) \ncong 0 \text{ mod3}$$
Proof:		According to Theorem 5: one third of elements are 0 mod3 $$q\{\tau_0^{0m3}\} = \frac{1}{3} q(\{U'\} - [1]) = \frac{2^0}{3^1} q(\{U'\} - [1]) \cong 0 \text{ mod3}$$ Elements that are not 0 mod3 have their respective n_{1x} represented in the set $q\{\tau_0^{\neg 0m3}\}$ $$q\{\tau_0^{\neg 0m3}\} = q(\{U'\} - [1]) - q(\tau_0^{0m3}) = q(\{U'\} - [1]) - \frac{1}{3} q(\{U'\} - [1]) = \frac{2^1}{3^1} q(\{U'\} - [1]) \ncong 0 \text{ mod3}$$
P2.b.4.2	At t=1	$$q\{\tau_1^{0m3}\} = \frac{2^1}{3^2} q(\{U'\} - [1])\ \&\ q\{\tau_1^{\neg 0m3}\} = \frac{2^2}{3^2} q(\{U'\} - [1])$$
Proof		According to Theorem 5: one third of elements are 0 mod3 $$q\{\tau_1^{0m3}\} = \frac{1}{3} q\{\tau_0^{\neg 0m3}\} = \frac{1}{3} . \frac{2^1}{3^1} q(\{U'\} - [1]) = \frac{2^1}{3^2} q(\{U'\} - [1])$$

		Elements that are not 0 mod3 have their respective n_{1x} represented in the set $q\{\tau_1^{\neg 0m3}\}$ $$q\{\tau_1^{\neg 0m3}\} = q\{\tau_1\} - q\{\tau_1^{0m3}\} = \frac{2^2}{3^2} q(\{U'\} - [1])$$
P2.b.4.3	Mathematical Induction	$$q\{\tau_t^{0m3}\} = \frac{1}{3} q\{\tau_{t-1}^{\neg 0m3}\} = \frac{2^t}{3^{t+1}} q(\{U'\} - [1])$$ $$\& \; q\{\tau_t^{\neg 0m3}\} = \frac{2^1}{3} q\{\tau_{t-1}^{\neg 0m3}\} = \frac{2^{t+1}}{3^{t+1}} q(\{U'\} - [1])$$
Proof:		Assumed case for mathematical induction
P2.b.4.4		$$q\{\tau_{t+1}^{0m3}\} = \frac{2^{t+1}}{3^{t+2}} q(\{U'\} - [1])$$ $$\& \; q\{\tau_{t+1}^{\neg 0m3}\} = \frac{2^{t+2}}{3^{t+2}} q(\{U'\} - [1])$$
Proof:		Using P2.b.4.4 $$q\{\tau_{t+1}^{\neg 0m3}\} = \frac{2}{3} q\{\tau_{t+1}^{\neg 0m3}\} = \frac{2}{3} \frac{2^{t+1}}{3^{t+1}} q(\{U'\} - [1])$$ $$= \frac{2^{t+2}}{3^{t+2}} q(\{U'\} - [1])$$ $$q\{\tau_{t+1}^{0m3}\} = q\{\tau_t\} - q\{\tau_{t+1}^{\neg 0m3}\}$$ $$= \frac{2^t}{3^{t+1}} q(\{U'\} - [1]) - \frac{2^{t+2}}{3^{t+2}} q(\{U'\} - [1])$$ $$= \frac{2^t}{3^{t+2}} q(\{U'\} - [1])(3-1)$$ $$q\{\tau_{t+1}^{0m3}\} = \frac{2^{t+1}}{3^{t+2}} q(\{U'\} - [1])$$
P2.b.5.1		$$q(\{U'\} - [1]) = \sum_{t=0}^{t\to\infty} q\{\tau_t^{0m3}\}$$
Proof:		$$\sum_{t=0}^{t\to\infty} q\{\tau_t^{0m3}\} = q\{\tau_0^{0m3}\} + q\{\tau_1^{0m3}\} + q\{\tau_2^{0m3}\} + q\{\tau_3^{0m3}\}$$ $$+ q\{\tau_4^{0m3}\} + q\{\tau_5^{0m3}\} + q\{\tau_6^{0m3}\} + q\{\tau_7^{0m3}\} \ldots$$ $$\sum_{t=0}^{t\to\infty} q\{\tau_t^{0m3}\} = q(\{U'\} - [1])(\frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \frac{2^6}{3^7}$$ $$+ \frac{2^7}{3^8} \ldots)$$ $$\text{let } \frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \frac{2^6}{3^7} + \frac{2^7}{3^8} + \frac{2^8}{3^9} + \frac{2^9}{3^{10}} + \cdots = s$$ $$(\frac{2}{3}(\frac{2^0}{3^1} + \frac{2^1}{3^2} + \frac{2^2}{3^3} + \frac{2^3}{3^4} + \frac{2^4}{3^5} + \frac{2^5}{3^6} + \frac{2^6}{3^7} + \frac{2^7}{3^8} + \frac{2^8}{3^9} + \cdots)) = s - \frac{1}{3}$$

		$s = 1$ $$\sum_{t=0}^{t\to\infty} q\{\tau_t^{0m3}\} = q(\{U'\} - [1])$$
P2.b.0	THEN	$n_s \not\equiv\not\equiv n_s$
Proof:		Using P2.b.5.1 upon applying reverse transformation, the total number of elements that are 0 mod3 is equal to total number of elements in $\{U'\} - [1]$ implying all the elements of $\{U'\} - [1]$ reach 0 mod3. Using P2.b.2.2: none of the elements that are 0 mod3 loop. None of the elements can loop under given transformational conditions.

■

Alternatively, one could prove that $n_s \equiv\equiv n_s | n_s \neq n_x$ for any and all arbitrary element/s by just using Corollary 5 encountering similar expression mentioned in proof of P2.b.5.1

However, in the negative domain loop exists, example:$-7 \equiv -5 \equiv -7$, but we don't care as it is out of domain of the conjecture.

Possible solutions at $t \to \infty$ may be represented as; $n_s \equiv\equiv \infty \vee n_s \equiv\equiv n_s \vee n_s \equiv n_s$

$$s = p \vee q \vee r |\ p = (n_s \equiv\equiv \infty), q = (n_s \equiv\equiv n_s | n_s \neq 1), r = (n_s \equiv n_s = 1)$$

Using P1.0, P2.b.0 & P2.a.0, we know

$$\forall n_s \not\equiv \infty \vee n_s \not\equiv\not\equiv n_s \vee n_s \equiv n_s = 1$$

$$\dashv p \dashv q \vdash s = r$$

$$\forall \lim_{t\to\infty} n_s \equiv\equiv 1$$

Thus, the conjecture is true.

■

References: Also known as 3*n* + 1 problem, the 3*n* + 1 conjecture, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem. The sequence of numbers involved is sometimes referred to as the hailstone sequence, hailstone numbers or hailstone numerals

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