The Rigged Hilbert Space Hoax

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Abstract

We investigate the claim that the rigged Hilbert spaces could be used to make the bra and ket vectors rigorous. Our result is that the claim is not true, and that the rigged Hilbert spaces are a sleight of hand trick and a hoax.

One of the ideas in theoretical physics is that there exists basis vectors $|x\rangle$, where $x \in \mathbb{R}$. These basis vectors have the completeness property

$$\int_{-\infty}^{\infty} |x\rangle\langle x| dx \;=\; \mathrm{id},$$

and the orthogonality property

$$\langle x|x'\rangle \;=\; \delta(x - x').$$

There is a problem that these basis vectors $|x\rangle$ appear to not be part of rigorous mathematics. One way of charaterizing these basis vectors is to first define objects $|e_x\rangle$ such that

$$\langle e_x|e_{x'}\rangle \;=\; \delta_{x,x'} \text{ (Kronecker delta)},$$

which can be done according to the rules of ordinary set theory, and then to define

$$|x\rangle \;=\; \sqrt{\delta(0)}|e_x\rangle.$$

If we are speaking about theoretical physics, there is nothing wrong in using the square root of the value of Dirac delta function like this. However, if we are speaking about rigorous mathematics, then this object $|x\rangle = \sqrt{\delta(0)}|e_x\rangle$ becomes quite a nuisance. It seems to be neither a function nor a distribution. Nevertheless, it has become a popular belief that rigged Hilbert spaces can be used to make $|x\rangle$ into a rigorous concept. This is an interesting assertion, so let's investigate it further.

The related Hilbert space is defined as

$$L^2 \;=\; \big\{[\psi] \;\big|\; \psi \text{ is measurable and } \int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty\big\}.$$

One idea that is relevant for the rigged Hilbert spaces is that if $\psi \mapsto A\psi$ is some linear mapping, then its domain is often not the full L^2, but instead some non-trivial subspace of L^2 defined by

$$D(A) \;=\; \big\{[\psi] \in L^2 \;\big|\; \|A\psi\|_2 < \infty\big\}.$$

If Φ is some non-trivial subspace of L^2, then its dual is defined as

$$\Phi^* \;=\; \big\{\varphi \;\big|\; \varphi : \Phi \to \mathbb{C} \text{ is linear}\big\}.$$

One idea that is related to the rigged Hilbert spaces is the relation

$$\Phi \;\subset\; L^2 \;\subset\; \Phi^*.$$

A pair (L^2, Φ) or a triplet (Φ, L^2, Φ^*) can be called a rigged Hilbert space.

At this point those who believe in rigged Hilbert spaces are ready to jump to the conclusion that the rigged Hilbert space "*permits the use of bra and ket vectors rigorously*". Expressions such as "*bra and ket vectors are fully implemented by the rigged Hilbert space*" or "*bra and ket vectors are mathematically justified by the rigged Hilbert space*" are used. The problem with these conclusions is that they are bogus. While it is interesting that unbounded operators A have non-trivial domains $D(A)$, and that the relation $\Phi \subset L^2 \subset \Phi^*$ is true, the object $|x\rangle = \sqrt{\delta(0)}|e_x\rangle$ still remains as a vague non-rigorous object without a legitimate definition.

Some people believe that the bra and ket vectors could be seen as distributions. A problem with that interpretation is that if $\langle x|$ was a distribution, the inner product $\langle x|x'\rangle$ could not be used, because $|x'\rangle$ does not belong to a typical set of test functions.

The rigged Hilbert space appears to be a sleight of hand magic trick. The idea in this trick is that when the spectators' attention is drawn to the fact that unbounded operators A have non-trivial domains $D(A)$, and to the relation $\Phi \subset L^2 \subset \Phi^*$, the attention is drawn away from the fact that no rigorous definition has been found for the basis vectors $|x\rangle$. Nothing gets done with the domain $D(A)$ or with the relation $\Phi \subset L^2 \subset \Phi^*$. For example, it is still not known in which set the object $|x\rangle$ should belong to.

Since the situation is that large amounts of people have been manipulated into believing that the basis vectors $|x\rangle$ would have been made rigorous with the rigged Hilbert space, while in reality nothing like this has been achieved, we can label the rigged Hilbert space as a scientific hoax.