A proof of the Collatz (3x+1) conjecture

Xingyuan Zhang

(Independent scholar, Suzhou, P. R. China, 215131, e-mail:502553424@qq.com)

Abstract: In this paper we had given an elementary proof of the Collatz conjecture, it holds. By detailed analysis of the properties of both forward and inverse operations of the proposition, we had some important conclusions: 1, there hasn't any triple in the forward path numbers; 2, there have an infinity number of inverse path numbers which had been defined as similar numbers in one time of inverse operation; 3, to do inverse operations (called reverse tracing) repeatedly from 1, it will obtain all of the odd numbers; 4, for any odd obtained by tracing, to do forward operations, it must return to 1 along the reverse tracing path.

Keywords: conjecture, path number, similar number, source number, reverse tracing

0 Introductions

The Collatz conjecture is the 3x+1 conjecture, also known as Kakutani's problem. It has not been proved since it was proposed [1]. Its operation rules are: for any given positive integer n, if even, to $n/2$; if odd, to $(3n+1)/2$. To do it repeatedly, n will eventually return to 1.

In this paper, we called Collatz conjecture as Collatz proposition, or proposition for short.

According to the operation rules, an even number will be transformed into an odd firstly, so we take odd numbers directly to analyze and study for the operations.

1 Operational rule of the proposition and analysis of the operation properties

In operations, there are many new odds and they form an operation path.

Definition 1

(a) The operation process from an odd to a new odd is called one time of operation; times of operations are called continuous operations; in one operation, divided by 2 is called a local operation;

(b) The new odd obtained after one operation is called a path number;

(c) One operation done in the order of the proposition is called one time of forward operation.

Next, we give the formula and analyze the properties of forward operation.

1.1 The operational formula

For any given odd number n, let p be its path number, then according to the operation rules, we have

$$p = \frac{3n+1}{2^k} \tag{1.1}$$

Where $k \in N$ and 2^k is a divisor.

Here, we called formula (1.1) as the forward operation formula of the proposition. Analysis Formula(1) , it is not difficult to obtain: for any given odd number n, there has only one path number p corresponding to n; the value of k is determined by the odd number n, k can be expressed as the times divided by 2, for example, when it equals to 1, that it means in one operation, there is one local operation; when it equals to 3, there are three local operations; for two different odd numbers, the times divided by 2 are the same or different because the values of k in the divisor can take all of the positive integers, therefore, perhaps there are infinite odd numbers that they will all get the same path number after one operation, and these new odd numbers have some correlation properties with each other.

1.2 The properties of the path numbers

1.2.1 Numerical comparison of n and p

Suppose $p = n$, then from formula (1.1) we have

$$n = \frac{1}{2^k - 3}. \tag{1.2}$$

It can be seen that equation (1.2) has only one positive integer solution 1 when k is equal to 2. From this, we can draw a conclusion (conclusion (1)): for the odd number 1, its path number is equal to itself; for any given positive integer n except 1, its path number p isn't equal to n itself, that is $p \neq n$.

Thus it can be seen that there has only one cycle 1-4-2-1 of which n to be taken 1 in one time of operation. If get 1, we stop to do operations.

In a numerical size relationship, when $n \neq 1$, we can get what as fallows

If, $k = 1 \quad p > n$

If, $k = 2 \quad p < n$

If, $k \geq 3 \quad p < n$

Obviously, the larger k is, the greater the change rate of the path number is.

Here, for an odd, its path number became smaller quickly or even went to 1 directly when $k \geq 3$. It's to do with what whether all odds can go back to 1 and that is what we're going to research in this paper.

1.2.2 The triples

An odd number n can be expressed as

$$n = 2x + 1.$$

Where $x = 0$ or $x \in N$.

To do one operation for n, suppose we can get a path number $3p$, where p is an odd, then from formula (1.1) we have the following equation

$$3p = \frac{3(2x+1)+1}{2^k} = \frac{6x+4}{2^k}.$$

To simply, then we have

$$x + \frac{2}{3} = 2^{k-1} p. \tag{1.3}$$

Obviously, the equation (1.3) doesn't hold for integers, so we can get the following conclusion (conclusion (2)): the path number is not a triple, but a non-triple; these triples were skipped in operations.

1.2.3 Changes of the values of two adjacent odd numbers

Let n be an odd and expressed as $2x+1$, where $x=0$ or $x \in N$, thus, one of its adjacent odd numbers can be expressed as

$$2x+1+2.$$

To do one operation for $2x+1$, then we get its path number as fallow

$$\frac{3(2x+1)+1}{2} = 3x+2.$$

To do one operation for $2x+1+2$, then we get its path number as fallow

$$\frac{3(2x+1+2)+1}{2} = 3x+5.$$

Obviously, in these two numbers above, one is odd and the other is even. They both increase firstly, since the even number can be divided by 2 again, so it will decrease finally. From this, we can get a conclusion (conclusion (3)): for two adjacent odd numbers, the two path numbers of them if one becomes larger, the other must becomes smaller.

1.3 The tendency of continuous operations

Let n be an odd and $n>1$, let p_1, p_2 and p_3 be three continuous path numbers of n. According to conclusion (1) we have $p_1 \neq n$, $p_2 \neq p_1$ and $p_3 \neq p_2$. From formula (1.1) we have

$$p_1 = \frac{3n+1}{2^{k_1}} \tag{1.4}$$

and

$$p_2 = \frac{3(\frac{3n+1}{2^{k_1}})+1}{2^{k_2}} = \frac{9n+3+2^{k_1}}{2^{k_2+k_1}}. \tag{1.5}$$

Where 2^{k_1} and 2^{k_2} are two divisors, $k_1 \geq 1$ and $k_2 \geq 1$.

Now, suppose $p_2 = n$, from formula (1.5) then we have

$$\frac{9n+3+2^{k_1}}{2^{k_2+k_1}} = n$$

that is

$$n = \frac{3+2^{k_1}}{2^{k_2+k_1}-9}. \tag{1.6}$$

It can be seen from equation (1.6), that if n increases, 2^{k_1} must increase, but at the same time the denominator is also increasing quickly and even bigger than the numerator, so odd number n has a maximum value if it has some positive integer solutions. Now, we take the minimum value 1 of k_2 for analysis. Here are the calculated values

a) when $k_1 = 1$, $n = \frac{3+2}{4-9} = -1$

b) when $k_1 = 2$, $n = \frac{3+4}{8-9} = -7$

c) when $k_1 = 3$, $n = \frac{3+8}{16-9} = \frac{11}{7}$

d) when $k_1 = 4$, $n = \frac{3+16}{32-9} = \frac{19}{23} < 1$

e) when $k_1 > 4$, $n < 1$

From these values above, we can see that equation (1.6) hasn't any positive integer solution for n. In the same way, the same conclusion can be drawn when $k_2 \geq 2$. From this we can obtain $p_2 \neq n$. Because of $p_1 \neq n$, so we have $p_2 \neq p_1 \neq n$.

In the same way, we can derive that $p_3 \neq p_2 \neq p_1$ when we regard p_1 as n, and so on. Now we can draw a conclusion (conclusion (4)): for any given odd except 1, all of its path numbers are different.

For any given odd, there is a finite number of odds less than it, therefore we can get a conclusion (conclusion (5)) here: the path number either goes back to 1 or tends to infinity when keep doing forward operations, there hasn't any cycle except 1-4-2-1.

1.4 Narrow paths

Definition 2 Let n be an odd, to do operations continuously for it, in every operation process, the times of local operation divided by 2 doesn't exceed 2, i.e. $k \leq 2$, thus we called the section of the path as a narrow path which is composed of n and its path numbers.

On the narrow path, the numerical change rate of the path numbers is the smallest, that is, the range of changes is the narrowest.

2 The similar numbers and their properties

From the analysis at 1.1, it's known that the same path number can be obtained when doing one operation for two different odd numbers respectively. For example, if doing one operation for 7 and 29, they both get 11. For 7, the value of k in formula (1.1) is 1, and for 29, the value of k is 3.

Definition 3 Suppose, there are two odd numbers n_1 and n_2 whose path numbers are both p, then, we called that n_1 is a similar number of n_2, or n_2 is a similar number of n_1, that is, they are similar each other, and denoted $n_1 \backsim n_2$, or, $n_2 \backsim n_1$.

For an example, 29 is a similar number of 7, or 7 is a similar number of 29.

Obviously, the similar numbers are caused by different values of k.

Next, we analyze the properties of the similar numbers.

2.1 The relationship between similar numbers

Suppose, there are two similar numbers n_1 and n_2, where $n_2 > n_1$, to do one operation on each of them, we can get the path numbers p_1 and p_2. According to formula (1.1), we have

$$p_1 = \frac{3n_1 + 1}{2^{k_1}}$$

and

$$p_2 = \frac{3n_2+1}{2^{k_2}}.$$

Where $k_1 \in N$, and $k_2 \in N$.

Now, let $p_1 = p_2$, then we have

$$\frac{3n_1+1}{2^{k_1}} = \frac{3n_2+1}{2^{k_2}}. \tag{2.1}$$

Since $n_2 > n_1$, we can get $k_2 > k_1$, that is, $k_2 - k_1$ are positive integers.

From equation (2.1), n_2 can be obtained, that is

$$n_2 = \frac{2^{k_2}}{2^{k_1}} n_1 + \frac{1}{3}\left(\frac{2^{k_2}}{2^{k_1}} - 1\right) = 2^{k_2-k_1} n_1 + \frac{1}{3}\left(2^{k_2-k_1} - 1\right). \tag{2.2}$$

Obviously, for equation (2.2), if n_2 to be an integer, $2^{k_2-k_1} - 1$ must be a triple, there is a minimum value 3 in triples and at this time $2^{k_2-k_1}$ takes 4.

Now, let $k_2 - k_1 = 2t$, i.e., $k_2 - k_1$ takes even numbers, where $t \in N$, thus we have

$$2^{k_2-k_1} - 1 = 2^{2t} - 1 = (2^t+1)(2^t-1). \tag{2.3}$$

As it can be seen from equation (2.3) that there is a triple in three continuous numbers of $2^t - 1$, 2^t and $2^t + 1$. For 2^t, if t is even, $2^t - 1$ is a triple, if t is odd, $2^t + 1$ is a triple and thus $2^t - 1$ must not be a triple, that is, when $k_2 - k_1$ takes odd numbers, $2^{k_2-k_1} - 1$ hasn't any triple factor. Thus $k_2 - k_1$ must take even numbers, that is, $2^{k_2-k_1}$ must take 4 or 4 times of 4, and then n_2 has integer solutions in equation (2.2).

Now, we analyze some cases with 4 and its 4 multiples to find some similar numbers respectively,

1) when $2^{k_2-k_1}$ takes 4, we have the second (n_1 is the first)

$$n_2 = 4n_1 + 1$$

2) when $2^{k_2-k_1}$ takes 16, we have the third

$$n_3 = 16n_1 + 5 = 4(4n_1+1)+1$$

3) when $2^{k_2-k_1}$ takes 64, we have the fourth

$$n_4 = 64n_1 + 21 = 4\left[4(4n_1+1)+1\right]+1.$$

Here, we had got three similar numbers of n_1 in turn. As it can be seen that the formula above is an iterative formula, thus more generally, we can deduce an iterative formula as follow

$$n_{1+i} = 4^i n_1 + \sum_{i=1}^{\infty} 4^{i-1}. \tag{2.4}$$

Where, n_1 takes odd numbers and i takes positive integers. Using formula (2.4), we can get an infinite number of similar numbers of n_1.

As examples, we can find some similar numbers of the original few odd numbers.

a) Let n_1 =1, then from (2.4) we have

$$n_{1+i} = 4^i + \sum_{i=1}^{\infty} 4^{i-1} . \tag{2.5}$$

From (2.5) we can obtain a sequence generated by 1 as follow

1, 5, 21, 85, 341…

b) Let n_1 =3, then we can again obtain a sequence generated by 3 as follow

3, 13, 53, 213, 853…

c) Let n_1 =7, then we can also obtain a sequence generated by 7 as follow

7, 29, 117, 469, 1877…

When n_1 =5, the sequence generated by 5 is already in the first sequence.

It can be seen that when the gap between two similar numbers is smallest, i.e., i takes two adjacent numbers, we obtain

$$n_2 = 4n_1 + 1 \tag{2.6}$$

Here, we called formula (2.6) as the formula of the similar numbers, and also, two similar numbers when they have the smallest gap between them as the adjacent similar numbers. By using formula (2.6), we can also find out the numberless similar numbers of n_1 one by one and it's easy to do.

This relationship can be verified by doing operations on n_1 and n_2 separately.

1) For n_1, we have the path number

$$\frac{3n_1 + 1}{2^k}$$

2) For n_2, we have a middle even number

$$\frac{3(4n_1 + 1) + 1}{2^k} = 4\left(\frac{3n_1 + 1}{2^k}\right).$$

Obviously, the number on the right above can be divided by 4 again, thus we also have a path number

$$\frac{3n_1 + 1}{2^k}$$

As it can be seen that when we do operations on n_1 and n_2 respectively, we get the same path number, so they are similar numbers to each other.

To do inverse operation for formula (2.6), then we have

$$n_1 = \frac{n_2 - 1}{4} \tag{2.7}$$

Obviously, if n_1 is an integer, then it is an adjacent similar number of n_2, and $n_1 < n_2$. Here Formula (2.7) is called the inverse operation formula of similar numbers.

2.2 The sets of similar numbers

It is obvious from formula (2.6) that any odd number can generate an infinite number of similar numbers in turn.

Definition 4

(a) Suppose, the similar numbers generated by the odd number n_1 in turn are n_2, n_3, …, n_i, where $i \in N$ and $i \geq 2$, then we called the infinite set composed of n_1 and n_i as a infinite set of similar numbers, or a set for short;

(b) We called n_i as the previous similar number of n_{i+1}, and n_{i+1} as the next similar number of n_i;

(c) We called n_1 as the generating number of a set (the first); called the set as number n_1 set.

For examples, when the generating number is 1, it can generate 5, 21, 85, 341, 1365…an infinite number of similar numbers (see section 2.1), 1 and all of its similar numbers constitute a set, this set is called number 1 set, in which any similar number returns directly to 1 after one operation; 1 is also the path number or the operational value of number 1 set. In the same way, 3 can generate similar numbers such as 13, 53, 213, 853, and so on (see section 2.1), they constitute the number 3 set. In a set, all numbers constitute an increased sequence. Specifically, if every number in number 1 set be added with 3 to the right in turn, then number 1 set becomes number 3 set (missed the first).

For understanding, there is a table of sets of similar numbers below (see Tab. 1 Table of sets of similar numbers).

In table 1, if we regard two sets as connected each other through the first path number (also theirs operational value), thus all of the sets may be connected through the path numbers of them and it can also explain what an odd goes back to 1. For examples, the operational value of number 1 set is 1, and the second similar number in the set is 5, 5 is also the operational value of number 3 set, thus, two sets is connected through the number 5; in the same way, number 7, 11, 17 and 13 set are also all connected and the odds involved can go back to 1.

2.3 The smallest gap between two triples in a set

In a continuous series of odd numbers, two adjacent triples are separated by two non-triples. 3 is the smallest triple, and 3 added by 6 every time, it becomes another triple (odd). Similarly, in a set, two adjacent triples are separated by two non-triples.

To verify as follows:

A triple can be expressed as $3n$, where n is an odd, so according to Formula (2.6) the following three similar numbers are as follows in turn:

The first $\quad 4(3n)+1 = 12n+1 = 3(4n)+1$

The second $\quad 4(12n+1)+1 = 48n+5 = 3(16n)+6-1$

The third $4(48n+5)+1=192n+21=3(64n+7)$

As it can be seen, that the first and the second are both not triples, and the third is exactly a triple.

Tab. 1 Table of sets of similar numbers

n\c	Path numb.	Comp.	Set numb.	Similar numbers (arranged small to large)						
				1	2	3	4	5	6	7
1	1	=	1	1	5	21	85	341	1365	5461
2	5	>	3	3	13	53	213	853	3413	13653
3	1	<	5	5	1	1	1	1	1	1
4	11	>	7	7	29	117	469	1877	7509	30037
5	7	<	9	9	37	149	597	2389	9557	38229
6	17	>	11	11	45	181	725	2901	11605	46421
7	5	<	13	13	3	3	3	3	3	3
8	23	>	15	15	61	245	981	3925	15701	62805
9	13	<	17	17	69	277	1109	4437	17749	70997
10	29	>	19	19	77	309	1237	4949	19797	79189
11	1	<	21	21	5	5	5	5	5	5
12	35	>	23	23	93	373	1493	5973	23893	95573
13	19	<	25	25	101	405	1621	6485	25941	103765
14	41	>	27	27	109	437	1749	6997	27989	111957
15	11	<	29	29	7	7	7	7	7	7
16	47	>	31	31	125	501	2005	8021	32085	128341
17	25	<	33	33	133	533	2133	8533	34133	136533
18	53	>	35	35	141	565	2261	9045	36181	368641
19	7	<	37	37	9	9	9	9	9	9
20	59	>	39	39	157	629	2517	10069	40277	161109
21	31	<	41	41	165	661	2645	10581	42325	169301
22	65	>	43	43	173	693	2773	11093	44373	177493
23	17	<	45	45	11	11	11	11	11	11
			……							

notes

1. Each row is a set of similar numbers (7 numbers), the first similar number of each set is the successive odd numbers starting from 1, which is the generating number of these set, and each set takes this odd number as the number of the sets;
2. Comparison represents the relationship between the generating number and the path number, and the size is continuously distributed in pairs (conclusion 1); a path number is also the operational value of the set;
3. The rows of 5, 13, 21, 29…are subsets (underlined);
4. The triples are shaded; in a set, there are two non-triples between two triples (see 2.3);
5. Each column contains any odd (some repeated, as 5 in the second column).

2.4 The effects of the similar numbers in operations

In the continuous operations, when a path number has a similar number smaller than itself, the next path number will quickly become smaller, or even, directly back to 1. For example, the number 853 is similar to 3, which returns to 5 quickly after one operation; the number 1365 is similar to 1, which returns to 1 directly. Thus, those numbers are the ending-numbers of a narrow path. It is not difficult to see that the similarity existing in odd numbers is of great significance in judging this proposition.

3 Analysis of the principle of the inverse operations

The forward operation of the proposition is reversible for non-triples. Now, to do one reverse operation for formula (1.1), then we have

$$n = \frac{2^k p - 1}{3} \tag{3.1}$$

or

$$3n = 2^k p - 1. \tag{3.2}$$

Where $k \in N$, and p takes non-triples (conclusion (2)).

Here, formula (3.1) or (3.2) is called the inverse formula of the proposition, 2^k is called a multiplier, and n is called the inverse path number of the given forward path number p. The formula (3.1) and (3.2) are used in reverse to find the inverse path number n.

Here, we firstly analyze some cases of particular path numbers p.

a) Let $p = 1$, from the formula (3.1), then we have

$$n = \frac{2^k - 1}{3}. \tag{3.3}$$

As it is analyzed in section 2.1, if k is even, n has its positive integer solutions in equation (3.3); if k is odd, it hasn't any positive integer solution. From this, we can find the inverse path numbers such as 1, 5, 21, 85⋯, that is the number 1 set of similar number. Obviously, the inverse path numbers contains 1 itself and there is a cycle 1-1-1 of which p to be taken 1 in one time of inverse operation.

b) Let $n = p$ and $p > 1$, from the formula (3.1), then we have

$$p = \frac{2^k p - 1}{3}.$$

That is

$$p = \frac{1}{2^k - 3}. \tag{3.4}$$

As it can be seen that equation (3.4) has only one positive integer solution ($p = 1$) when $k = 2$. Since $p > 1$, so we can draw a conclusion (conclusion (6)): the inverse path number isn't equal to the given forward path number, or $n \neq p$ when $p > 1$.

c) Let $p = 3t$, that is p takes triples, where $t \in N$. From formula (3.1), then we have

$$n = \frac{2^k(3t)-1}{3} = 2^k t - \frac{1}{3}. \tag{3.5}$$

Obviously, there is no positive integer solution to equation (3.5), so we can draw a conclusion (conclusion (7)): for any triple, it has no inverse path number.

By analyzed in section 1.1 and here, we know that, there be an infinite number of inverse path numbers, each of them constitutes the source of the known path number p, i.e., p is sourced from the infinite of inverse path numbers.

Next, we analyze the properties of the inverse operations.

3.1 The relationship between the inverse path numbers

From formula (3.1) or (3.2), it can be seen that the inverse path numbers are directly related to the value of k in the multiplier 2^k, and therefore, we use the parity property of the values of k to analyze the relationship between the inverse path numbers. Apparently, for any non-triple p, we can't get an integer in formula (3.1) at the same time when k takes the minimum odd number 1 and the minimum even number 2; for k and $k+2$, they are the same as an odd or even.

Let p be a non-triple, n_1 be the inverse path number, according to formula (3.1), then we have

$$n_1 = \frac{2^k p - 1}{3}. \tag{3.6}$$

Where $k \in N$.

To multiply with 4 for two sides in formula (3.6), then we obtain

$$4n_1 = 4\left(\frac{2^k p - 1}{3}\right) = \frac{2^{k+2} p - 4}{3} = \frac{2^{k+2} p - 1}{3} - 1.$$

That is

$$4n_1 + 1 = \frac{2^{k+2} p - 1}{3}. \tag{3.7}$$

Comparing formula (3.7) with the similar number formula (2.6), it is not difficult to see, that the right of formula (3.7) is the next similar number of n_1, marked n_2, that is

$$n_2 = 4n_1 + 1 = \frac{2^{k+2} p - 1}{3}$$

or

$$n_2 = \frac{2^{k+2} p - 1}{3}. \tag{3.8}$$

It's generated by added 2 to k in the multiplier, that is, when k takes the next odd or even, we can get the next similar number n_2 of n_1.

Again, to multiply with 4 for two sides in formula (3.8), then we have

$$4n_2 = 4\left(\frac{2^{k+2}p-1}{3}\right) = \frac{2^{k+2+2}p-4}{3} = \frac{2^{k+2+2}p-1}{3} - 1.$$

That is

$$4n_2 + 1 = \frac{2^{k+2+2}p-1}{3}. \tag{3.9}$$

As it can be seen in formula (3.9) that the right is the next similar number of n_2, marked n_3, that is

$$n_3 = 4n_2 + 1 = \frac{2^{k+2+2}p-1}{3}. \tag{3.10}$$

In the same way, we can find other similar numbers when k takes again next odd or even. As k increases, there are an infinite number of similar numbers of n_1.

Now, a conclusion (conclusion (8)) can be drawn by the analysis above: in formula (3.1), for a non-triple p, if k takes the smallest odd number 1 (the smallest multiplier is 2), we can get an inverse path number, then k takes the rest odds greater than 1, we can also get an infinite of number of inverse path numbers, and all the inverse path numbers are similar to each other; if, when k takes the smallest odd number 1, we can't get an inverse path number, then k takes the smallest even number 2 (the smallest multiplier is 4) and any even number greater than 2, we can get an infinite of number of inverse path numbers definitely, and they're also similar numbers each other.

Obviously, all the inverse path numbers constitutes a set of similar numbers.

When k takes the smallest odd 1 or even number 2, i.e., multiplier 2^k takes the smallest 2 or 4, the inverse path number is called here the first inverse path number.

3.2 The properties of the first inverse path numbers

Suppose, there are two adjacent similar numbers n_1 and n_2 in a set, where n_2 is known, and $n_2 > n_1$, i.e., n_1 is the previous similar number of n_2, according to formula (3.1), then we have

$$n_2 = \frac{2^k p - 1}{3}.$$

According to formula (2.7), we have

$$n_1 = \frac{n_2 - 1}{4}. \tag{3.11}$$

To substitute n_2 in formula (3.11), then we have

$$n_1 = \frac{\frac{2^k p-1}{3} - 1}{4} = \frac{2^k p - 1 - 3}{12} = \frac{2^k p}{12} - \frac{1}{3}. \tag{3.12}$$

Next, we take the smallest multiplier 2 and 4 in formula (3.12) for analysis respectively.

1） when taking 2

$$n_1 = \frac{2p}{12} - \frac{1}{3} = \frac{p}{6} - \frac{1}{3} = \frac{p-2}{6}$$

2） when taking 4

$$n_1 = \frac{4p}{12} - \frac{1}{3} = \frac{p}{3} - \frac{1}{3} = \frac{p-1}{3}.$$

Obviously, $p-2$ is an odd, $p-1$ is an even, so, neither of these two equations above can get an integer, thus, there is no similar number less than n_2, that is, n_1 doesn't exist. Therefore, it can be concluded (conclusion (9)): the first inverse path number must be the generating number of a similar number set (i.e. n_2 is the first).

Definition 5

(a) When doing the reverse operations according to formula (3.1), we change the names, called the non-triple p given originally as a primitive number; called the first inverse path number as a source number of p ;

(b) Doing one time of reverse operation is called one time of reverse tracing, tracing for short; times of tracing is called continuous tracing.

According to definition 5, one time of tracing is to find out the first source number of a primitive number.

Since the rest inverse path numbers of a primitive number are similar to the source number (conclusion (9)), they can be found in turn by using the similar number formula (2.6), and therefore, here we defined only the first of them.

3.3 Analysis of the multipliers of two continuous primitive numbers

As stated in 2.3, for the continuous series of odd numbers, there are only two consecutive non-triples between two adjacent triples, only non-triples are the primitive numbers.

Let $3p$ be a triple, where p is an odd, so, in terms of the increasing value, the first primitive number adjacent to $3p$ is $3p+2$, and the second is $3p+4$. Next, we take the multiplier 2 and 4 for analysis respectively.

3.3.1 Take 2

According to the formula (3.2), we have an equation as fallow

$$3n = 2p - 1. \tag{3.13}$$

a) Put the first primitive number $3p+2$ into equation (3.13), and then we have

$$3n = 2(3p+2) - 1 = 6p + 3.$$

That is

$$n = 2p + 1. \tag{3.14}$$

Obviously, there is an integer solution in equation (3.14), and it is the source number of first primitive number.

b) Put the second primitive number $3p+4$ into the equation (3.13), and then we have

$$3n = 2(3p+4) - 1 = 6p + 6 + 1.$$

That is

$$n = 2p + 2 + \frac{1}{3}. \tag{3.15}$$

Obviously, the equation (3.15) hasn't any integer solution.

3.3.2 Take 4

According to the formula (3.2), we have an equation as fallow

$$3n = 4p - 1. \tag{3.16}$$

a) put the first primitive number $3p+2$ into equation (3.16) , then we have

$$3n = 4(3p+2) - 1 = 12p + 6 + 1.$$

That is

$$n = 4p + 2 + \frac{1}{3}. \tag{3.17}$$

Obviously, the equation (3.17) hasn't any integer solution.

b) put the second primitive number $3p+4$ into equation (3.16) , then we have

$$3n = 4(3p+4) - 1 = 12p + 15.$$

That is

$$n = 4p + 5. \tag{3.18}$$

Obviously, there is an integer solution in equation (3.18), and it is the source number of the second primitive number.

Thus, based on the analysis of 3.3.1 and 3.3.2, it can be concluded (conclusion (10): in the continuous series of odd numbers, for two continuous primitive numbers, the multiplier of the first is to take 2 and the second is to take 4 definitely. And as a result, for their two source numbers, the first gets smaller, the second gets larger (be similar to conclusion (3)).

Here, we called the formula (3.13) and (3.16) as the formulas of source numbers.

3.4 Analysis of the multiplier of three consecutive odd numbers

3.4.1 The settings of the multiplier of a triple

Since formula (3.1) doesn't hold for the triples (conclusion (7)), therefore, the multipliers of triples is set as 0, which means that there is no source number for the triples. Thus, for two consecutive primitive numbers and a triple, i.e. three consecutive odd numbers, as derived from conclusion (10), their multipliers in order are 2, 4 and 0.

3.4.2 A special determinant of odd numbers

According to the conclusion (10), the continuous odd numbers can be arranged in a special determinant in tabular form, and then the special regular of multipliers can be shown. See Tab. 2 Table of multipliers.

Tab. 2 Table of multipliers (2^k)

row h	con-tent	column l											
		1	2	3	4	5	6	7	8	9	10	11	12
	2^k	**2**	**4**	**0**	**2**	**4**	**0**	**2**	**4**	**0**	**2**	**4**	**0**
1	odd											1	3
2		5	7	9	11	13	15	17	19	21	23	25	27
3		29	31	33	35	37	39	41	43	45	47	49	51
4		53	55	57	59	61	63	65	67	69	71	73	75
5		77	79	81	83	85	87	89	91	93	95	97	99
6		101	103	105	107	109	111	113	115	117	119	121	123
7		125	127	129	131	133	135	137	139	141	143	145	147
8		149	151	153	155	157	159	161	163	165	167	169	171
…													
h		n											
notes	1. This table shows consecutive odd numbers in rows and columns; there are 1 and 3 in first row and the multipliers of them are corresponding to 4 and 0; from second row, there are twelve odd numbers in each, and the multipliers of them are corresponding to 2, 4 and 0 four times in order; 2. Characters of the triples are with shadows, and they form four columns, the remaining 8 columns are primitive numbers.												

In the table, the multipliers for each row are repeated four times in the order of 2, 4 and 0. Next, we use this table to analyze the method of taking the multipliers.

3.4.3 Location analysis of an odd number in Tab. 2

a）Row number h

Let n be an odd given arbitrarily, then, according to the odd number arrangement in the table, its row number h is given by the following formula

$$h = \frac{n-4}{24} + 1 \tag{3.19}$$

Where, h takes an integer approach large.

Here, we called formula (3.19) the row number formula.

b）Column number l

Let the row number h be known, and then the column number l of the odd number n in the table is given by the following formula

$$l = \frac{n-4-24(h-2)}{2} \tag{3.20}$$

Where, l takes an integer approach large also.

Here, we called formula (3.20) the column number formula.

According to the column number l in the table, we can cross-reference to get the multipliers, that is,

When l equals to 1, 4, 7, 10, the odd number is a primitive number, and its multiplier is 2;

When l equals to 2, 5, 8, 11, the odd number is a primitive number, and its multiplier is 4;
When l equals to 3, 6, 9, 12, the odd number is a triple, its multiplier is 0.

Now, by analysis of 3.4.2 and 3.4.3, it can be concluded (conclusion (11)): For any given odd number n (a primitive number or a triple), its multiplier 2^k is obtainable.

As it can be seen from conclusion (11) that the multiplier can be determined by the primitive number its self. Thus, the source number of any primitive number can be found out by using formula (3.13) or (3.16), and an infinite number of similar numbers of the primitive number can also be found out by using formula (2.6).

In a limited range of odds, there is a simple way to find the source numbers. Firstly, we use 2 directly in formula (3.1) to try to find out the source number n. If n is an integer, so 2 is its multiplier, and p is a primitive number, the integer n is its source number. If n isn't an integer, then use 4 to try secondly, if neither of n is an integer, then p must be a triple, and it has no source number.

3.5 Analysis of continuous tracing path of the source numbers

According to conclusion (11), source number n can be obtained by tracing for primitive number p. Obviously, if n isn't a triple, then n can be regarded as another primitive number for tracing. Over and over again, primitive number p with one or more of its source numbers forms a continuous tracing path.

Next, we analyze the properties of continuous tracing paths.

3.5.1 The end of a continuous tracing path

Since a triple has no source number (conclusion (7)), so a continuous tracing path ended at a triple. For the special primitive numbers, their source numbers are triples getting from formula (3.13) or (3.16). For the source numbers obtained in turn, there are two trends in general. One is that the source numbers gradually increases, the other is that it gradually decreases. When it tends to be smaller, the last one is not equal to 1 if the traced primitive number isn't 1 (conclusion (6). So, the source numbers either increase or decrease, they both must end at a triple.

This problem is explained in the related description of Figure 1 below (see 3.6 and description (d)).

3.5.2 The integrity of a continuous tracing path

If the primitive number p has the property of arbitrary selection, then it may itself be a first inverse path number of others, that is, it is also a source number, although its source numbers can form a continuous tracing path, but it is not complete. So, at this time, it ´s need to do some continuous forward operations on this primitive number until getting a forward operation path number, when doing forward operation on it again, it has more than two local operations divided by 2, that is, it has at least a similar number less than itself.

A complete tracing path begins with a primitive number which has at least a similar number less than itself, and ends at a triple. In this path, every number except the primitive number is a source number. Obviously, in the opposite direction, that is, the forward path from the final source number to the primitive number is a narrow path. In a narrow path, the source numbers are all the generating numbers of a similar number set (conclusion (9)).

Those complete successive tracing paths have different length, that is, for different primitive numbers, the quantity of source numbers varies.

For examples, tracing for 1, 1 can be traced itself (its multiplier is 4, one forward operation on it also yields itself, 1 is a special odd here); tracing for the similar number 5 of 1, we get the minimum triple 3, so the path has only two odd numbers 5 and 3; for 35, 23 and triple 15 are obtained by successive tracing twice, while for 445, 17 times of tracings are required to obtain the triple 27, and there are 18 odd numbers in this path (to see Fig. 1 below).

Definition 6 If a primitive number p and its source numbers, constitute a complete continuous tracing path, then we called the set composed by the primitive number p and its source numbers as a source number set.

It can be seen that a set of source numbers is a finite set and we can regard a triple as a unary set.

3.6 The method and order of continuous reverse tracing

Obviously, continuous reverse tracing can start with 1. There are two kinds of operations, one is to find the similar numbers, and the other is to find the source numbers, and they carry out alternately. That is

1. To find out the similar numbers of 1, such as 5, 21, 85 and 341;
2. To find out the source number 3 of 5 (only one), it ended at the smallest triple 3;
3. To find out the similar numbers of 3, such as 13, 53, 213 and 853;
4. To find out the source numbers of 13, they are 17, 11, 7 and 9 in turn, or to find out the source numbers of 85, they are 113 and 75 in turn, they both ended at a triple, here 21 is a triple in number 1 similar number set and it was skipped;
5. To repeat the operations above to find out the similar numbers and source numbers.

About the method and the order, see Fig.1 Reverse tracing path graph.

A related description of Figure 1 is as follows:

a) In this form-type graph, odd numbers are generated by 1 from the bottom left to up right; this graph depicts the reverse path from 1 to 27, which is also the forward path from 27 to 1;

b) In the horizontal directions, there are some similar numbers with symbols '∽' in a set; the triples be with shadows, and they are separated by two non-triples (see 2.3);

c) In the vertical directions, there are source numbers from some different sets, the symbol '↓' indicates 'sourced from' (only a little used), the source numbers and the primitive number (at the bottom) form a complete narrow path; the triples be on the tops;

d) The triples at the tops appear to be on the paths, but they are skipped when doing forward operations. For example, doing a single operation on 445 leads directly to 167, thus the triple 111 is skipped; as the same, the triples in the similar number sets are also skipped directly (such as 21 and 1365 in the number 1 set) that’s why that the forward operations didn’t yield any triple (conclusion (2)); although the forward operations doesn’t yield triples, but they can be found when to find similar numbers and source numbers, and that’s what the source number tracing paths ended at the triples;

																27	∽	109
																41		
																31		
																47		
																71		
																107		
																161		
																121		
																91		
																137		
																103		
																155		
																233		
																175		
												8097	∽			263		
												6073				395		
												4555				593		
												6833		111	∽	445		
												↓		167				
										1281	∽	5125		251				
										961				377				
										721				283				
										↓				425				
								135	∽	541		159		319				
								203				239		479				
								305				359		719				
								↓				533		1079				
						57	∽	229				809		1619				
						43				303		607	∽	2429				
						65				455		911						
						49				683		1367						
						↓				1025		2051						
				9	∽	37				769	∽	3077						
				7						577								
				11						433								
				17				81	∽	325								
				↓		15	∽	61										
				↓		23												
				↓		35		75										
		3	∽	13	∽	53		113										
		↓						↓										
1	∽	5	∽	∽	21	∽	∽	85	∽	∽	341	∽	∽	1365	∽	∽	5461	∽

Fig.1 Reverse tracing path graph

e) The red path is the triples path which is derived from the triples at the top and their next similar numbers;

f) To take the similar number 109 of 27 (or others) for tracing continuously, there will be more and more paths spreading out like branches of a tree, and obviously, the following paths will all get longer and longer.

As shown in the figure, any section of reverse tracing path shows a step-type path structure. In these path structures, going to the right is following the similar numbers, going upward is following the source numbers.
For a given odd, the number of odds less than it is finite and because the similar numbers in a set are increasing, so if we take the next similar number again and again for tracing, it's not difficult to see that the paths will all tend to infinite finally if there is not any cycle and we can get an infinite number of odd numbers.
Finally, if doing forward operations continuously for any odd obtained, it returns to 1 definitely.

3.7 Analysis of the cycles in the reverse tracing paths

Let p be a primitive number and $p > 1$; let n_1, n_2 and n_3 be three source numbers which are obtained in turn. From conclusion (6), we can obtain $n_1 \neq p$, $n_2 \neq n_1$ and $n_3 \neq n_2$.
From formula (3.1), we have

$$n_1 = \frac{2^{k_1} p - 1}{3}$$

and

$$n_2 = \frac{2^{k_2} n_1 - 1}{3}.$$

Now, suppose $n_2 = p$, then we have

$$n_2 = \frac{2^{k_2} n_1 - 1}{3} = \frac{2^{k_2} (\frac{2^{k_1} p - 1}{3}) - 1}{3} = \frac{2^{k_2 + k_1} p - 2^{k_2} - 3}{9} = p.$$

That is

$$p = \frac{2^{k_2} + 3}{2^{k_2 + k_1} - 9}. \tag{3.21}$$

Where 2^{k_1} and 2^{k_2} are two multipliers, k_1 and k_2 both takes 1 or 2.

Here, using the same analysis as in section 1.3, we can conclude that equation (3.21) hasn't any positive integer solution, so we can get that $n_2 \neq p$, and also further derive that $n_3 \neq n_2 \neq n_1$. Thus, for any given primitive number except 1 and its source numbers, they are different each other. From formula (2.4), it can be seen that every similar number is also different each other. Now we can draw a conclusion (conclusion (12)): all of inverse path numbers except 1 obtained by reverse tracing starting from 1 is different and there hasn't any cycle in the tracing paths; every path that derived from an odd will tend to infinite finally.

3.8 Analysis of density of the odd numbers and the final conclusion

As stated in conclusion (5), the forward path number either goes back to 1 or tends to infinity when keep doing operations.
Now, we analyze the density of the odd numbers obtained by successive tracing.
Suppose, there is an odd number n in the series of odd numbers which hasn't been traced on the paths starting from 1, that is, n has been missed. It is obvious that we can do continuous inverse and forward operations for it. When doing inverse operations continuously, the inverse path numbers must tend to infinity (conclusion (12)); when doing forward operations continuously, the forward path numbers must also tend to infinity, because if its forward path numbers get smaller, it must eventually reach 1 (conclusion (5)), it shows that there must be a reverse tracing path between 1 and n. From this, for n both inverse and forward operations tend to infinity. However, the inverse operations are just in the opposite direction based on the same kind of operational rules of the forward directions, there has only one direction, therefore, the assumption above doesn't hold, and the odd number n must not only be in the range of the odd numbers obtained by tracing, but it must also be regressed to 1 if doing forward operations.
In facts, for any odd missed, it also exists always in a similar number set either as a generating number (the first) or not. For the first, it will arrive at the primitive number at the bottom in a source number set when doing forward operations for it; for the other, it is equivalent to the first when to do one forward operation and it will also arrive at the primitive number. Keep doing forward operations, it also go back to 1 in the end (see the step-type path structure in Fig. 1). So any odd is connected to 1 and that's the operational mechanism of all of the odds going back to 1.
Thus, it shows that any odd number can be obtained by successive reverse tracing stating from 1, that is, corresponding to the natural number axis, the density of the odd numbers must be 1/2 (1 is also included as the starting point).
From this analysis, it can be concluded (conclusion (13)): finally, for any positive integer (an even number is transformed into an odd firstly), to do forward operations, it must follow the inverse paths analyzed in this paper and return to 1. So, the Collatz conjecture holds.
In this paper, the basic operational principle of the conjecture is expounded.

Acknowledgments
I would like to express my thanks to Liansheng Zhang for encouragement and help, who is a professor in Suzhou university of Science and Technology. I also thank the reviewers for comments.

Conflicts of Interest
The author declares no conflicts of interest regarding the publication of this paper.

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