Proving the Collatz Conjecture

by Jim Rock

Abstract. Collatz sequences are formed by dividing an even number by two until it is odd. Then multiply by three and add one to get an even number. The Collatz conjecture states that if this process is repeated you always get back to one. Using geometric series summations we prove that a connected Collatz Structure exists, which contains all positive integers exactly once. The terms of the Collatz Structure are joined together via the Collatz algorithm. Thus, every positive integer forms a Collatz sequence with unique terms terminating in the number one.

History. The Collatz conjecture was made in 1937 by Lothar Collatz. Through 2017 the conjecture has been checked for all starting values up to $(87)(2^{60})$, but very little progress has been made toward proving the conjecture. Paul Erödos said about the Collatz conjecture: "Mathematics may not be ready for such problems." https://en.wikipedia.org/wiki/Collatz_conjecture

Introduction. **Note! This paper is hard to follow the first time you read it. But your effort is rewarded.**

The Collatz Structure (displayed in the diagram below) consists of horizontal branches and vertical towers. Vertical arrows ↓ represent descending Collatz towers, where each term is half the previous term. Horizontal arrows ← indicate the Collatz algorithm is applied to move from term to term in the branch. We show how different integer types fit in the Collatz Structure (Section 1) exactly once (Section 2). In section 3 we define the binary series of branches, which are used in section 4 to show that all positive integers are in the branches and towers. We address the issue of unending Collatz sequences in Section 5. Appendix 1 proves there can be no more than two consecutive even integers in a branch. Appendix 2 gives vertical tower details. Appendix 3 provides details about the Collatz Structure.

Section 1

Defining and populating the Collatz Structure

Collatz Structure Branches and Towers ↓ indicates a descending Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40←	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10←	3																				
8																									
4																									
2																									
1																									

The **Trunk Tower** is the left-most tower, where each term is a power of two 2^s, $s=0,1,2,3\ldots$. A Collatz sequence can begin anywhere within the Collatz Structure and eventually by applying the Collatz algorithm a 2^s term in the **Trunk Tower** will be reached. From there we repeatedly divide by two until the base term 1 is reached. Every Collatz sequence terminates at the **Trunk Tower** base term 1.

Notice that every **red tower** base term is of the form $24m+4$, $24m+10$, or $24m+22$. The rest of the **red tower** terms alternate between $12k+8$ terms $20, 80$ in **blue** and $24k+16$ terms $40, 160$ in **brown.**

We trace a **red tower** from its ***n-th*** term $24k_n+16 \rightarrow 12k_n+8 \rightarrow 6k_n+4 = 24k_{n-1}+16$ $(k_n = 4k_{n-1}+2)$…to its ***first*** (base) term. $24k_2+16 \rightarrow 12k_2+8 \rightarrow 6k_2+4 = 24k_1+16$ $(k_2 = 4k_1+2) \rightarrow 12k_1+8 \rightarrow \underline{6k_1+4}$.

If $k_1=4m$, $\underline{6k_1+4} = 24m+4$. **If** $k_1=4m+1$, $\underline{6k_1+4} = 24m+10$. **If** $k_1=4m+3$, $\underline{6k_1+4} = 24m+22$.

Every $24k+16$ term can be written as $4^j a$, $j = 1,2,3\ldots$ $a = 24m+4, 24m+10,$ **or** $24m+22, m=0,1,2,3\ldots$

The Collatz Structure starts with the **Trunk Tower.** Each $(4^j)(4), j=1,2,3...$ Trunk Tower term is the last term in a branch. At every ***a=24m+4, 24m+10***, and ***24m+22*** base term in the Trunk Tower branches is a $4^j a, j=1,2,3...$ secondary **red tower**. Each of these $4^j a$ terms in the secondary **red towers** is the last term in a branch. At every ***a=24m+4, 24m+10,*** and ***24m+22*** base term in these secondary branches is a $4^j a$ secondary **red tower**. Each $4^j a$ is the last term in a branch. This process is repeated indefinitely.

Note that ***24k+16*** terms, which are divisible by eight are the last term in a branch. All the other even terms that appear in the middle of a branch ***24m+4→12m+2, 24m+10,*** or ***24m+22,*** have even factors of at most four or two. In appendix 1 we show there can be no more than two consecutive even terms in a branch. Since they are divisible by eight, *24k+16* terms must appear at the end of a branch. We will show in section 4 that there are no unending branches.

Collatz Structure Branches and Towers ↓ indicates an descending Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40←	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10←	3																				
8																									
4																									
2																									
1																									

The successor of any odd term is an even term ***2j+1→6j+4*** that leaves a reminder of one when divided by three. The **green** first terms in a branch are of the form ***6j+3.*** They all divisible by three, as are all other terms in a **green tower.** They are of the form $(2^s)(6j+3)$ ***s=1,2,3…*** No odd term can precede a ***6j+3*** term in a branch. An odd term can only precede an even term that leaves a remainder of one when divided by three ***(5→16).*** Each term above the ***6j+3*** base term in a **green tower** is also of the form ***24k, 24k+6, 24k+12,*** or ***24k+18.*** The exact relation between the two form types is shown below. *** Note: every positive integer is of the form $(2j+1)(2^k)$. Take any positive integer and repeatedly divide by two until the remainder is odd.

$$^{***}(2j+1)(24)(2^k) = (2^{k+3})(6j+3) \quad j=0,1,2,3\ldots\ k=0,1,2,3\ldots$$
$$24k+6 = (2)(6j+3),\ (j=2k)\quad j=0,2,4\ldots\ k=0,1,2,3\ldots$$
$$24k+12=(4)(6j+3),\ (j=k)\quad j=0,1,2,3\ldots\ k=0,1,2,3\ldots$$
$$24k+18=(2)(6j+3),\ (j=2k+1)\ j=1,3,5\ldots\ k=0,1,2,3\ldots$$

Terms of the form ***24k+2s, 0 ≤ s ≤ 11,*** and ***6j+t, t=1,3,5*** fit within the Collatz Structure as follows:

24k **green tower**
24k+2 **successor of *24j+4, j=2k***
24k+4 **red tower base middle of a branch**
24k+6 **green tower**
24k+8 **red tower successor of *24j+16, j = 2k***
24k+10 **red tower base middle of a branch**
24k+12 **green tower**
24k+14 **successor of *24j+4, j = 2k+1***
24k+16 **red tower end of a branch**
24k+18 **green tower**
24k+20 **red tower successor of *24j+16, j = 2k+1***
24k+22 **red tower base middle of a branch**
6j + 1 **middle of a branch**
6j + 3 **green tower and beginning of a branch**
6j + 5 **middle of a branch**

Section 2

No individual term appears more than once in the Collatz structure. There can be no duplicate terms in a branch. All the predecessors of a duplicate pair of terms would be duplicates. This would require ***24h+3, 24h+9,*** or ***24h+15*** to be a duplicate term, and those terms only appear at the beginning of a branch. ***24h+21* have a *24(3h+2)+16*** term as an immediate successor without duplicates. All terms in the Trunk Tower are unique. That makes all terms in secondary towers and all terms in branches that terminate in secondary towers unique. Thus, no individual term appears more than once in the Collatz structure.

Section 3

We define the branch binary series, and provide examples. It will be used to prove all positive integers are in the Collatz Structure, and that there are no unending Collatz sequences.

The ***6j+3*** branch first terms are sub-divided into four types: ***24h+3, 24h+9, 24h+15* and *24h+21,*** $h \geq 0$.
A branch **binary series** counts the number of divisions by two on its **red tower base** terms: ***24m+4 (2), 24m+10 (1),*** and ***24m+22 (1)***. Only ***24h+3, 24h+9,* and *24h+15*** first terms appear in branches with binary series. These three groups of branches are characterized by their first term ***24h+3, 24h+9*** or ***24h+15*** and a binary series of ***1'*s** and ***2'*s** (see ***2,1,1,2*** below) counting the divisions by two on their **red tower base** terms ***24m+4 (2), 24m+10 (1),*** or ***24m+22 (1)*** and a last term ***24k+16.*** The **length *r*** of its binary series is the number of **red tower base** terms in a branch.

If the sum of ***r 1's*** and ***2's*** in the binary series is ***s,*** there are three different formulas for the first terms of branches that have the same binary series.

$$24h+3+(p-1)(24)(2^s),$$
$$24h+9+(p-1)(24)(2^s),$$
$$24h+15+(p-1)(24)(2^s),$$
(zero length binary series) $24h+21+(p-1)(24)(2^0), p=1,2,3\ldots\ 0 \leq h < 2^s.$

Each individual value of ***h*** is part of a different group of branches with the same binary series.

All branches end with $24k+16+(p-1)(24)(3^{r+1}),\ 0 \leq k < 3^{r+1},\ r \geq 0,\ p=1,2,3\ldots.$

We have 3 branches with the binary series ***(2,1,1,2)*** counting divisions by two on their **red tower** base terms.

The first branch is ***9, 28(2), 14, 7, 22(1), 11, 34(1), 17, 52(2), 26, 13, 40.***
The second branch is ***1545, 4636(2), 2318, 1159, 3478(1), 1739, 5218(1), 2609, 7828(2), 3914 1957, 5872.***
The third branch is ***3081, 9244(2), 4622, 2311, 6934(1), 3467, 10402(1), 5201, 15604(2), 7802, 3901, 11704.***

The sum of this binary series is six. These are a series of branches whose first terms differ by $(24)(2^6)=1536$.

The first term sequence is $9+(p-1)(24)(2^6)$ ***p=0,1,2,… 9, 1545, 3081,…***

The length of this binary series is four. There are five applications of $2j+1 \rightarrow 6j+4$ to the odd terms in the branches.

These are a series of branches whose last terms differ by $(24)(3^5)=5832$.

The last term sequence is $40+(p-1)(24)(3^5)$ ***p=0,1,2,… 40, 5872, 11704,…*** .

Apply the Collatz algorithm to the first term ***24h+q, q=3,9 or 15*** of a branch with a binary series of **length *r.*** If ***s*** divisions by two on even terms and ***r+1*** applications of $2j+1 \rightarrow 6j+4$ to odd terms result in a last term of ***24k+16,*** then for $24h+q+(p)(24)(2^s)$, ***p=0,1,2,… s*** divisions by two on even terms and ***r+1*** applications of $2j+1 \rightarrow 6j+4$ to odd terms will produce a branch last term of $24k+16+(p)(24)(3^{r+1})$. ***p=0,1,2,…***

Dividing by two ***s*** times eliminates the 2^s term from $(p)(24)(2^s)$. Applying $2j+1 \rightarrow 6j+4$ to $24h+q+(p)(24)(2^s)$ multiplies $(p)(24)(2^s)$ by three. $24h+q+(p)(24)(2^s) \rightarrow 72h+3q+1+(p)(24)(2^s)(3)$.

Starting with $(p)(24)(2^s)$ ***s*** divisions by two on even terms and ***r+1*** applications of $2j+1 \rightarrow 6j+4$ to odd terms gives $(p)(24)(3^{r+1})$.

Section 4

All positive integers appear in branches or towers. Note! The proofs in sections 4.2 and 4.3 are exactly the same as section 4.1. You could skip to section 4.4 after which there is a summary.

Branches come in a group of four with first terms of the form ***24h+3, 24h+9, 24h+15,*** or ***24h+21***[1] and a ***24k+16*** last term ***h = 0,1,2,3,… k = 0,1,2,3,…***. **Branch segments start in the middle of branches.**

Branch segments come in two groups of four. The first group has first terms of the form ***24h+1, 24h+7, 24h+13***[2]***,*** or ***24h+19.*** The second group has first terms of the form ***24h+5***[3]***, 24h+11, 24h+17,*** or ***24h+23.*** Both groups have a ***24k+16*** last term. ***h = 0,1,2,3,… k = 0,1,2,3,…***

24h+21[1]***→ 24(3h+2)+16*** is the first term in the branch group with an empty **length *r = 0*** binary series.

24h+13[2]***→24(3h+1)+16*** and ***24h+5***[3]***→24(3h)+16*** are first terms in the branch segment groups with empty **length *r = 0*** binary series.

The proportion of each of the other nine odd term forms in a branch or branch segment with a binary series **length *r = 1, 2, 3,…*** is given by a formula using the binary series length. The formula for ***24h+9*** is $3^{r-1}/2^{2r}$.

Each formula generates a geometric series. Each series term is the proportion of a terms with a binary series of **length *r = 1,2,3,…*** ***1/4, 3/16, 9/64…*** for ***24h+9,*** which sums to ***1*** (the total proportion of ***24h+9*** terms). All elements of the nine odd term forms are in branches with binary series.

Section 4.1 *24h+3, 24h+9,* and *24h+15* for all values of *h* are the terms of the Collatz structure branches with binary series of every combination of *1's* and *2's* for every value of *r.*

Theorem 4.1.1: All ***24h+3*** are terms in branches of the Collatz structure.

Lemma 4.1.1: The first two ***24h+3*** term binary series are ***(1)*** for ***h=2,4,6,… 1/2***[1], ***(1,2)*** for ***h=3, 11, 19,...*** and ***(1,2,...)*** for all other binary series with ***h*** an odd number.

For ***h=2n, 24h+3 = 48n+3→144n+10(1)→72n+5→(24n)(9)+16.***
For ***h=8n+3, 192n+75→576n+226(1)→288n+113→864n+340(2)→216n+85→(27n+10)(24)+16***
For ***h=2n+1, 48n+27→144n+82(1)→72n+41→216n+124(2)→…***
By Lemma 4.1.1 The binary series proportion for ***r=2*** is ***(1,2)=1/2***^{3}[4] ***= 1/8***[2]***=*** $3^{r-2}/2^{2r-1}$ ***s=3***[4] **binary series sum (Section 3).**
Assume the proportion of ***24h+3*** terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

The ***r+1*** position in the branch binary series can be ***(1)*** or ***(2).*** The proportion of ***24h+3*** terms of binary series **length *r+1*** is: $(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.
The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1}) / (3^{r-1}/2^{r+1}) = 3/4$[3]
The total proportion of ***24h+3*** terms in the Collatz structure is ***1/2***[1]***+(1/8)***[2]***/(1 – 3/4***[3]***) = 1.***

All ***24h+3*** terms are in branches of the Collatz structure.

Theorem 4.1.2: All ***24h+9*** are terms in branches of the Collatz structure.

Lemma 4.1.2.: The first ***24h+9*** terms' binary series is ***(2)*** for ***h=3,7,11,… 1/4*** of all the terms.
All other binary series begin with ***(2,...)***.

For ***h=4n+3, 24h+9 = 96n+81→288n+244(2)→72n+61→(9n+7)(24)+16.***
24h+9→72h+28(2)→18h+7→…
By Lemma 4.1.2 The binary series for ***r =1*** is ***(2) = 1/2***^{2}[1] $= 3^{r-1}/2^{2r}$.
Assume the proportion of ***24h+9*** terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The proportion of ***24h+9*** terms of binary series **length** $r+1 = (1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r}) / (3^r/2^{2r+2}) = 3/4$[2]

The total proportion of ***24h+9*** terms in the Collatz structure is ***(1/4***[1]***)/(1 – 3/4***[2]***) = 1.***

All ***24h+9*** terms are in branches of the Collatz structure.

Theorem 4.1.3: All $24h+15$ are terms in branches of the Collatz structure.

Lemma 4.1.3: The first $24h+15$ term binary series is $(1,1)$ for $h=3,7,11,\ldots$ $1/4$ of all the terms.
All other binary series with an odd number of terms begin with $(1,1,\ldots)$.

For $h=4n+3$, $24h+15 = 96n+87 \rightarrow 288n+262(1) \rightarrow 144n+131 \rightarrow 432n+394(1) \rightarrow 216n+197 \rightarrow (24)(27n+24)+16$
$24h+15 \rightarrow 72h+46(1) \rightarrow 36h+23 \rightarrow 108h+70(1) \rightarrow 54h+35 \rightarrow \ldots$

By Lemma 4.1.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2 = 1/4$[1]$= 3^{r-2}/2^{2r-2}$.

Assume the proportion of $24h+15$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The proportion of $24h+15$ terms of binary series **length** $r+1$ is
$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.
The ratio between successive terms is $(3^{r-2}/2^{2r-2}) / (3^{r-1}/2^{2r}) = 3/4$[2].
The total proportion of $24h+15$ terms in the Collatz structure is $1 = (1/4$[1]$)/(1 - 3/4$[2]$)$.
All $24h+15$ terms are in branches of the Collatz structure.

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Collectively all $24h+3$, $24h+9$, $24h+15$, are first terms in finite branches with binary series of all 2^r combinations of *1's* and *2's* for every value of r. As shown in section 4.4, all $24h+16$ are last terms in finite branches with binary series of all 2^r **combinations of** *1's* and *2's* for all r. **There are no unending** $24h+3$, $24h+9$, $24h+15$ **branches.**

Section 4.2 $24h+1$, $24h+7$, and $24h+19$ are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of r.

Theorem 4.2.1: All $24h+19$ terms are in branches of the Collatz structure.

Lemma 4.2.1: The first two $24h+19$ term binary series are (1) for $h=2,4,6,\ldots$ $1/2$[1] **of all terms**
$(1,2)$ for $h = 5,13,21\ldots$ and $(1,2,\ldots)$ for all other binary series with h an odd number.

For $h=2n$, $24h+19 = 48n+19 \rightarrow 144n+58(1) \rightarrow 72n+29 \rightarrow (24)(9n+3)+16$.

For $h=8n+5$, $192n+139 \rightarrow 576n+418(1) \rightarrow 288n+209 \rightarrow 864n+628(2) \rightarrow 216n+157 \rightarrow (27n+24)(24)+16$

For $h=2n+1$, $48n+27 \rightarrow 144n+82(1) \rightarrow 72n+41 \rightarrow 216n+124(2) \rightarrow \ldots$

By Lemma 4.2.1 The binary series for $r=2$ is $(1,2)=1/2^3 = 1/8$[2]$= 3^{r-2}/2^{2r-1}$

Assume the proportion of $24h+19$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

The proportion of $24h+19$ terms of binary series **length** $r+1$ is: $(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.
The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1}) / (3^{r-1}/2^{r+1}) = 3/4$[3]
The total proportion of $24h+19$ terms in the Collatz structure is $1/2$[1]$+(1/8)$[2]$/(1 - 3/4$[3]$) = 1$.

All $24h+19$ terms are in branches of the Collatz structure.

Theorem 4.2.2: All $24h+1$ terms in branches of the Collatz structure.

Lemma 4.2.2.: The first $24h+1$ term binary series is (2) for $h=2, 6, 10,\ldots$ $1/4$ of all the terms.
All other binary series begin with $(2,\ldots)$.

For $h=4n+2$ $24h+1 = 96n+49 \rightarrow 288n+148(2) \rightarrow 72n+37 \rightarrow (24)(9n+4)+16$
$24h+1 \rightarrow 72h+4(2) \rightarrow 18h+1 \rightarrow \ldots$

By Lemma 4.2.2 The binary series for $r=1$ is $(2) = 1/2^2$[1]$= 3^{r-1}/2^{2r}$.

Assume the proportion of $24h+1$ terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The proportion of $24h+1$ terms of binary series **length** $r+1 = (1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r}) / (3^r/2^{2r+2}) = 3/4$[2]

The total proportion of $24h+1$ terms in the Collatz structure is $(1/4$[1]$)/(1 - 3/4$[2]$) = 1$.

All $24h+1$ terms are in branches of the Collatz structure.

Theorem 4.2.3: All $24h+7$ terms are in branches of the Collatz structure.

Lemma 4.2.3: The first $24h+7$ term binary series is $(1,1)$ for $h=2,6,10,\ldots$ $1/4$ of all $24h+7$ terms.
All other binary series with an odd number of terms begin with $(1,1,\ldots)$.

For $h=4n+2$, $24h+7 = 96n+55\rightarrow 288n+166(1)\rightarrow 144n+83\rightarrow 432n+250(1)\rightarrow 216n+125\rightarrow(24)(27n+15)+16$
$24h+7\rightarrow 72h+22(1)\rightarrow 36h+11\rightarrow 108h+34\rightarrow 54h+17(1)\rightarrow\ldots$

By Lemma 4.1.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2 = 1/4$[1] $= 3^{r-2}/2^{2r-2}$.

Assume the proportion of $24h+7$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The proportion of $24h+7$ terms of binary series **length** $r+1$ is
$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.
The ratio between successive terms is $(3^{r-2}/2^{2r-2}) / (3^{r-1}/2^{2r}) = 3/4$[2].
The total proportion of $24h+7$ terms in the Collatz structure is $1 = (1/4$[1]$)/(1 - 3/4$[2]$)$.
All $24h+7$ terms are in branches of the Collatz structure.

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Collectively all $24h+1$, $24h+7$, and $24h+19$ are first terms in finite branch segments with binary series of all 2^r combinations of *1's* and *2's* for every value of r. **There are no unending $24h+1$, $24h+7$, or $24h+19$ branch segments.**

Section 4.3 $24h+11$, $24h+17$, and $24h+23$ are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of r.

Theorem 4.3.1: All $24h+11$ terms are in branches of the Collatz structure.

Lemma 4.3.1: The first two $24h+11$ term binary series are (1) for $h=1,3,5,\ldots$ $1/2$[1] **of all $24h+11$ terms** $(1,2)$ for $h = 5,13,21\ldots$ and $(1,2,\ldots)$ for all other value of h.

For $h=2n+1$, $24h+11 =48n+35\rightarrow 144n+106(1)\rightarrow 72n+53\rightarrow(24)(9n+6)+16$.

For $h=8n+8$, $192n+2\,03\rightarrow 576n+610(1)\rightarrow 288n+305\rightarrow 864n+916(2)\rightarrow 216n+229\rightarrow(27n+28)(24)+16$

$24h+11\rightarrow 72h+34(1)\rightarrow 36h+17\rightarrow 108h+52(2)\rightarrow\ldots$

By Lemma 4.2.1 The binary series for $r=2$ is $(1,2)=1/2^3 = 1/8$[2]$= 3^{r-2}/2^{2r-1}$

Assume the proportion of $24h+11$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

The proportion of $24h+11$ terms of binary series **length** $r+1$ is: $(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1}) / (3^{r-1}/2^{r+1}) = 3/4$[3]

The total proportion of $24h+11$ terms in the Collatz structure is $1/2$[1]$+(1/8)$[2]$/(1 - 3/4$[3]$) = 1$.

All $24h+11$ terms are in branches of the Collatz structure.

Theorem 4.3.2: The proportion of $24h+17$ in branch segments with binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

Lemma 4.3.2.: The first $24h+17$ term binary series is (2) for $h=4, 8, 12,\ldots$ $1/4$ of all the terms.
All other binary series begin with $(2,\ldots)$.

For $h=4n+4$ $24h+17 = 96n+113\rightarrow 288n+340(2)\rightarrow 72n+85\rightarrow(24)(9n+10)+16$
$24h+17\rightarrow 72h+52(2)\rightarrow 18h+13\rightarrow\ldots$

By Lemma 4.2.2 The binary series for $r=1$ is $(2) = 1/2^2$[1] $= 3^{r-1}/2^{2r}$.

Assume the proportion of $24h+17$ terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The proportion of $24h+17$ terms of binary series **length** $r+1 = (1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r}) / (3^r/2^{2r+2}) = 3/4$[2]

The total proportion of $24h+17$ terms in the Collatz structure is $(1/4$[1]$)/(1 - 3/4$[2]$) = 1$.

All $24h+17$ terms are in branches of the Collatz structure.

Theorem 4.3.3: All *24h+23* terms are in branches of the Collatz structure.

Lemma 4.3.3: The first *24h+23* term binary series is *(1,1)* for *h=2,6,10,… 1/4* of all the terms.
All other binary series with an odd number of terms begin with *(1,1,...)*.

For *h=4n+4, 24h+23=96n+119→288n+358(1)→144n+179→432n+538(1)→216n+269→(24)(27n+33)+16*
24h+23→72h+70(1)→36h+35→108h+106→54h+53(1)→...

By Lemma 4.1.3.1 The binary series for *r=2* is *(1,1)* $= 1/2^2 = 1/4$[1] $= 3^{r-2}/2^{2r-2}$.

Assume the proportion of *24h+23* terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The proportion of *24h+23* terms of binary series **length** *r+1* is
$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.
The ratio between successive terms is $(3^{r-2}/2^{2r-2}) / (3^{r-1}/2^{2r}) = 3/4$[2].
The total proportion of *24h+23* terms in the Collatz structure is $1 = (1/4$[1]$)/(1 - 3/4$[2]$)$.
All *24h+23* terms are in branches of the Collatz structure.

***

Collectively all *24h+11, 24h+17,* and *24h+23* are first terms in finite branch segments with binary series of all 2^r combinations of *1's* and *2's* for every value of *r*. **There are no unending *24h+11, 24h+17,* or *24h+23* branch segments.**

Section 4.4 *24k+16* are the last terms of finite branches with binary series of every combination of *1's* and *2's* for every value of *r*.

The formula (section 3) for the last term in a group of branches with the same binary series of **length** *r* is $24k+16+(p-1)(24)(3^{r+1})$ *p=1,2,3…*

The proportion of *24k+16* terms in branches with a binary series **length** *r* is $2^r/3^{r+1}$. Proof by induction.
24h+21→ 24(3h+2)+16 is the formula for a branch with **length** *r = 0* binary series. The proportion of *24k+16* terms in branch with an empty binary series is *1/3*[1]. is true For *r = 0.* $1/3 = 2^0/3^{0+1} = 2^r/3^{r+1}$.

Assume the proportion of *24k+16* terms in branches with a binary series of **length** $r \geq 0$ is $2^r/3^{r+1}$.
2^r is the number of different branch binary series of **length** *r.* There are *r+1* applications of *2j+1→6j+4* .
2^{r+1} is the number of branch binary series of **length** *r+1.* There are *r+2* applications of *2j+1→6j+4*.

Thus, the proportion of *24k+16* terms in branches with a binary series of **length** *r+1* is *2/3*[2] the proportion of *24k+16* terms in branches with a binary series of **length** *r.* $(2/3)(2^r/3^{r+1}) = 2^{r+1}/3^{r+2}$. The total proportion is $1 = (1/3$[1]$)/(1 - 2/3$[2]$)$. **All *24k+16*** are last terms of finite branches with binary series of every combination of *1's* and *2's* for every value of *r*.

Section 4 Summary. All positive integers are in branches or towers of the Collatz structure.

The terms of the form *24h+3*[3]*, 24h+11*[4], and *24h+19*[4] have proportion formulas $3^{r-2}/2^{2r-1}$ and are the first terms in **branch**[3] or **branch segments**[4] with binary series of *(1), (1,2)* and *(1,2,…)*.

The terms of the form *24h+7*[4]*, 24h+15*[3]*,* and *24h+23*[4] have proportion formulas $3^{r-2}/2^{2r-2}$ and are the first terms in **branch**[3] or **branch segments**[4] with binary series of *(1,1)* and *(1,1,…)*.

The terms of the form *24h+1*[4]*, 24h+9*[3]*,* and *24h+17*[4] have proportion formulas $3^{r-1}/2^{2r}$ and are the first terms in **branch**[3] or **branch segments**[4] with binary series of *(2)* and *(2,…)*.

The terms of the form *24h+5*[4]*, 24h+13*[4]*,* and *24h+21*[3] are the first terms in **branch**[3] or **branch segments**[4] with an empty **length** *r = 0* binary series. *24h+5→24(3h)+16. 24h+13→24(3h+1)+16. 24h+21→ 24(3h+2)+16.*

The proportion formulas create geometric series that all sum to *1 (100%)*. All odd terms are in branches. All even terms are connected with one of these odd terms. All even terms are also in branches and/or towers.
(2n+1→6n+4) 24m+4 (n = 4m)→12m+2 (24j+2, m=2j, 24j+14, m=2j+1), 24m+10 (n = 4m+1), 24m+16 (n = 4m+2), and 24m+22 (n = 4m+3) are in branches.

All $(2^s)(6j+3)$ *24k, 24k+6, 24k+12,* and *24k+18* terms are in **green towers.**
All *24k+16→12k+8 (24j+8, k=2j, 24j+20, k=2j+1)* terms are in **red towers.**

All terms *24k+2s,* and *24k+2(s+1), k=0,1,2,…* $0 \leq s \leq 11$, are in the branches or towers.

Section 5

There are no unending or circular Collatz sequences.

A circular Collatz sequence could not contain any $6j+3$ terms. The only predecessors of $6j+3$ terms are of the form $(2^s)(6j+3)$ and they cannot be in a circular sequence. They have no predecessors but themselves. No $6j+1$ or $6j+5$ terms can be in a circular Collatz sequence. They are all in branches, which contain $6j+3$ terms. All even terms are in branches or towers. Therefore, there are no circular Collatz sequences.

To prove there are no unending Collatz sequences we need to define a new item that is a part of all Collatz sequences. An L_8 begins with a $24k+16$ ***(280)*** term in a secondary tower. The Collatz algorithm is applied until the **red tower base** term ***(70)*** appears**.** The Collatz algorithm is applied to the branch segment until a $24k+16$ term ***(160)*** appears in an adjoining tower. Thus, we have an L_8**.** It has an **L** shape and joins two $24k+16$ terms both divisible by eight. The adjoining L_8 is between ***160*** and ***16.*** We have reached the Trunk Tower.

The process stops.

280
140
160←53← 106←35←70
80
40
20
16 ←5 ←10

Definition of an L_8 chain binary series A chain of adjoining L_8 moves through Collatz Structure until reaching a $24k+16$ Trunk Tower term. An L_8 chain binary series is made of the number of divisions by two in each **red tower base** term in the individual L_8 of the L_8 chain. The length of an L_8 chain binary series is the number of **red tower base** terms in the individual L_8 of the L_8 chain.

The above L_8 chain has a binary series of ***(1,1,1).***

Theorem 5.1 The proportion of first term $24k+16$ terms in L_8 chains of binary series length $r \geq 0$ is $2^r/3^{r+1}$**.**

We prove theorem 5.1 by induction.

The first term of a tower branch with no binary series is $24h+21+((24)(3h+2)+16)(4^{(3)(p-1)} - 1)/3$.
The last term is $(24k+16)(4^{(3)(p-1)})$ (Appendix 2).

All L_8 chain first term $24k+16$ terms with no binary series are in the Trunk Tower.
Set $h = 0, k=2, p = 1, 2, 3, \ldots$

$$4^3 \leftarrow 21,\ \ 4^6 \leftarrow 21+(64)(21),\ 4^9 \leftarrow 21+(64)(1365), \ldots$$

1/3[1] of the Trunk Tower *24k+16* terms are the first terms in L_8 chains with no binary series.
The proportion formula $2^r/3^{r+1}$ is true for **length** $r = 0$. $1/3 = 2^0/3^{0+1} = 2^r/3^{r+1}$**.**
Assume the proportion of $24k+16$ first terms in L_8 chains with a binary series of **length** $r \geq 0$ is $2^r/3^{r+1}$**.**

2^r is the number of different L_8 chains binary series of **length** r**.** There are $r+1$ applications of $2j+1 \rightarrow 6j+4$.
2^{r+1} is the number of L_8 chains binary series of **length** $r+1$**.** There are $r+2$ applications of $2j+1 \rightarrow 6j+4$.
Thus, the proportion of first $24k+16$ terms in L_8 chains with a binary series of **length** $r+1$ is *2/3*[2] the proportion of first $24k+16$ terms in L_8 chains with a binary series of **length** r**.** $(2/3)(2^r/3^{r+1}) = 2^{r+1}/3^{r+2}$**,** which is the proportion of first $24k+16$ terms in L_8 chains with a binary series of **length** $r+1$. The total proportion is ***1 = (1/3*[1]*)/(1 – 2/3*[2]*).***

Every last term of a branch is the first term of an L_8 chain. Each L_8 chain binary series is of finite length, but there is no longest L_8 chain binary series. There are no unending L_8 chains. They would never reach a Trunk Tower term and could not be part of L_8 chain of $24k+16$ first term proportion geometric series sum.

We have shown the all positive integers are in the Collatz structure only once. Thus, every positive integer forms a Collatz sequence with unique terms terminating in the number one.

Appendix 1. **A branch cannot have more than two consecutive even terms.**

$6n+1 \rightarrow 18n+4$

If $n = 4j$, $18n+4 = 72j+4$ $(24m+4,\ m=3j) \rightarrow 36j+2 \rightarrow 18j+1$.

If $n = 4j+1$, $18n+4 = 72j+22$ $(24m+22,\ m=3j) \rightarrow 36j+11$.

If $n = 4j+2$, $18n+4 = 72j+40$ $(24m+16,\ m=3j+1)$ Last term in the branch.

If $n = 4j+3$, $18n+4 = 72j+58$ $(24m+10,\ m=3j+2) \rightarrow 36j+29$

$6n+3 \rightarrow 18n+10$.

If $n = 4j$, $18n+10 = 72j+10$ $(24m+10,\ m=3j) \rightarrow 36j+5$.

If $n = 4j+1$, $18n+10 = 72j+28$ $(24m+4,\ m=3j+1) \rightarrow 36j+14 \rightarrow 18j+7$

If $n = 4j+2$, $18n+10 = 72j+46$ $(24m+22,\ m=3j+1) \rightarrow 36j+23$.

If $n = 4j+3$, $18n+10 = 72j+64$ $(24m+16,\ m=3j+2)$ Last term in the branch.

$6n+5 \rightarrow 18n+16$.

If $n = 4j$, $18n+16 = 72j+16$ $(24m+16,\ m=3j)$ Last term in the branch.

If $n = 4j+1$, $18n+16 = 72j+34$ $(24m+10,\ m=3j+1) \rightarrow 36j+17$.

If $n = 4j+2$, $18n+16 = 72j+52$ $(24m+4,\ m=3j+2) \rightarrow 36j+26 \rightarrow 18j+13$.

If $n = 4j+3$, $18n+16 = 72j+70$ $(24m+22,\ m=3j+2) \rightarrow 36j+35$.

Appendix 2. **The repeating binary series structure of towers.**

Within a tower if the sum of ***r 1's*** and ***2's*** in the binary series of a branch is ***s***, there are three groups of branches having the same binary series ***24h+3, 24h+9,*** and ***24h+15***.

The first begins with $24h+3+(2^s)(24k+16)(4^{(x)(p-1)}-1)/3^{r+1}$, $h=0,1,2,3,\dots$, $x=3^{r+1}$, $p=1,2,3\dots$ and ends with $(24k+16)(4^{(x)(p-1)})$, $x=3^{r+1}$, $p=1,2,3\dots$ where ***24h+3*** becomes ***24k+16*** after ***r+1* applications of** $2j+1\rightarrow 6j+4$ applied to ***24h+3*** and its odd successors and ***s*** divisions by two applied to ***72h+10*** and its even successors.

and applied to $(2^s)(24k+16)(4^{(x)(p-1)}-1)/3^{r+1}$ becomes $(24k+16)(4^{(x)(p-1)}-1)$.

This gives $(24k+16)+(24k+16)(4^{(x)(p-1)}-1) = (24k+16)(4^{(x)(p-1)})$.

A branch with no binary series starts with $24h+21+((24)(3h+2)+16)(4^{(3)(p-1)}-1)/3$
and ends with $((24)(3h+2)+16)(4^{(3)(p-1)})$.

The other two groups that begin with ***24h+9…*** and ***24h+15…*** have the same form as ***24h+3***...

Link between the formulas for branch and tower first terms.

For some *t*, $24h+3+(t-1)(24)(2^s) = 24h+3+(2^s)(24k+16)(4^{(x)(p-1)}-1)/3^{r+1}$.

For $x=3^{r+1}$ every power of three in $4^{(x)(p-1)}-1 = (3+1)^{(x)(p-1)}-1$ has a coefficient divisible by 3^{r+1}.
$(24k+16)(4^{(x)(p-1)}-1)/3^{r+1}$ is a multiple ***24.*** The same is true for the forms beginning with ***24h+9…, 24h+15…,*** and ***24h+21…***

Each tower's branch binary series structure is a microcosm of the total branch binary series structure**.**

$x=3^{r+1}4^{(x)(p-1)}$ vs 3^{r+1}. $((24)(3h+2)+16)(4^{(x)(p-1)})$ vs $24k+16+(p-1)(24)(3^{r+1})$ $p=1,2,3\dots$

In each case the last terms of tower branches with the same binary series occur in intervals of 3^{r+1}. $2^r/3^{r+1}$ is the proportion of the 2^r last terms of tower branches with a binary series of **length *r***.

For length $r \geq 0$ $1/3+2/9+4/27\dots = 1$ is the total tower proportion.

There are tower branches with binary series of all 2^r combinations of *r 1's* and *2's* for every value of *r*.
The first branch with a binary series of length *r* comes within the first 3^{r+1} branches in the tower.

Appendix 3. **Collatz Structure Details.**

Groups of similar Collatz sequence segments. If a Collatz sequence segment has a first term ***a*** and a last term ***b*** with ***r,*** $2j+1\rightarrow 6j+4$ and ***s*** divisions by two, there is a series of Collatz sequence segments containing the same number of terms and the same number of adjoining L_8 of the same size and structure with a first term $a+(p-1)(24)(2^s)$ and last term $b+(p-1)(24)(3^r)$, $p=1,2,3\dots$

The average branch binary series length: $3r=(1)(3/4)+(2)(9/16)+(3)(27/64)+\dots$ $3r-(3)(3/4)r = 3$, $r=4$.
The binary series usage factor is three. Three lengths ***3r*** are being calculated. ***3/4*** is the proportion of length one. ***9/16*** of length two…Multiply the equation by ***3/4*** and subtract. $3r-(3)(3/4)r = 3/4 + 9/16 + \dots = 3$.

The average branch binary series sum: ***((2,1,1,1)+(2,2,1,1)+(2,1,1,1))/3 = (5+6+5)/3 = 4.333…***
There are twice as many binary series components with one division by two ***24j+10 (1), 24j+22 (1)*** than there are components with two divisions by two ***24j+4 (2).*** Three binary series of length four with twice as many ***1's*** as ***2's*** make up the computation.

A circular sequence ***1→4→2→1→4→2→1→4→2→1…*** **can be used to generate a sequence of arbitrary length with the same number and positions of** ***2j+1→6j+4*** **and divisions by two. The binary series of length** ***s*** **is** ***(2,2,2,…) s*** **times.**

$1+(2^{2s})(24)(p-1)$ is the beginning term and $1+(3^{s})(24)(p-1)$ end term.

For $s=3$, $p=2$, ***1537→4612[2]→2306→1153→3460[2]→1730→865→2596[2]→1298→649***

$1+(2^{2s})(24) = 8n_1+1 \to 24n_1+4 \to 12n_1+2 \to 8n_2+1 \to 24n_2+4 \to 12n_2+2 \to \ldots \to 8n_s+1 \to 24n_s+4 \to 12n_s+2 \to 8n_{s+1}+1 = 1+(3^s)(24)$

Thanks for your interest in this paper. If you wish to make comments send them to Jim Rock at collatz3106@gmail.com**.**