Certain summation formulas involving generalized harmonic numbers

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Abstract: Here we aim at presenting further interesting identities about certain finite or infinite series involving generalized harmonic numbers.

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1. Introduction

The number pi is defined by

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right) \tag{1}$$

The nth generalized harmonic number of order $p \in \mathbb{N}$ is defined by

$$H_n^{(p)} = \sum_{k=1}^{n}\frac{1}{k^p} \;,\quad H_0^{(p)} = 0 \;,\; n \in \mathbb{N}\cup\{0\} \,,\; p \in \mathbb{N} \tag{2}$$

when $p = 1$ we have $H_n = H_n^{(1)}$, the nth harmonic number.

The inverse tangent function is defined by

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt \;,\; x \in \mathbb{R} \tag{3}$$

$$\tan^{-1} x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1} \;,\; |x| < 1 \tag{4}$$

In this note we give some formulas involving generalized harmonic numbers and inverse tangent function.

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2. Formulas

Entry 1. for $m \in \mathbb{N}$, $|u| < 1$, we have

$$\sum_{k=1}^{m}\frac{1}{\sqrt{2k-1}}\tan^{-1}\left(\frac{u}{\sqrt{2k-1}}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^n}H_m^{(n)}\right)\frac{(-1)^{n-1}u^{2n-1}}{2n-1} \tag{5}$$

Example 1: for $m \in \mathbb{N}$, we have

$$\frac{\pi}{6\sqrt{3}} + \sum_{k=2}^{m}\frac{1}{\sqrt{6k-3}}\tan^{-1}\left(\frac{1}{\sqrt{6k-3}}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^n}H_m^{(n)}\right)\frac{(-1)^{n-1}3^{-n}}{2n-1} \tag{6}$$

Entry 2. for $m \in \mathbb{N}$, $|u| < 1$, we have

$$\sum_{k=1}^{m}\tan^{-1}\left(\frac{u}{2k-1}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(2n-1)} - \frac{1}{2^{2n-1}}H_m^{(2n-1)}\right)\frac{(-1)^{n-1}u^{2n-1}}{2n-1} \tag{7}$$

Example 2: for $m \in \mathbb{N}$, we have

$$\frac{\pi}{6} + \sum_{k=2}^{m} \tan^{-1}\left(\frac{1}{(2k-1)\sqrt{3}}\right) = \sqrt{3} \sum_{n=1}^{\infty}\left(H_{2m}^{(2n-1)} - \frac{1}{2^{2n-1}} H_{m}^{(2n-1)}\right) \frac{(-1)^{n-1} 3^{-n}}{2n-1} \tag{8}$$

Example 3: for $m \in \mathbb{N}$, we have

$$\frac{\pi}{4} + \sum_{k=2}^{m} \tan^{-1}\left(\frac{5(2k-1)}{24k(k-1)+5}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(2n-1)} - \frac{1}{2^{2n-1}} H_{m}^{(2n-1)}\right) \frac{(-1)^{n-1}\left(2^{-2n+1} + 3^{-2n+1}\right)}{2n-1} \tag{9}$$

Entry 3. for $m \in \mathbb{N}$, $|r| < 1$, $0 < \theta < \pi/2$, we have

$$\sum_{k=1}^{m} \tan^{-1}\left(\frac{r\sin\theta}{2k-1+r\cos\theta}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^{n}} H_{m}^{(n)}\right) \frac{(-1)^{n-1} r^{n} \sin(n\theta)}{n} \tag{10}$$

Example 4. for $m \in \mathbb{N}$, we have

$$\frac{\pi}{6} + \sum_{k=2}^{m} \tan^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{4(2k-1)\sqrt{2}+\sqrt{6}-\sqrt{2}}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^{n}} H_{m}^{(n)}\right) \frac{(-1)^{n-1} 2^{-n/2}}{n} \sin\left(\frac{5n\pi}{12}\right) \tag{11}$$

Entry 4. for $m \in \mathbb{N}$, $0 < x^2 + y^2 < 1$, we have

$$\sum_{k=1}^{m} \tan^{-1}\left(\frac{y}{2k-1+x}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^{n}} H_{m}^{(n)}\right) \frac{(-1)^{n-1}}{n} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} (-1)^{k} x^{n-2k-1} y^{2k+1} \tag{12}$$

Example 5: for $m \in \mathbb{N}$, we have

$$\frac{\pi}{4} + \sum_{k=2}^{m} \tan^{-1}\left(\frac{1}{4k-3}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^{n}} H_{m}^{(n)}\right) \frac{2^{-n}}{n} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} (-1)^{k} = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^{n}} H_{m}^{(n)}\right) \frac{2^{-n/2}}{n} \sin\left(\frac{n\pi}{4}\right) \tag{13}$$

Example 6: for $m \in \mathbb{N}$, we have

$$\frac{\pi}{12} + \sum_{k=2}^{m} \tan^{-1}\left(\frac{1}{(2k-1)\left(\sqrt{3}+1\right)+1}\right) = \sum_{n=1}^{\infty}\left(H_{2m}^{(n)} - \frac{1}{2^{n}} H_{m}^{(n)}\right) \frac{(-1)^{n-1}}{n} \left(\frac{\sqrt{2}}{\sqrt{3}+1}\right)^{n} \sin\left(\frac{n\pi}{4}\right) \tag{14}$$

3. References

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