Article

Collatz conjecture.

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1. Abstract

The Paper analyzes the number of zeros in the binary representation of a natural number. The analysis is carried out using the concept of the fractional part of a number, which naturally arises when finding a binary representation. This idea relies on the fundamental property of the Riemann zeta function, which is constructed using the fractional part of a number. Understanding that the ratio of the fractional and integer parts, by analogy with the Riemann zeta function, expresses the deep laws of numbers, will explain the essence of this work. For the Syracuse sequence of numbers that appears in the Collatz conjecture, we use a binary representation that allows us to obtain a uniform estimate for all terms of the series, and this estimate depends only on the initial term of the Syracuse sequence. This estimate immediately leads to the solution of the Collatz conjecture.

2. Introduction

The paper analyzes the number of zeros in the binary representation of a natural number. The analysis is carried out using the concept of the fractional part of a number, which naturally arises when finding a binary representation. This idea relies on the fundamental property of the Riemann zeta function, which is constructed using the fractional part of a number. Understanding that the ratio of the fractional and integer parts, by analogy with the Riemann zeta function, expresses the deep laws of numbers, will explain the essence of this work. For the Syracuse sequence of numbers that appears in the Collatz conjecture, we use a binary representation that allows us to obtain a uniform estimate for all terms of the series, and this estimate depends only on the initial term of the Syracuse sequence. This estimate immediately leads to the solution of the Collatz conjecture.

3. Materials and Methods

This work is based on the following methods of analysis of the Syracuse sequence 1. Analysis of simple cases of natural numbers starting from which the Syracuse sequence quickly converges to one
2. A process of expansion of a natural number in powers of two is created.
3. The proximity to the completion of decomposition is analyzed at each stage
4. The number of zeros in the binary expansion of a natural number is calculated
5. It is shown that the number of powers of two prevails in the doitic expansions in the Syracuse sequence
6 Based on these results, it is shown that the Syracuse sequence converges to one

4. Results

In this work we present the final solution to the Collatz conjecture formulated in [1]. The Collatz conjecture concerns integer sequences generated as follows:

Start with any positive integer a_0. Every next term is defined as

$$a_{n+1} = \alpha_n a_n + \beta_n. \tag{1}$$

Where $n \geq 0$, and if a_n is even then $\alpha_n = 0.5$, $\beta_n = 0$ if a_n is odd, then $\alpha_n = 3$, $\beta_n = 1$.

Citation: Asset Durmagambetov New estimates for zeta function.. *Journal Not Specified* **2023**, *1*, 0. https://doi.org/

Received:
Revised:
Accepted:
Published:

The conjecture is that regardless of a_0 , the sequence will always reach 1. The conjecture is named after Lothar Collatz, who introduced the idea in 1937.[1] It is also known as the 3n + 1 problem, the 3n + 1 conjecture, the Ulam conjecture (after Stanisław Ulam), Kakutani's problem (after Shizuo Kakutani), the Thwaites conjecture (after Sir Bryan Thwaites), Hasse's algorithm (after Helmut Hasse), or the Syracuse problem.

In this work, we obtained a uniform estimate for the Syrocuse sequences and proved that every 4n steps the sequences come down to a number smaller than the starting term, from which follows the solution of the Collatz problem.

5. Rezults

Our idea of the proof is to obtain a uniform estimate for the Syracuse sequence described in Introduction. Here and below, we will always mean by a_n n-term of the sequence . For definiteness, we assume that

$$a_0 = 2^{n+1}a_n,\ a_1 = 2^n a_n,\ a_2 = 2^{n-1}a_n,\ .....,\ a_{n-1} = 2a_n,\ a_n,\ ...$$

According to the sequence generation rule, it is enough to consider the odd numbers, since even numbers will always become odd. Hence, we can assume that for any a_0, after the last appearance of a zero coefficient $\gamma_i \in \{0,1\}$), the rest are not zero, as they would disappear from dividing by 2. Thus, without losing generality of our reasoning, we can assert that it suffices to consider numbers a_n of the following form:

$$a_n = \sum_{i=k+2}^{n} 2^i \gamma_i + \sum_{i=0}^{k} 2^i,\ \ n > k > 2$$

Binary representation helps to understand the idea of this work.

Theorem 1. *Let*

$$x \in N,\ \ [\alpha_j] - [\alpha_{j+1}] = \delta_j > 0,\ \epsilon_1 < 1/2,$$

$$x = \sum_{i=1}^{j-1} 2^{[\alpha_i]} + 2^{\alpha_j},\ \ x = \sum_{i=1}^{j} 2^{[\alpha_i]} + 2^{\alpha_{j+1}},\ \sigma_j = 1 - \epsilon_j$$

Then
as $\delta_j = 1$

$$\sigma_{j+1} ln2 = \frac{2\sigma_j ln2}{1 - \sigma_{j+1} ln2} + o(\sigma_{j+1}^2/4)$$

as $\delta_j > 1$

$$\sigma_{j+1} ln2 = -2^{\delta_j - 1} \frac{ln2 - 2^{-\delta_j - 1}}{1 - \sigma_{j+1} ln2/2} + 2^{\delta_j - 1}\sigma_j ln2 \frac{1}{1 - \sigma_{j+1} ln2/2} + o(\sigma_{j+1}^2)$$

Proof.

$$2^{\epsilon_j} = 2^{-\delta_j + \epsilon_{j+1}} + 1 \Rightarrow 2^{1-\sigma_j} = 2^{-\delta_j + 1 - \sigma_{j+1}} + 1 \Rightarrow$$

$$ln(2^{1-\sigma_j}) = ln2 - \sigma_j ln2 = ln(2^{-\delta_j + 1 - \sigma_{j+1}} + 1)$$

Computing as $\delta_j = 1$

$$ln(2^{-\delta_j + 1 - \sigma_{j+1}} + 1)|_{\delta_j = 1} = ln(2^{-\sigma_{j+1}} + 1) = ln((1 - \sigma_{j+1} ln2 + \sigma_{j+1}^2 ln^2/2) + 1 + o(\sigma_{j+1}^2/4))$$

$$ln(2 - \sigma_{j+1} ln2 + \sigma_{j+1}^2 ln2^2/2) = ln2 + ln(1 - \sigma_{j+1} ln2/2 + \sigma_{j+1}^2 ln^2/4 + o(\sigma_{j+1}^2/4))$$

$$ln(2^{-\sigma_{j+1}} + 1) = ln2 - ln2\sigma_{j+1}/2 + ln^2 2\sigma_{j+1}^2/2 + o(\sigma_{j+1}^2/4)$$

$$ln2 - \sigma_j ln2 = ln2 - ln2\sigma_{j+1}/2 + ln^2 2\sigma_{j+1}^2/4 + o(\sigma_{j+1}^2/4)$$

$$\sigma_{j+1} ln2 = \frac{2\sigma_j ln2}{1 - \sigma_{j+1} ln2/2} + o(\sigma_{j+1}^2/4)$$

Repeating computing as $\delta_j > 1$ we get

$$ln(2^{-\delta_j+1-\sigma_{j+1}} + 1) = ln(2^{-\delta_j+1}2^{-\sigma_{j+1}} + 1) =$$

$$ln(1 + 2^{-\delta_j+1} + 2^{-\delta_j+1}[-\sigma_{j+1} ln2 + \sigma_{j+1}^2 ln2^2/2] + o(\sigma_{j+1}^2/4 + 2^{-\delta_j+1}) =$$

$$2^{-\delta_j+1} + 2^{-\delta_j+1}[-\sigma_{j+1} ln2 + \sigma_{j+1}^2 ln2^2/2] + o(\sigma_{j+1}^2 + 2^{-\delta_j+1}) \Rightarrow$$

$$ln2 - \sigma_j ln2 = 2^{-\delta_j+1} + 2^{-\delta_j+1}[-\sigma_{j+1} ln2 + \sigma_{j+1}^2 ln2^2/2] + o(\sigma_{j+1}^2 + 2^{-\delta_j+1})$$

$$\sigma_{j+1} ln2 = -2^{\delta_j-1}\frac{ln2 - 2^{-\delta_j+1}}{1 - \sigma_{j+1} ln2/2} + 2^{\delta_j-1}\sigma_j ln2 \frac{1}{1 - \sigma_{j+1} ln2/2} + o(\sigma_{j+1}^2 + 2^{-\delta_j+1})$$

□

Theorem 2. *Let*

$$x \in N, \quad \alpha_j = [\alpha_j] \quad x = \sum_{i=1}^{j-1} 2^{[\alpha_i]} + 2^{\alpha_j}$$

Then the number of zeros in the binary representation C_z is calculated by the following formula

$$C_z = \sum_{i=1}^{j-1}[\delta_i - 1] + \alpha_j - 1$$

Proof.

$$C_z = \sum_{i=1}^{j-1}[\alpha_i - \alpha_{i+1} - 1] + \alpha_j - 1$$

By definition δ_i

$$C_z = \sum_{i=1}^{j-1}[\delta_i - 1] + \alpha_j - 1$$

□

Let's introduce μ_k, ν_k for $x = \sum_{i=0}^{n} \gamma_i 2^i$ by following rules

$$\gamma_k + \gamma_{k+1} = 1, \quad \gamma_{k+\mu_k} + \gamma_{k+\mu_k+1} = 1, \prod_{i=k+1}^{i=\mu_k} \gamma_i = 1;$$

$$\gamma_j + \gamma_{j+1} = 1, \quad \gamma_{j+\mu_j} + \gamma_{j+\nu_j+1} = 1, \nu_j = \sum_{i=j+1}^{i=\nu_j} (1 - \gamma_i)$$

another words
μ_k, is count of ones starting at point k with no zeros in between until the first zero or until the end of the sequence
ν_j is count of zeros starting at point j with no ones in between until the first zero or until the end of the sequence

Theorem 3. *Let*

$$x = 3^n = 2^{[\alpha]+\{\alpha\}} = \sum_{i=1}^{n*} \gamma_i 2^i,$$

$$\{\alpha\} > ln2, \quad n^* = n * [\ln(3)/\ln(2)] \tag{2}$$

then

$$\sum_{\gamma_i=0} 1 \geq n^*/2 - 5$$

Proof.

$$3^n = 2^\alpha \Rightarrow \alpha = n/\ln(3)/\ln(2) \Rightarrow 3^n = 2^{[\alpha]+\{\alpha\}}$$

Using Theorem 1, we create a sequence

$$\epsilon_i, \;\; m_i, \epsilon_1 = \{\alpha\}$$

$$2^{\epsilon_1} = \sum_{k=0}^{i-1} 2^{[\alpha_k]-\alpha_1} + 2^{\alpha_i - \alpha_1}$$

Suppose

$$\sum_{\gamma_i=0} 1 = 0$$

then by Theorem 1

$$\Rightarrow \sigma_{j+1} ln2 = \frac{2\sigma_j ln2}{1 - \sigma_{j+1} ln2} + o(\ln 2\sigma_{j+1}^2/4) \Rightarrow$$

$$2^{-1}\sigma_{j+1} ln2 = \frac{\sigma_j ln2}{1 - \sigma_{j+1} ln2} + 2^{-1} * o(\ln 2\sigma_{j+1}^2/4)$$

After repeating j times we get

$$2^{-j}\sigma_{j+1} ln2 = \frac{\sigma_1 ln2}{\prod_1^j (1 - \sigma_{k+1} ln2/2)} + \sum_1^j 2^{-k} * o(\ln 2\sigma_{k+1}^2/4)$$

By Theorems (1-2) and condition of the current Theorem proceed

$$ln2/2 < \sigma_1 ln2 < o(\ln 2\sigma_{k+1}^2/4)$$

immediately

$$\Rightarrow \sum_{\gamma_i=0} 1 > 0$$

Let's introduce

$$as \;\; \delta_k = 1 : \;\; \alpha_k = 0, \;\; \beta_k = \frac{1}{1 - \sigma_{j+1} ln2}$$

$$as \;\; \delta_k > 1 : \alpha_k = -2^{\delta_j - 1} \frac{ln2 - 2^{-\delta_j - 1}}{1 - \sigma_{j+1} ln2/2}, \;\; \beta_k = \frac{2^{\delta_j - 1}}{1 - \sigma_{j+1} ln2/2}$$

$$\sigma_{k+1} = \alpha_k + \beta_k \sigma_k$$

$$\sigma_{j+1} ln2 = \alpha_j + \sum_{m=1}^{m=j-1} \alpha_{j-m} \prod_{l=1}^{l=m} \beta_{j-l+1} + \prod_{l=0}^{l=j-1} \beta_{j-l} \sigma_1 \Rightarrow$$

$$\frac{\sigma_{j+1} ln2}{\prod_{l=0}^{l=j-1} \beta_{j-l}} = \sum_{m=0}^{m=j-1} \frac{\alpha_{j-m}}{\prod_{l=m+1}^{l=j-1} \beta_{j-l}} + \sigma_1 \Rightarrow$$

By condition the theorem

$$\frac{\sigma_{j+1} ln2}{\prod_{l=0}^{l=n-1}\beta_{n-l}} - \sum_{m=0}^{m=n-1}\frac{\alpha_{j-m}}{\prod_{l=m}^{l=n-1}\beta_{j-l}} + \sigma_1 \Rightarrow$$

$$\frac{\sigma_n ln2}{\prod_{l=0}^{l=n-1}\beta_{n-l}} - \sum_{m=0}^{m=n-1}\frac{ln2 - 2^{-\delta_j+1}}{\prod_{l=m+1}^{l=n-1}\beta_{j-l}} = \sigma_1 \Rightarrow$$

Suppose $\delta_j = 2i \in (1, n) \Rightarrow$

$$\frac{\sigma_n ln2}{\prod_{l=0}^{l=n-1}\beta_{n-l}} + \sum_{m=0}^{m=n-1}\frac{ln2 - 1/2}{2^m} > \sigma_1 \Rightarrow$$

$$2(ln2 - 1/2) > \sigma_1 \Rightarrow$$

$\exists\delta_j > 2 \Rightarrow$ statement of Theorem 56

□ 57

Theorem 4. *Let*

$$a_n = \sum_{i=0}^{n}\gamma_i 2^i, \quad n > 1000, \quad \gamma_i \in \{0, 1\}$$

then

$$a_{8n} < a_n$$

Proof. In more detail, the estimation process consists of replacing 3^l in a_{n+l} by formula 7 which does not contain powers of the triple which allows one to evaluate the resulting terms of the Syracuse sequence. as a result, we get the following estimate. Let's introduce operators defined formulas

$$Pf = f/2, \; Tf = 3f + 1, \; Zf = 3f$$

Let's consider all possible scenarios of the behavior of the Syracuse sequence, the same possible scenarios can be written in the following form

$$a_{n+n} = T_1T_2.....T_n a_n$$

$$T_i \in \{P, T\}, \; R_i \in \{Z, P\}, \; , \; a_{n+n} = R_1R_2.....R_n a_n + A$$

Let's introduce

$$m = \sum_{R_i=Z} 1$$

and compute

$$\sum_{R_i=P} 1 = n - m + m = n$$

By rules of Collatz we have after 2n steps

$$a_{n+n} = 3^m/2^n a_n + B_n$$

where

$$A_j = \sum_{R_i=Z,\, i=1,j} 1, \;\; C_j = -\sum_{R_i=Z, i=1,j} 1 - \sum_{R_i=P,\, i=1,j} 1$$

$$B_n = \sum_{j=1,n} 3^{A_j} 2^{C_j}$$

$$B_n \le \sum_{j=1,n} 3^j/2^j < 23^n/2^n \le 2(3/4)^n a_n$$

$$A = a_{2n} = 3^m(a_n * 2^{-n} + B_n) = (a_n * 2^{-n} + B_n)3^m$$

$$A = \sum_{i=0}^{[\alpha_1]} \gamma_i 2^i, \ \ \gamma_i \in \{0,1\}, \ \ \alpha_1 = m * ln3/ln2 + ln(2^{-n}a_n)$$

Let

$$m^* \ \ is \ \ count \ \ of \ \ non \ zeros \ of \ \ \gamma_i$$

$$l^* \ \ is \ \ count \ \ of \ \ \ zeros \ \ of \ \ \gamma_i$$

by theorem 2 we will have

$$m^* \leq [\alpha_1]/2 + 5 = [m \ln 3/\ln 2]/2 + 5$$

$$l^* \geq [\alpha_1/2 - 5 = [m \ln 3/\ln 2]/2 - 5$$

After $[\alpha_1]$ steps applying rules of Collatz we have

$$a_{2n+[\alpha_1]} \leq 3^m 3^{\alpha_1/2} 2^{-\alpha_1} 2^5 * 3^5 (a_n * 2^{-n} + B_n) = 3^m q_1 * a_n$$

where

$$q_1 = 3^m 3^{\alpha_1/2} 2^{-\alpha_1} 2^5 * 3^5$$

Repeating the process 3 times and using $n > 1000 \ \Rightarrow q_3 < 1 \ \ \Rightarrow \ a_{8n} < a_n$ □

Theorem 5. *Let*

$$a_n = \sum_{i=0}^{n} \gamma_i 2^i, \ \ n > 1000, \ \ \ \gamma_i \in \{0,1\}$$

then for a_n Collatz conjecture is true

Proof. Proof follows from theorem 1-7
□

6. Conclusions

Our assertion proves that after 2n of steps the sequence comes to a number less than the start one, from which follows the solution of the Collatz conjecture.

References

1. O'Connor, J.J., Robertson, E.F. "Lothar Collatz". St Andrews University School of Mathematics and Statistics, Scotland.2006.