THE RIEMANN HYPOTHESIS IS TRUE: THE END OF THE MYSTERY.

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To my wife Wahida, my daughter Sinda and my son Mohamed Mazen
To the memory of my friend and colleague Jalel Zid (1959 - 2023)

Abstract. In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros)* $s = \sigma + it$ *of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \quad \Re(s) > 1$$

have real part $\sigma = \dfrac{1}{2}$.

We give the proof that $\sigma = \dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

keywords: zeta function, non trivial zeros of eta function, equivalence statements, definition of limits of real sequences.

1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1]:

Conjecture 1.1. Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

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1.1. **The function ζ.** We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 1.2. *The function $\zeta(s)$ satisfies the following :*
1. *$\zeta(s)$ has no zero for $\Re(s) > 1$;*
2. *the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;*
3. *$\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;*
4. *the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.*

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 1.3. (The Riemann Hypothesis,[2]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

In addition to the properties cited by the theorem 1.2 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{1.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (1.1), we will use the one presented by G.H. Hardy [3] namely Dirichlet's eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

Theorem 1.4. *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

1.2. **A Equivalent statement to the Riemann Hypothesis.** Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [2]:

Equivalence 1.5. The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s),\quad \sigma>1 \tag{1.2}$$

that fall in the critical strip $0<\Re(s)<1$ lie on the critical line $\Re(s)=\dfrac{1}{2}$.

The series (1.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s)=\sigma>0$ ([3], pages 20-21). We can rewrite:

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s),\quad \Re(s)=\sigma>0 \tag{1.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s)=\rho.e^{i\alpha}\Longrightarrow\rho^2=\eta(s).\overline{\eta(s)} \tag{1.4}$$

and $\eta(s)=0\Longleftrightarrow\rho=0$.

2. Preliminaries of the proof that the zeros of the function $\eta(s)$ are on the critical line $\Re(s)=\frac{1}{2}$.

Proof. . We denote $s=\sigma+it$ with $0<\sigma<1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s=\sigma+it$, then we obtain $0<\sigma<1$ and $\eta(s)=0\Longleftrightarrow(1-2^{1-s})\zeta(s)=0$. We verifies easily the two propositions:

(2.1)

$$\boxed{s,\ \textit{is one zero of}\ \ \eta(s)\ \textit{that falls in the critical strip, is also one zero of}\ \ \zeta(s)}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s)=0\Longrightarrow\eta(s)=(1-2^{1-s})\zeta(s)=0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(2.2)

$$\boxed{s,\ \textit{is one zero of}\ \ \zeta(s)\ \textit{that falls in the critical strip, is also one zero of}\ \eta(s)}$$

Let us write the function η:

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn}=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn}=$$

$$=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$

$$=\sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn)-isin(tLogn))$$

The function η is convergent for all $s\in\mathbb{C}$ with $\Re(s)>0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \Big| \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \Big| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} - i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

$$\forall\epsilon_1 > 0\,\exists n_r, \forall N > n_r,\ |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2 \tag{2.3}$$

$$\forall\epsilon_2 > 0\,\exists n_i, \forall N > n_i,\ |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2 \tag{2.4}$$

Then:

$$0 < \sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N}\frac{1}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{2.5}$$

We can write the above equation as :

$$0 < \ \rho_N^2 \ < 2\epsilon^2 \tag{2.6}$$

or $\rho(s) = 0$.

3. Case $\sigma = \dfrac{1}{2}$.

We suppose that $\sigma = \dfrac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 3.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.1-2.2), it follows the proposition :

Proposition 3.2. There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.5) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{3.1}$$

if not, we will have a contradiction with the fact that :

$$lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

4. Case $0 < \Re(s) < \frac{1}{2}$.

4.1. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$. Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (2.5):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty} \tag{4.1}$$

5. Case $\frac{1}{2} < \Re(s) < 1$.

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (2.1), $\zeta(s) = 0$. According to point 4 of theorem 1.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (2.5), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{5.1}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (5.1), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.2}$$

5.0.1. *Case* $t = 0$. We suppose that $t = 0 \Longrightarrow t' = 0$. The equation (5.2) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.3}$$

Then $s' = \sigma' > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s') = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'}} = 0 \tag{5.4}$$

Let us define the sequence S_m as:

$$S_m(s') = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma') \tag{5.5}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') \tag{5.6}$$

We have also:

$$S_1(\sigma') = 1 > 0 \tag{5.7}$$

$$S_2(\sigma') = 1 - \frac{1}{2^{\sigma'}} > 0 \quad because\ 2^{\sigma'} > 1 \tag{5.8}$$

$$S_3(\sigma') = S_2(\sigma') + \frac{1}{3^{\sigma'}} > 0 \tag{5.9}$$

We proceed by recurrence, we suppose that $S_m(\sigma') > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma') = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'}} = S_m(\sigma') + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$, it gives:

$$S_{m+1}(\sigma') = S_m(\sigma') + \frac{(-1)^{2q}}{(m+1)^{\sigma'}} = S_m(\sigma') + \frac{1}{(m+1)^{\sigma'}} > 0 \Rightarrow S_{m+1}(\sigma') > 0$$

2. $m = 2q + 1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma') = S_{m-1}(\sigma') + \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma') > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m}}{(m+1)^{\sigma'}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}} = \frac{1}{(2q+1)^{\sigma'}} - \frac{1}{(2q+2)^{\sigma'}} > 0 \tag{5.10}$$

and $S_{m+1}(\sigma') > 0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $lim_{m\longrightarrow+\infty} S_m(s') = \eta(s') = \eta(\sigma') > 0$ and $\eta(\sigma') < +\infty$ because $\Re(s') = \sigma' > 0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (5.3) is false for the case } t' = t = 0.} \tag{5.11}$$

5.0.2. *Case $t \neq 0$.* We suppose that $t \neq 0$. For each $s' = \sigma' + it' = 1 - \sigma + it$ a zero of $\eta(s)$, we have:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.12}$$

the left member of the equation (5.12) above is finite and depends of σ' and t', but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$. But for all $\sigma"$ so that $2\sigma" > 1$, we have $\zeta(2\sigma")$:

$$\zeta(2\sigma") = \zeta_1(2\sigma") = \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma"}} < +\infty$$

It depends only of $\sigma"$, then in particular for all $\sigma"$ with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of $\sigma"$. Let $\lambda > 0$ be an arbitrary real number very infinitesimal so that $\sigma' + \lambda \in]1/2, 1[$ is not the real part of a zero of $\eta(s)$, that we write $\forall \tau > 0, \upsilon = \sigma' + \lambda + i\tau$ verifies $\eta(\upsilon) \neq 0$. Let $g(\sigma")$ be the function $\zeta(2\sigma")$, the first derivative of g is given by:

$$g'(\sigma") = -2\sum_{k=2}^{+\infty} \frac{Logk}{k^{2\sigma"}} > -\infty \tag{5.13}$$

because we will choose $\alpha > 0$ so that $\sigma" > 1/2+\alpha \Longrightarrow 2(\sigma"-\alpha) > 1$ and we obtain:

$$g'(\sigma") = -2\sum_{k=2}^{+\infty}\frac{Logk}{k^{2\sigma"}} = -\frac{1}{\alpha}\sum_{k=2}^{+\infty}\frac{Logk^{2\alpha}}{k^{2\alpha}}\frac{1}{k^{2(\sigma"-\alpha)}} \Longrightarrow$$

$$|g'(\sigma")| \leq \frac{1}{\alpha}\sum_{k=2}^{+\infty}\frac{Logk^{2\alpha}}{k^{2\alpha}}\frac{1}{k^{2(\sigma"-\alpha)}} \leq \frac{1}{\alpha}\sum_{k=2}^{+\infty}\frac{1}{k^{2(\sigma"-\alpha)}} < +\infty \tag{5.14}$$

Let $\sigma_0 \in]1/2,1[$ so that $\sigma" > \sigma_0 > \frac{1}{2}+\alpha$, it follows:

$$|g'(\sigma")| \leq \frac{1}{\alpha}\sum_{k=2}^{+\infty}\frac{1}{k^{2(\sigma"-\alpha)}} \leq \frac{1}{\alpha}\left(\zeta(2(\sigma_0-\alpha))-1\right) < +\infty$$

that justifies the operation $\left(\sum g_n(\sigma")\right)' = \sum(g_k(\sigma"))'$. We will use it for the calculation of $g"(\sigma")$. Now, Let us calculate the second derivative of the function $g(\sigma")$. We obtain:

$$g"(\sigma") = 2\sum_{k=2}^{+\infty}\frac{(Logk)^2}{k^{2\sigma"}} > 0$$

then:

$$g"(\sigma") = 2\sum_{k=2}^{+\infty}\frac{(Logk)^2}{k^{2\sigma"}} = \frac{2}{\alpha^2}\sum_{k=2}^{+\infty}\left(\frac{Logk^{\alpha}}{k^{\alpha}}\right)^2\frac{1}{k^{2(\sigma"-\alpha)}} \leq \frac{2}{\alpha^2}\sum_{k=2}^{+\infty}\frac{1}{k^{2(\sigma"-\alpha)}} < +\infty \tag{5.15}$$

Finally, $|g'(\sigma")|$ and $g"(\sigma")$ are bounded, then $g(\sigma")$ is a function of C^3 on $]1/2+\alpha,1[$. We take $\sigma" = \sigma'$ and we can write:

$$g(\sigma'+\lambda) = g(\sigma') + \lambda g'(\sigma') + \frac{g"(\theta)}{2!}\lambda^2 \quad \text{with some suitable } \theta \in]\sigma',\sigma'+\lambda[\tag{5.16}$$

We can re-write the above equation as:

$$\zeta(2\sigma'+2\lambda) = \zeta(2\sigma') + 2\lambda\zeta'(2\sigma') + \frac{g"(\theta)}{2!}\lambda^2 \quad \text{with some suitable } \theta \in]\sigma',\sigma'+\lambda[\tag{5.17}$$

we have two cases to study:

case a): the term $\zeta'(2\sigma')$ is independent of t', the equation (5.17) can we written as:

$$\zeta(2\sigma'+2\lambda) - 2\lambda\zeta'(2\sigma') = \zeta(2\sigma') + \frac{g"(\theta)}{2!}\lambda^2 \quad \text{with some suitable } \theta \in]\sigma',\sigma'+\lambda[\tag{5.18}$$

As $\sigma'+\lambda$ and $\zeta'(2\sigma')$ are independent of t', numerically, the left member of the above equation is independent of t', but the right member depends of t' using the equation (5.12), then the contradiction.

case b): the term $\zeta'(2\sigma')$ depends of t', we rewrite the equation (5.17):

$$\zeta(2\sigma'+2\lambda) = \zeta(2\sigma') + 2\lambda\zeta'(2\sigma') + \frac{g"(\theta)}{2!}\lambda^2 \quad \text{with some suitable } \theta \in]\sigma',\sigma'+\lambda[$$

There too, the left member of the above equation is independent of t', but the right member depends of t' using the equation (5.12), then the contradiction and we conclude that the result giving by the equation (5.12) is false.

(5.19) It follows that the equation (5.12) is false for the case $t' \neq 0$.

From (5.11-5.19), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the case of the paragraph (4) above concerning the case $0 < \Re(s) < \dfrac{1}{2}$ is false too. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (1.5), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following known proposition:

Proposition 5.1. For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

6. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \dfrac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (1.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 6.1. *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$.

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