The Riemann Hypothesis Is True: The End of The Mystery (V5)

Abdelmajid Ben Hadj Salem

To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract. In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \quad \Re(s) > 1$$

have real part $\sigma = \dfrac{1}{2}$. In this note, I give the proof that $\sigma = \dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

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1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1]:

Conjecture 1.1. Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

1.1. The function zeta(s)

We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \ \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 1.2. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;

2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;

3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;

4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 1.3. (The Riemann Hypothesis,[2]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

In addition to the properties cited by the theorem 1.2 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0, 1\}$:

$$\zeta(1 - s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{1.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (1.1), we will use the one presented by G.H. Hardy [3] namely Dirichlet's eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

Theorem 1.4. *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

1.2. A Equivalent statement to the Riemann Hypothesis

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [2]:

Equivalence 1.5. *The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :*

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{1.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (1.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{1.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{1.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2. Preliminaries of the proof that the zeros of the function eta(s) are on the critical line $\Re(s) = 1/2$

We begin by recalling some definitions:

- Let a_n be a sequence of real or complex numbers. A necessary and sufficient condition for the sequence to converge is that for any $\epsilon > 0$ there exists an integer $n_0 > 0$ such that:

$$| a_p - a_q | < \epsilon$$

holds for all integers p and q greater than n_0. This is the Cauchy criterion.

- An infinite series $\sum_{n=1}^{+\infty} a_n$ converges if and only if for any $\epsilon > 0$ there exists an integer $n_0 > 0$ satisfying $\mid a_q + \ldots + a_p \mid < \epsilon$ for all integers p and q greater than n_0. The last condition can also be written as :

$$\mid \sum_{n=1}^{n=q-1} a_n \mid < \epsilon$$

- An infinite series $\sum_{n=1}^{+\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{+\infty} \mid a_n \mid$ converges.

Proof. . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1 - 2^{1-s})\zeta(s) = 0$. We verify easily the two propositions:

$$\boxed{s, \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)} \tag{2.1}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{s, \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)} \tag{2.2}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \mid \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \mid < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s)=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^s}=\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma}-i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

with $s=\sigma+it$ and $t\neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s)=0$, with $0<\sigma<1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s)=0=\eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma}=0$$

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}=0$$

Using the definition of the limit of a sequence, we can write:

$$\forall\epsilon_1>0\,\exists n_r,\forall N>n_r,\;|\,\Re(\eta(s)_N)\,|<\epsilon_1\Longrightarrow\Re(\eta(s)_N)^2<{\epsilon_1}^2 \tag{2.3}$$

$$\forall\epsilon_2>0\,\exists n_i,\forall N>n_i,\;|\,\Im(\eta(s)_N)\,|<\epsilon_2\Longrightarrow\Im(\eta(s)_N)^2<{\epsilon_2}^2 \tag{2.4}$$

Then:

$$0<\sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma}<\epsilon_1^2$$

$$0<\sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma}<\epsilon_2^2$$

Taking $\epsilon=\epsilon_1=\epsilon_2$ and $N>max(n_r,n_i)$, we get by making the sum member to member of the last two inequalities:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}<2\epsilon^2 \tag{2.5}$$

We can write the above equation as :

$$0<\ \rho_N^2\ <2\epsilon^2 \tag{2.6}$$

or $\rho(s)=0$.

3. Case $\Re(s)=1/2$

We suppose that $\sigma=\dfrac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 3.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.1-2.2), it follows the proposition :

Proposition 3.2. *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.5) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \quad (3.1)$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

4. Case $0 < \Re(s) < 1/2$

4.1. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$.

Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (2.5):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} = -\infty} \quad (4.1)$$

5. Case $1/2 < Re(s) < 1$

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (2.1), $\zeta(s) = 0$. According to point 4 of theorem 1.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (2.5), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{5.1}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (5.1), it follows that :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty} \tag{5.2}$$

5.0.1. Case $t = 0$. We suppose that $t = 0 \Longrightarrow t' = 0$. The equation (5.2) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.3}$$

Then $s' = \sigma' > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s') = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'}} = 0 \tag{5.4}$$

Let us define the sequence S_m as:

$$S_m(s') = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma') \tag{5.5}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') \tag{5.6}$$

We have also:

$$S_1(\sigma') = 1 > 0 \tag{5.7}$$

$$S_2(\sigma') = 1 - \frac{1}{2^{\sigma'}} > 0 \quad because\ 2^{\sigma'} > 1 \tag{5.8}$$

$$S_3(\sigma') = S_2(\sigma') + \frac{1}{3^{\sigma'}} > 0 \tag{5.9}$$

We proceed by recurrence, we suppose that $S_m(\sigma')>0$.

1. $m=2q\Longrightarrow S_{m+1}(\sigma')=\displaystyle\sum_{n=1}^{m+1}\frac{(-1)^{n-1}}{n^{s'}}=S_m(\sigma')+\frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$, it gives:

$$S_{m+1}(\sigma')=S_m(\sigma')+\frac{(-1)^{2q}}{(m+1)^{\sigma'}}=S_m(\sigma')+\frac{1}{(m+1)^{\sigma'}}>0\Rightarrow S_{m+1}(\sigma')>0$$

2. $m=2q+1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma')=S_{m-1}(\sigma')+\frac{(-1)^{m-1}}{m^{\sigma'}}+\frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma')>0$, let $T=\dfrac{(-1)^{m-1}}{m^{\sigma'}}+\dfrac{(-1)^m}{(m+1)^{\sigma'}}$, we obtain:

$$T=\frac{(-1)^{2q}}{(2q+1)^{\sigma'}}+\frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}}=\frac{1}{(2q+1)^{\sigma'}}-\frac{1}{(2q+2)^{\sigma'}}>0 \tag{5.10}$$

and $S_{m+1}(\sigma')>0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $lim_{m\longrightarrow+\infty}S_m(s')=\eta(s')=\eta(\sigma')>0$ and $\eta(\sigma')<+\infty$ because $\Re(s')=\sigma'>0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (5.3) is false for the case } t'=t=0.} \tag{5.11}$$

5.0.2. Case $t\neq 0$. Great effort has been put to find regions inside the critical strip where there are no zeros of the function $\zeta(s)$. The classical zero-free region is of the form $\sigma>1-1/(R_0log|t|)$, where R_0 is a positive constant. The best known result of this form is due to H. Kadiri [4]:

Theorem 5.1. ***(Kadiri, 2005)*** *$\zeta(s)$ does not vanish in the region:*

$$\Re(s)\geq 1-\frac{1}{R_0log|\Im(s)|},\ |\Im(s)|\geq 2 \quad with\ R_0=5.69693 \tag{5.12}$$

In the equation (5.2), we have used $s'=\sigma'+it'$ where we can consider that $t'>2$, with $2>2\sigma'>1$ and $\sigma'\in]1/2,1[$. The same equation expresses that $\eta(s')=0\Longrightarrow\zeta(s')=0$, but it does not give any obstruction that $s'=\sigma'+it'$ could be in the zero-free region of the function ζ defined by the last theorem above so that:

$$\sigma'\geq 1-\frac{1}{R_0log|t'|}>1-\frac{1}{R_0log2}\approx 0.74\Longrightarrow 2>2\sigma'>1,\quad t'>2$$

Then the contradiction, it follows that the equation (5.2) is false and $\eta(s')$ does not vanish for $\sigma'\in]1/2,1[$ and:

$$\boxed{\text{The equation (5.2) is false for the case } t'=t\neq 0.} \tag{5.13}$$

From (5.11) and the equation above, we conclude that the function $\eta(s)$ has no zeros for all $s'=\sigma'+it'$ with $\sigma'\in]1/2,1[$, it follows that the case of the paragraph (4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false too. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (1.5), it follows that **the Riemann hypothesis is verified**. $\square$

From the calculations above, we can verify easily the following known proposition:

Proposition 5.2. *For all* $s = \sigma$ *real with* $0 < \sigma < 1$, $\eta(s) > 0$ *and* $\zeta(s) < 0$.

6. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (1.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 6.1. *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ *with* $s = \sigma + it$ *lie on the vertical line* $\Re(s) = \frac{1}{2}$.

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Abdelmajid Ben Hadj Salem
Residence Bousten 8, Bloc B, Av. Mosquee Raoudha,
Soukra, 1181 Soukra Raoudha,
Tunisia
e-mail: abenhadjsalem@gmail.com