Proof of the Riemann Hypothesis

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Abstract

In this working paper I try to prove the Riemann hypothesis

we have established the equalities (see On the Riemann Hypothesis version 3)

$$2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} = (2n+\varepsilon)^3 - \varepsilon s(2n+\varepsilon)^2 + \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon) + C'(s) + o(1) \qquad (7)$$

$$2s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$= (2n+\varepsilon)^s - \varepsilon s(2n+\varepsilon)^{(s-1)} + \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)^{(s-2)} + C'(s)(2n+\varepsilon)^{(s-3)} + o((2n+\varepsilon)^{(s-3)}) \qquad (8)$$

Let the sequence U such that $\forall\, n \in \mathbb{N}^*\ \ U(n,s) = 2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} - \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)$

So $\forall\, n \in \mathbb{N}^*\ \ U(n,s) = (2n+\varepsilon)^3 - \varepsilon s(2n+\varepsilon)^2 + C'(s) + o(1)$

Since $\eta(1-s) = 0$ we have

$\forall\, n \in \mathbb{N}^*\ \ U\big(n,(1-s)\big) = (2n+\varepsilon)^3 - \varepsilon(1-s)(2n+\varepsilon)^2 + C'(1-s) + o(1)$

So $\forall\, n \in \mathbb{N}^*\ \ U(n,s) + U\big(n,(1-s)\big) = 2(2n+\varepsilon)^3 - \varepsilon(2n+\varepsilon)^2 + C'(s) + C'(1-s) + o(1)$

Since $\eta(\bar{s}) = 0\ and\ \eta(1-\bar{s}) = 0$ we have

$\forall\, n \in \mathbb{N}^*\ \ U(n,\bar{s}) + U\big(n,(1-\bar{s})\big) = 2(2n+\varepsilon)^3 - \varepsilon(2n+\varepsilon)^2 + C'(\bar{s}) + C'(1-\bar{s}) + o(1)$

So $\forall\, n \in \mathbb{N}^*\ \ \Big(U(n,s) + U\big(n,(1-s)\big)\Big) - \Big(U(n,\bar{s}) + U\big(n,(1-\bar{s})\big)\Big)$

$= (C'(s) + C'(1-s)) - \big(C'(\bar{s}) + C'(1-\bar{s})\big) + o(1)$

Let the sequence V such that

$\forall\, n \in \mathbb{N}^*\ \ V(n,s) = \Big(U(n,s) + U\big(n,(1-s)\big)\Big) - \Big(U(n,\bar{s}) + U\big(n,(1-\bar{s})\big)\Big)$

We have $\forall\, n \in \mathbb{N}^*\ \ V(n,s) = \big(C'(s) + C'(1-s)\big) - \Big(C'(\bar{s}) + C'(1-\bar{s})\Big) + o(1)$

So $\lim_{n\to+\infty} V(n,s) = \big(C'(s) + C'(1-s)\big) - \Big(C'(\bar{s}) + C'(1-\bar{s})\Big)$

So $\lim_{n\to+\infty}\big(V((n+1),s) - V(n,s)\big) = 0$

$\forall\, n \in \mathbb{N}^*\ V\big((n+1),s\big) - V(n,s) = \Big[\big(U((n+1),s) - U(n,s)\big) + \Big(U\big((n+1),(1-s)\big) - U(n,(1-s))\Big)\Big]$

$-\Big[\big(U((n+1),\bar{s}) - U(n,\bar{s})\big) + \Big(U\big((n+1),(1-\bar{s})\big) - U\big(n,(1-\bar{s})\big)\Big)\Big]$

For each $n \in \mathbb{N}^*$ let's calculate $\left(U((n+1),s) - U(n,s)\right)$

$\forall\, n \in \mathbb{N}^*\ U\big((n+1),s\big) - U(n,s)$

$= 2s(2(n+1)+\varepsilon)^{(3-s)} \sum_{k=0}^{n} \frac{1}{(2k+1)^{(1-s)}} - 2s(2n+\varepsilon)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

$-\frac{1}{2}s(s-1)\varepsilon^2\big((2(n+1)+\varepsilon) - (2n+\varepsilon)\big)$

$= 2s(2n+2+\varepsilon)^{(3-s)} \sum_{k=0}^{n} \frac{1}{(2k+1)^{(1-s)}} - 2s(2n+\varepsilon)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} - s(s-1)\varepsilon^2$

$= 2s(2n+2+\varepsilon)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} - 2s(2n+\varepsilon)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} + 2s(2n+2+\varepsilon)^{(3-s)} \times \frac{1}{(2n+1)^{(1-s)}}$

$-s(s-1)\varepsilon^2$

$= \big((2n+2+\varepsilon)^{(3-s)} - (2n+\varepsilon)^{(3-s)}\big)2s \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} + 2s(2n+2+\varepsilon)^{(3-s)} \times (2n+1)^{(s-1)} - s(s-1)\varepsilon^2$

We have

$(2n+2+\varepsilon)^{(3-s)} = (2n+\varepsilon)^{(3-s)} \left(1 + \frac{2}{2n+\varepsilon}\right)^{(3-s)}$

$= (2n+\varepsilon)^{(3-s)} \left(1 + \frac{2(3-s)}{2n+\varepsilon} + \frac{4(3-s)(2-s)}{2(2n+\varepsilon)^2} + \frac{8(3-s)(2-s)(1-s)}{6(2n+\varepsilon)^3} + o(\frac{1}{(2n+\varepsilon)^3})\right)$

$= (2n+\varepsilon)^{(3-s)} + 2(3-s)(2n+\varepsilon)^{(2-s)} + 2(3-s)(2-s)(2n+\varepsilon)^{(1-s)}$

$+\frac{4}{3}(3-s)(2-s)(1-s)(2n+\varepsilon)^{(-s)} + o((2n+\varepsilon)^{(-s)})$

So $\big((2n+2+\varepsilon)^{(2-s)} - (2n+\varepsilon)^{(2-s)}\big)2s \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} =$

$\Big[\ 2(3-s)(2n+\varepsilon)^{(2-s)} + 2(3-s)(2-s)(2n+\varepsilon)^{(1-s)} + \frac{4}{3}(3-s)(2-s)(1-s)(2n+\varepsilon)^{(-s)} + o((2n+\varepsilon)^{(-s)})\ \Big] \times$
$\Big[\ (2n+\varepsilon)^{s} - \varepsilon s(2n+\varepsilon)^{(s-1)} + \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)^{(s-2)} + C'(s)(2n+\varepsilon)^{(s-3)} + o((2n+\varepsilon)^{(s-3)})\Big]$

$= 2(3-s)(2n+\varepsilon)^2 - 2\varepsilon s(3-s)(2n+\varepsilon) + s(s-1)(3-s)\varepsilon^2 + 2(3-s)(2-s)(2n+\varepsilon) - 2\varepsilon s(3-s)(2-s)$

$+\frac{4}{3}(3-s)(2-s)(1-s) + o(1)$

$= 2(3-s)(2n+\varepsilon)^2 + \big(-2\varepsilon s(3-s) + 2(3-s)(2-s)\big)(2n+\varepsilon) + s(s-1)(3-s)\varepsilon^2 - 2\varepsilon s(3-s)(2-s)$

$+\frac{4}{3}(3-s)(2-s)(1-s) + o(1)$

So $\big((2n+2+\varepsilon)^{(2-s)} - (2n+\varepsilon)^{(2-s)}\big)2s \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

$= 2(3-s)(2n+\varepsilon)^2 + \big(-2\varepsilon s(3-s) + 2(3-s)(2-s)\big)(2n+\varepsilon) + s(s-1)(3-s)\varepsilon^2 - 2\varepsilon s(3-s)(2-s)$

$+\frac{4}{3}(3-s)(2-s)(1-s) + o(1) \qquad (9)$

We have

$(2n+2+\varepsilon)^{(3-s)} \times (2n+1)^{(s-1)} = (2n+1+(1+\varepsilon))^{(3-s)} \times (2n+1)^{(s-1)}$

$= (2n+1))^{(3-s)} \left(1 + \frac{(1+\varepsilon)}{2n+1}\right)^{(3-s)} \times (2n+1)^{(s-1)}$

$$= (2n+1)^2\left(1+\frac{(3-s)(1+\varepsilon)}{2n+1}+\frac{(3-s)(2-s)(1+\varepsilon)^2}{2(2n+1)^2}+o\left(\frac{1}{(2n+1)^2}\right)\right)$$

$$= (2n+1)^2+(3-s)(1+\varepsilon)(2n+1)+\frac{1}{2}(3-s)(2-s)(1+\varepsilon)^2+o(1)$$

$$= (2n+\varepsilon+(1-\varepsilon))^2+(3-s)(1+\varepsilon)(2n+\varepsilon+(1-\varepsilon))+\frac{1}{2}(3-s)(2-s)(1+\varepsilon)^2+o(1)$$

$$= (2n+\varepsilon)^2+2(1-\varepsilon)(2n+\varepsilon)+(1-\varepsilon)^2+(3-s)(1+\varepsilon)(2n+\varepsilon)+(3-s)(1+\varepsilon)(1-\varepsilon)$$

$$+\frac{1}{2}(3-s)(2-s)(1+\varepsilon)^2+o(1)$$

$$= (2n+\varepsilon)^2+\big((3-s)(1+\varepsilon)+2(1-\varepsilon)\big)(2n+\varepsilon)+(1-\varepsilon)^2+(3-s)(1-\varepsilon^2)$$

$$+\frac{1}{2}(3-s)(2-s)(1+2\varepsilon+\varepsilon^2)+o(1)$$

$$= (2n+\varepsilon)^2+\big((3-s)(1+\varepsilon)+2(1-\varepsilon)\big)(2n+\varepsilon)+(1-\varepsilon)^2+(3-s)-(3-s)\varepsilon^2$$

$$+\frac{1}{2}(3-s)(2-s)+(3-s)(2-s)\varepsilon+\frac{1}{2}(3-s)(2-s)\varepsilon^2+o(1)$$

$$= (2n+\varepsilon)^2+\big((3-s)(1+\varepsilon)+2(1-\varepsilon)\big)(2n+\varepsilon)+\frac{1}{2}(3-s)(-s)\varepsilon^2+(3-s)(2-s)\varepsilon$$

$$+\frac{1}{2}(3-s)(4-s)+(1-\varepsilon)^2+o(1)$$

$$= (2n+\varepsilon)^2+\big((3-s)(1+\varepsilon)+2(1-\varepsilon)\big)(2n+\varepsilon)-\frac{1}{2}s(3-s)\varepsilon^2+(3-s)(2-s)\varepsilon$$

$$+\frac{1}{2}(3-s)(4-s)+(1-\varepsilon)^2+o(1)$$

so

$$2s(2n+2+\varepsilon)^{(2-s)}\times(2n+1)^{(s-1)}$$

$$= 2s(2n+\varepsilon)^2+2s\big((3-s)(1+\varepsilon)+2(1-\varepsilon)\big)(2n+\varepsilon)-s^2(3-s)\varepsilon^2+2s(3-s)(2-s)\varepsilon$$

$$+s(3-s)(4-s)+2s(1-\varepsilon)^2+o(1) \qquad (10)$$

From equalties (9) and (10) we deduce that

$$U\big((n+1),s\big)-U(n,s)$$

$$= 6(2n+\varepsilon)^2+\Big(-2\varepsilon s(3-s)+2(3-s)(2-s)+2s\big((3-s)(1+\varepsilon)+2(1-\varepsilon)\big)\Big)(2n+\varepsilon)$$

$$+s(s-1)(3-s)\varepsilon^2-2\varepsilon s(3-s)(2-s)+\frac{4}{3}(3-s)(2-s)(1-s)-s^2(3-s)\varepsilon^2+2s(3-s)(2-s)\varepsilon$$

$$+s(3-s)(4-s)+2s(1-\varepsilon)^2-s(s-1)\varepsilon^2+o(1)$$

$$= 6(2n+\varepsilon)^2+2\big(-\varepsilon s(3-s)+(3-s)(2-s)+s(3-s)(1+\varepsilon)+2s(1-\varepsilon)\big)(2n+\varepsilon)$$

$$+\big(s(s-1)(2-s)-s^2(3-s)\big)\varepsilon^2+\frac{4}{3}(3-s)(2-s)(1-s)+s(3-s)(4-s)+2s(1-\varepsilon)^2+o(1)$$

$$= 6(2n+\varepsilon)^2+2\big(-\varepsilon s(3-s)+(3-s)(2-s)+s(3-s)+\varepsilon s(3-s)+2s(1-\varepsilon)\big)(2n+\varepsilon)$$

$$-s(s^2-3s+2+3s-s^2)\varepsilon^2+\frac{1}{3}[4(3-s)(2-s)(1-s)+3s(3-s)(4-s)]+2s(1-\varepsilon)^2+o(1)$$

$= 6(2n+\varepsilon)^2 + 2\big((3-s)(2-s) + s(3-s) + 2s(1-\varepsilon)\big)(2n+\varepsilon)$

$-2s\varepsilon^2 + \frac{1}{3}[4(3-s)(2-s)(1-s) + 3s(3-s)(4-s)] + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2\big(s^2 - 5s + 6 + 3s - s^2 + s(1-\varepsilon)\big)(2n+\varepsilon)$

$-2s\varepsilon^2 + \frac{1}{3}(3-s)[4(s^2 - 3s + 2) + 12s - 3s^2] + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2\big(-2s + 6 + 2s(1-\varepsilon)\big)(2n+\varepsilon) - 2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2(-2s\varepsilon + 6)(2n+\varepsilon) - 2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 - 4(s\varepsilon - 3)(2n+\varepsilon) - 2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 + o(1)$

Let $p(s) = -2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2$

So $\quad \forall\, n \in \mathbb{N}^*\ U\big((n+1),s\big) - U(n,s) = 6(2n+\varepsilon)^2 - 4(s\varepsilon - 3)(2n+\varepsilon) + p(s) + o(1)$

We have also

$\forall\, n \in \mathbb{N}^*\ U\big((n+1),(1-s)\big) - U(n,(1-s)) = 6(2n+\varepsilon)^2 - 4\big((1-s)\varepsilon - 3\big)(2n+\varepsilon) + p(1-s) + o(1)$

So $\big(U((n+1),s) - U(n,s)\big) + \Big(U\big((n+1),(1-s)\big) - U(n,(1-s))\Big)$

$= 12(2n+\varepsilon)^2 - 4(\varepsilon - 6)(2n+\varepsilon) + p(s) + p(1-s) + o(1)$

We have also

$\forall\, n \in \mathbb{N}^*\ \big(U((n+1),\bar{s}) - U(n,\bar{s})\big) + \Big(U\big((n+1),(1-\bar{s})\big) - U\big(n,(1-\bar{s})\big)\Big)$

$= 12(2n+\varepsilon)^2 - 4(\varepsilon - 6)(2n+\varepsilon) + p(\bar{s}) + p(1-\bar{s}) + o(1)$

So $\quad \forall\, n \in \mathbb{N}^*\ V\big((n+1),s\big) - V(n,s) = (p(s) + p(1-s)) - (p(\bar{s}) + p(1-\bar{s})) + o(1)$

Since $\quad \lim_{n\to+\infty}\big(V((n+1),s) - V(n,s)\big) = 0$

We deduce that $\quad (p(s) + p(1-s)) - \big(p(\bar{s}) + p(1-\bar{s})\big) = 0$

Let's calculate $(p(s) + p(1-s))$

Let $p(s) = -2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 = -\frac{1}{3}(s-3)(s^2+8) - 2s\varepsilon^2 + 2s(1-\varepsilon)^2$

$= -\frac{1}{3}(s^3 + 8s - 3s^2 - 24) - 2s\varepsilon^2 + 2s(1-\varepsilon)^2$

$= -\frac{1}{3}(s^3 - 3s^2) - \frac{8}{3}s + 8 - 2s\varepsilon^2 + 2s(1-\varepsilon)^2$

So $p(s) = -\frac{1}{3}(s^3 - 3s^2) - \frac{8}{3}s - 2s\varepsilon^2 + 2s(1-\varepsilon)^2 + 8$

$p(1-s) = -\frac{1}{3}((1-s)^3 - 3(1-s)^2) - \frac{8}{3}(1-s) - 2(1-s)\varepsilon^2 + 2(1-s)(1-\varepsilon)^2 + 8$

$p(s) + p(1-s) = -\frac{1}{3}(s^3 - 3s^2 + (1-s)^3 - 3(1-s)^2) - \frac{8}{3} - 2\varepsilon^2 + 2(1-\varepsilon)^2 + 16$

$s^3 - 3s^2 + (1-s)^3 - 3(1-s)^2 = s^3 - 3s^2 + 1 - 3s + 3s^2 - s^3 - 3s^2 + 6s - 3 = -3s^2 + 3s - 2$

$p(s) + p(1-s) = -\frac{1}{3}(-3s^2 + 3s - 2) - \frac{8}{3} - 2\varepsilon^2 + 2(1-\varepsilon)^2 + 16$

$p(s) + p(1-s) = (s^2 - s) - 2\varepsilon^2 + 2(1-\varepsilon)^2 + 16 + \frac{2}{3} - \frac{8}{3}$

$p(\bar{s}) + p(1-\bar{s}) = (\bar{s}^2 - \bar{s}) - 2\varepsilon^2 + 2(1-\varepsilon)^2 + 16 + \frac{2}{3} - \frac{8}{3}$

$(p(s) + p(1-s)) - \left(p(\bar{s}) + p(1-\bar{s})\right) = (s^2 - s) - (\bar{s}^2 - \bar{s})$

We deduce that $(s^2 - s) - (\bar{s}^2 - \bar{s}) = 0$

So $s^2 - s - \bar{s}^2 + \bar{s} = 0$

So $s^2 - \bar{s}^2 - (s - \bar{s}) = 0$

So $(s - \bar{s})(s + \bar{s}) - (s - \bar{s}) = 0$

So $(s - \bar{s})(s + \bar{s} - 1) = 0$

So $2ib(2a - 1) = 0$

Since $b \neq 0$ we have $2a - 1 = 0$

Thus $a = \frac{1}{2}$

So Riemann hypothesis is true