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No Odd Perfect Numbers Please!

Revised Version 4.0

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Dedicated to

My Wife Elisabetta

My Daughter Caterina

My Grand-Daughter Anna Isabel

Summary

In this article, we solve one of the oldest and most celebrated problems in number theory, namely the existence or nonexistence of odd perfect numbers. We know there is no number of this type having less than 100 digits. A number is said to be perfect if it is the sum of its proper divisors. Euclid in his *The Elements* ninth book gives a formula for all even perfect numbers. We answer the question of whether there exists an odd perfect number in the negative by proving a theorem asserting that the existence of such a number would lead to contradictions (proof by *reductio ad absurdum*). Somewhat remarkably, perhaps, this result is proved using only elementary methods. Hence, the popular conjecture that odd perfect numbers do not exist, no matter how large these numbers might be, is confirmed to be correct. Thus, one of the oldest and most celebrated questions in mathematics has now a definitive answer.

Keywords: ***Number Theory; Perfect Numbers; Euler's Sigma-Function; Mersenne's Primes***.

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1. Introduction

In this article, we solve one of the oldest and most celebrated problems in number theory, namely the existence or nonexistence of odd perfect numbers. It is not known whether any odd perfect number exists. We know only that none exists that are less than 10^{36}, a number having no less than 100 digits (this result was proven in 1967; see Guy [3], p. 66). A number is said to be perfect if it is the sum of its proper divisors. The theory of perfect even numbers is well known. Euclid in his *The Elements* ninth book gives a formula for all even perfect numbers. He proved that if $(2^p - 1)$ is prime, then $2^{p-1}(2^p - 1)$ is an even perfect number. The first four perfect (even) numbers – 6, 28, 496, and 8128 – were known to Euclid. Several centuries later Leonard Euler proved that every one of them is of this type (see Voigt [7]). Perfect numbers have seen a great deal of attention, ranging from very ancient numerology. The Pythagoreans equated the perfect number 6 to marriage, health, and beauty on account of the integrity and agreement of its parts (see Voigt [8]). Saint Augustine (among others, including the early Hebrews) considered 6 to be an ideal perfect number, since God fashioned the Earth in precisely these many days. Significantly, they were also important to the seventeenth century great mathematicians, such as Renée Descartes and Pierre de Fermat, whose investigations led the latter to the (little) theorem that bears his name. Such theorem states that if p is a prime number and n a positive integer then p is a divisor of $(n^p - n)$. In 1747 Leonard Euler showed that every even perfect number arises from an application of Euclid's rule. Primes of the form $(2^p - 1)$, defining the Euclid's rule for constructing even perfect numbers, are called Mersenne primes. Up to now, 51 have been found as part of the Great Internet Mersenne Prime Search (GIMPS; see http://www.mersenne.org/). Despite its ancient roots the subject of perfect numbers remains very much alive today, harbouring perhaps the "oldest unfinished project of mathematics" (c.f. Stan [6]). It is not known whether there exist infinitely many Mersenne primes and therefore we do not know whether there exist infinitely many (even) perfect numbers. Similarly, we do not know if near-perfect numbers - the sum of all proper divisors of a natural number N, except for one of them (c.f. Pollak and Shevelev [4]) – have an upper bound or not. As we have pointed out, equally mysterious, up to now, remains the question of whether there are any odd perfect numbers[2].

In this paper we answer the odd perfect numbers existence or nonexistence question by giving a proof of their nonexistence. Perhaps somewhat remarkably this result is proved using only elementary methods. In essence, our proof proceeds by contradiction showing that if one assumes that odd perfect numbers existed such assumption would lead to an absurd statement for the value of the Euler's *Sigma-Function* for odd perfect number. This should imply that their existence is logically impossible. Hence the popular conjecture that no odd perfect number is to be expected to exist, no matter how large such number might be, is confirmed to be correct. Thus, the Euclid's and Euler's results provide a complete characterisation of perfect numbers. However, there is still an open problem: whether the

[2] For a survey of the open problems concerning perfect numbers the Reader is referred to Guy [3], Ch. 4, or Sandor and Crstici [5], Ch. 1.

set of even perfect primes is finite or infinite, namely whether there are finitely or infinitely many Mersenne primes.

2. Odd Perfect Numbers Nonexistence Proof

Let N be a positive integer. Following the number theory literature, N is said (in increasing order of generality) to be perfect when $\sigma(N) = 2N$,

Definition 1. Leonard Euler introduced the concept of *Sigma-Function*, $\sigma(N)$, which sums the (positive) natural divisors of an integer N,

$$\sigma(N) \equiv \sum_{(d|N)} d \; ; \; d \cdot k = N; \; k, d \in \mathbb{N} \equiv \{0,1,2,3, \dots ,\}, d > 0, N \in \mathbb{N}, N > 0 \tag{1}$$

Where $(d|N)$ means the integer d divides N and runs over the positive divisors of N, including 1 and N itself; k is a positive integer solution and $\mathbb{N}$ denotes the set of natural numbers including zero. For example, $\sigma(11) = 1 + 11 = 12$ and $\sigma(15) = 1 + 3 + 5 + 15 = 24$.

■

The central reason for using the function $\sigma(N)$ is that it possesses some special properties. Among them $\sigma(M \cdot N) = \sigma(M) \cdot \sigma(N)$ whenever M and N are coprime (or relatively prime) numbers, namely their Greatest Common Divisor is equal to 1; $GCD(M, N) = 1$. Hence σ is completely determined when its value is known for every prime-power argument. This yields the following useful statement for the sum-of-divisors of N as,

Lemma 1 (see [7], th.4):

$$\sigma(N) = \prod_{i=1}^{m} \delta_i = \prod_{i=1}^{m} \frac{(p_i)^{\alpha_i+1} - 1}{p_i - 1}$$
$$\delta_i \equiv \sigma[(p_i)^{\alpha_i}] = 1 + (p_i)^1 + (p_i)^2 + \cdots + (p_i)^{\alpha_i}, p_i \in \mathbb{p}, \alpha_i \in \mathbb{N}, i = 1, m \tag{2}$$

where m is the number of prime factors decomposing N, $\mathbb{p}$ denotes the set of all prime numbers, $\{p_i\}$ is the set of prime divisors of N and α_i is the exponent of the highest power of the prime number p_i that divides N,

$$N = \prod_{i=1}^{m} (p_i)^{\alpha_i}, p_i \in \mathbb{p}, \alpha_i \in \mathbb{N}, i = 1, m, m \in \mathbb{N}, m > 0 \tag{3}$$

For example,

$$\sigma(11) = 1 + 11 = 12; \; \sigma(15) = 1 + 3 + 5 + 3 \cdot 5 = 24$$

$$N \equiv 11 = 1 \cdot 11\,, \alpha = 1, m = 2; N \equiv 15 = 1 \cdot 3 \cdot 5, \alpha = 1, m = 3 \tag{4}$$

■

Remark 1: $\sigma(N)$ can be either an even or an odd number. However, if $p_i = 2$, and there therefore N must be an even number (eq. 3), the term δ_i in eq. (2) is a sum of powers of 2 – which is an even number - plus 1. As a result, δ_i should be odd (see Appendix, addition rules (a)-(i) and (a)-(iii)).

For $p_i \geq 3$, δ_i should be an odd (resp. even) number when α_i is even (resp. odd) number. This latter implication can be obtained by adding recursively the powers of p_i, which are all odd numbers, to the initial even number $(1 + p_i)$ which makes it for an alternating sequence of odd-even integers, (*Even-Number* ⊕ *Odd-Number* ≙ *Odd-Number* and *Odd-Number* ⊕ *Odd-Number* ≙ *Even-Number*; see Appendix, Addition Rules (a)-(ii) and (a)-(iii)). Hence, $\sigma(N)$ is even if and only if there exists at least an exponent α_i of an odd prime p_i being odd. In fact, only one odd exponent α_i is sufficient to get an even number δ_i for $p_i \geq 3$, making $\sigma(N)$ necessarily an even number (see Appendix, Multiplication Rule (b)-(vi)).

Thus, $\sigma(N)$ being even is a necessary condition for a number N to be perfect so that the equation, $\sigma(N) = 2N$, becomes feasible, since $2N$ is an even number by construction. On the contrary, if $\sigma(N)$ were an odd number $\sigma(N) = 2N$ is ruled out. Moreover, $N = 2^{\alpha-1}$ can never be a perfect (even) number in that $\sigma(2^{\alpha-1}) = (2^{\alpha} - 1)$ is always an odd number. Hence, it should not come as surprise that any even perfect number should have a prime factor in it such as – the Mersenne's prime - $(2^{\alpha} - 1)$, with α being a prime, so that $N = 2^{\alpha-1}(2^{\alpha} - 1)$ and $\sigma[(2^{\alpha-1})(2^{\alpha} - 1)] = \sigma[2^{\alpha-1}]\sigma[(2^{\alpha} - 1)] = (2^{\alpha} - 1)2^{\alpha} = 2N$ with $(2^{\alpha} - 1)2^{\alpha}$ being always an even number and with $2^{\alpha-1}$ and $(2^{\alpha} - 1)$ being coprime, that is the only positive integer that is a divisor of both is 1.

■

The following elementary statement regarding odd numbers factor decomposition is useful as well,

Lemma 2: an odd number N_O is decomposed by odd prime factors only (denoted p_i^O),

$$N_O = \prod_{i=1}^{m} \left(p_i^O\right)^{\alpha_i}, \alpha_i \in \mathbb{N}, \alpha_i \geq 1, p_i^O \in \mathbb{p},\ p_i^O \geq 3, i = 1, m, N_O \in \mathbb{N}, N_O \geq 3 \tag{5}$$

Where $\mathbb{p}$ denotes the set of prime numbers.

Proof: we proceed by contradiction. Suppose that an odd number N_O were to include the integer 2 among its prime *m* factors,

$$N_O = (2)^{\alpha_1} \prod_{i=2}^{m} \left(p_i^O\right)^{\alpha_i}, \alpha_1 \geq 1$$

(6)

As the right-hand side product contains an even number, $(2)^{\alpha_1}$, by virtue of the multiplication rule b-(vi) (see Appendix), it should be an even number as well, thus contradicting the assumption that N_O is an odd number as claimed.

■

We now turn to the main goal of this paper by focusing on the issue of odd prime numbers nonexistence. We start with the formal definition of perfect number,

<u>Definition</u> 2: A number is perfect if its divisors add up to twice the number itself. Thus, if *N* is a perfect number, it must be the case that,

$$\sigma(N) = 2N, N \in \mathbb{N}, N > 0$$

(7)

Here is an example of a perfect (even) number,

$$\sigma(28) = 1 + 2 + 2^2 + 7 + 2 \cdot 7 + 28 = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \cdot 28$$
$$N \equiv 28 = 2^2 \cdot 7$$

(8)

Notice that example (8) implies that in eq. (4) we should have,

$$m = 2, \alpha_1 = 2; \alpha_2 = 1, p_1 = 2, p_2 = 7$$

(9)

■

One of the oldest (unsolved) problem in number theory is whether there exist an odd perfect number, N_O^*, which would yield,

$$\sigma(N_O^*) = 2N_O^*, N_O^* \in \mathbb{N}$$

(10)

We answer the question whether there exist such odd perfect number – e.g., eq. (10) holds for some odd natural number - in the negative by proving the following,

<u>Theorem 2</u>: No perfect number can be odd. Hence, eq. (10) cannot hold and therefore the Euler *Sigma-Function*, $\sigma(N_O)$, of any odd number, N_O, is never equal to twice its value,

$$\sigma(N_O) \neq 2N_O, \forall N_O \in \mathbb{N}, N_O \geq 1$$

(11)

<u>Proof</u>: we proceed in several steps by contradiction.

Let us consider a generic odd (non-prime) number, N_O^*, which we assume to be perfect (recall that no prime number can be perfect). We posit that N_O^* can be decomposed as follows,

$$N_O^* = \prod_{i=1}^{m} \left(p_i^O\right)^{\alpha_i}, p_i^O \geq 3,\ \alpha_i \geq 1, i = 1, m, m \geq 1$$

(12)

where the product $\prod_{i=1}^{m}\left(p_i^O\right)^{\alpha_i}$ represents the prime factor decomposition of N_O^*. By virtue of ***Lemma 2*** all prime factors, $p_i^O, i = 1, m$, should be odd.

Applying the *Sigma-Function* on both side of (12), according to eq. (2) in ***Lemma 1*** we get,

$$\sigma(N_O^*) = \prod_{i=1}^{m} \left[1 + \left(p_i^O\right)^1 + \left(p_i^O\right)^2 + \left(p_i^O\right)^3 + \cdots + \left(p_i^O\right)^{\alpha_i}\right]$$

(13)

Let us assume that N_O^* is an odd perfect number implying that eq. (10) should hold. By virtue of (13) we should then have,

$$\prod_{i=1}^{m} \delta_i \equiv \prod_{i=1}^{m} \left[1 + \left(p_i^O\right)^1 + \left(p_i^O\right)^2 + \left(p_i^O\right)^3 + \cdots + \left(p_i^O\right)^{\alpha_i}\right] = 2N_O^*$$

(14)

It suffices to consider three possibilities for the terms of the product on the left-hand side of eq. (14):

1) All terms δ_i of the product are odd numbers, which can happen if (and only if) all exponents $\alpha_i, i = 1, m$ are even numbers, as argued in ***Remark 1***;

2) At least two terms of the product, say $\delta_1$and δ_2, are even numbers, which is the case if (and only if) their corresponding exponent, α_1 and α_2, is an odd integer (see again ***Remark 1***);

3) One term of the product, say δ_1, is even with its greatest exponent α_1 being odd.

The first case entails that the product of all δ_i should be an odd number as well (see Appendix, Multiplication Rule (b)-(v)), and therefore $\sigma(N_O^*)$, the left-hand side of eq. (14), being odd, which contradicts the fact that $2N_O^*$ on the right-hand side is an even number.

The second case requires more elaboration. Suppose that not all δ_i are odd numbers. Hence, let assume without loss of generality that the first term and the second term of the product, δ_1 and δ_2, are even (therefore α_1 and $\alpha_2$must be odd; see again ***Remark 1***),

$$\delta_1 = [1 + (p_1^O)^1 + (p_1^O)^2 + (p_1^O)^3 + \cdots + (p_1^O)^{\alpha_1}] \triangleq \text{Even} - \text{Number}\,; \alpha_1 \triangleq \text{Odd} - \text{Number}$$
$$\delta_2 = [1 + (p_2^O)^1 + (p_2^O)^2 + (p_3^O)^3 + \cdots + (p_3^O)^{\alpha_2}] \triangleq \text{Even} - \text{Number}\,; \alpha_2 \triangleq \text{Odd} - \text{Number}$$

(15)

Hence, we can decompose δ_1 and δ_2, which are even numbers as assumed in (15), as follows,

$$\delta_1 = 2^{\rho_1} \cdot k_{O,1} \geq 4;\ \rho_1 \geq 1, k_{O,1} \geq 1, p_1^O \geq 3,\ \alpha_1 \geq 1$$
$$\delta_2 = 2^{\rho_2} \cdot k_{O,2} \geq 4;\ \rho_2 \geq 1, k_{O,2} \geq 1, p_2^O \geq 3,\ \alpha_2 \geq 1$$

(16)

with $k_{O,1}$ and $k_{O,2}$ being odd numbers. Notice that, $\delta_j = 1 + p_j^O = 4, j = 1,2$ is the minimum value in eq. (16) since $\alpha_j = 1$ and $p_j^O = 3, j = 1,2$ are the lowest possible values for the exponents, α_1 and α_2, and their factor primes, p_1^O and p_2^O. We can substitute the right-hand side of (16) in the left-hand side of (14), replacing the first and second term in the product $\prod_{i=1}^{m} \delta_i$ with $2^{\rho_1} \cdot k_{O,1}$ and $2^{\rho_2} \cdot k_{O,2}$ to get,

$$\left(2^{\rho_1} \cdot k_{O,1}\right)\left(2^{\rho_2} \cdot k_{O,2}\right) \prod_{i=3}^{m} \delta_i = 2N_O^*, \rho_1 \geq 1, \rho_2 \geq 1, m \geq 2$$

(17)

Dividing both sides by 2 we end up with

$$\left(2^{\rho_1 - 1} \cdot k_{O,1}\right)\left(2^{\rho_2} \cdot k_{O,1}\right) \prod_{i=3}^{m} \delta_i = N_O^*, \rho_1 \geq 1, \rho_2 \geq 1$$

(18)

Thus eq. (18) turns out to be a contradiction, in that 2^{ρ_2}, with $\rho_2 \geq 1$, must be even and thereby, as a result of Multiplication Rule (b)-(vi), its left-hand side must be an even number as well. Since the right-hand side, N_O^*, is (by definition) an odd number our claim is proved.

The third case deals with the assumption of only one term of the product $\prod_{i=1}^{m} \delta_i$, say δ_1, being even with the following decomposition assumed to hold,

$$\delta_1 = 2^{\rho_1} \cdot k_{O,1} \geq 4;\ \rho_1 \geq 1, k_{O,1} \geq 1, p_1^O \geq 3,\ \alpha_1 \geq 1$$

(19)

It is convenient to use decomposition (19) by distinguishing the following two cases,

$$\rho_1 > 1$$

(20a)

and,

$$\rho_1 = 1$$

(20b)

Let us consider the case (20a). Again, we can substitute the right-hand side of (19) in the left-hand side of (14), replacing the first term in the product $\prod_{i=1}^{m} \delta_i$ with its value $2^{\rho_1} \cdot k_{O,1}$, then dividing both side by 2 to get,

$$\left(2^{\rho_1 - 1} \cdot k_{O,1}\right) \prod_{i=2}^{m} \delta_i = N_O^*, \rho_1 > 1$$

(21)

which should yield a contradiction in that the left-hand side turns out to be even, since $2^{\rho_1 - 1}, \rho_1 > 1$ is certainly even, whereas the right-hand side, N_O^* , is (by definition) an odd number.

Let us consider the case (20b) which entails in (19) that,

$$\delta_1 = 2 \cdot k_{O,1} \geq 6;\ k_{O,1} \geq 3,$$

(22)

Substituting (22) into (16) and dividing by 2 we get

$$k_{O,1} \prod_{i=2}^{m} \delta_i = N_O^*$$

(23)

with

$$k_{O,1} \equiv \frac{1}{2} \delta_1$$

(23a)

Recalling the prime factor decomposition (12) for N_O^* we can write (23) as,

$$\frac{1}{2} \prod_{i=1}^{m} \delta_i = \prod_{i=1}^{m} \left(p_i^O\right)^{\alpha_i}$$

(24)

which is reduced to a more compact form by simple manipulation,

$$\prod_{i=1}^{m} \left[\frac{\delta_i}{\left(p_i^O\right)^{\alpha_i}}\right] = 2$$

(25)

As a result of ***Lemma 1*** each fraction in (25) turns out to be equal to

$$\frac{\delta_i}{(p_i^O)^{\alpha_i}} = \frac{(p_i^O)^{\alpha_i+1} - 1}{(p_i^O)^{\alpha_i}(p_i^O - 1)}, \forall\, i = 1, m$$

(25a)

Inserting (25a) in (25) we get,

$$\prod_{i=1}^{m}\left[\frac{(p_i^O)^{\alpha_i+1} - 1}{(p_i^O)^{\alpha_i}(p_i^O - 1)}\right] = 2$$

(26)

We decompose the left-hand side of (26) by separating the last fraction indexed by m,

$$\frac{P_{(-m)}^1}{P_{(-m)}^2}\frac{(p_m^O)^{\alpha_m+1} - 1}{(p_m^O)^{\alpha_m}(p_m^O - 1)} = 2$$

(27)

With

$$P_{(-m)}^1 \equiv \prod_{i=1}^{m-1}\left[(p_i^O)^{\alpha_i+1} - 1\right]; P_{(-m)}^2 \equiv \prod_{i=1}^{m-1}\left[(p_i^O)^{\alpha_i}(p_i^O - 1)\right], P_{(-m)}^1 > 1, P_{(-m)}^2 > 1$$

(28)

We rearrange eq. (27) so that we can get - as we will argue for eq. (32) below - a canonical linear Diophantine equation,

$$P_{(-m)}^1 (p_m^O)^{\alpha_m} p_m^O - 2P_{(-m)}^2 (p_m^O - 1)(p_m^O)^{\alpha_m} = P_{(-m)}^1$$

(29)

We want to prove that (29) cannot have integer solutions so that we obtain a contradiction. In this way we can argue that (26), and thereby (25), also do not have natural number solutions. As a result, we can claim that (14) does not hold either, thus completing our proof that N_O^* cannot be an odd perfect number, in that, if it were, it would lead to a contradiction.

It is evident that left-hand side of (29) is divisible by $(p_m^O)^{\alpha_m}$, therefore by dividing through both sides of (29) we obtain

$$P_{(-m)}^1 p_m^O - 2P_{(-m)}^2 (p_m^O - 1) = \frac{P_{(-m)}^1}{(p_m^O)^{\alpha_m}}$$

(30)

Hence $P_{(-m)}^1$, which is a positive integer, has to be divisible by $(p_m^O)^{\alpha_m}$ in that the left-hand-side of eq. (30) is an integer number, denoted k_m (see below),

$$\frac{P^1_{(-m)}}{(p^O_m)^{\alpha_m}} \equiv \frac{\prod_{i=1}^{m-1}\left[\left(p^O_i\right)^{\alpha_i+1}-1\right]}{(p^O_m)^{\alpha_m}} = k_m,\ k_m \in \mathbb{N},\ k_m \geq 1$$

(31)

Recall that eq. (30) has to have an integral solution in p^O_m, which implies that its left-hand side should be an integer. To match this latter, k_m should be an integer as well. Moreover, it must be positive in that the quotient $\frac{P^1_{(-m)}}{p^O_m}$ is positive by construction.

We shall treat eq. (30) as a linear Diophantine equation, with unknown $[x,y] = [p^O_m,(1-p^O_m)]$ and integral coefficients $[a,b] = \left[P^1_{(-m)}, 2P^2_{(-m)}\right]$, positing the following expression,

$$ax + by = k_m\ , a \in \mathbb{N}, b \in \mathbb{N}, [a,b] > [0,0]$$

(32)

We know that (32) has a solution if only if (cf. Andrews, [1], p. 44, th.2-4),

$$h_m \equiv GCD(a,b) |\ k_m, h_m \in \mathbb{N}, h_m \geq 1$$

(32a)

Namely, the greatest common divisor of $[a,b]$, h_m , also divides k_m. If this is the case, we can find a particular solution $[x_0, y_0]$, both integer numbers (at least one of them non-zero; cf. Andrews [1], p. 42 and example 2-8, p. 34), such that

$$ax_0 + by_0 = h_m, [x_0, y_0] \neq [0,0]$$

(32b)

Recall that we do not restrict $[x_0, y_0]$ to be non-negative integers; however, we know that x_0 must be positive whereas y_0 has to negative by virtue of their definition. Moreover, they can be found by the Euclidean algorithm (cf. Courant and Robbins [2], 1948, pp. 42-46 and p. 51).

Next step of the algorithm requires to find an integer g_m such that

$$k_m = h_m g_m, g_m \in \mathbb{N}, g_m \geq 1$$

(32c)

We let

$$x = x_0 g_m, y = y_0 g_m$$

(32d)

Clearly, given (32c) it turns out that (32d) is also a solution of eq. (32). We know that solution (32d) is constrained by the following restriction,

$$x + y = p^O_m + (1 - p^O_m) = 1$$

(33)

Hence substituting (32d) into (33) we get,

$$(x_0 + y_0)g_m = 1$$

(33a)

which entails that both (integer) terms on the left-hand side of (33a) should be equal to 1,

$$(x_0 + y_0) = 1, g_m = 1$$

(34)

Since (34) dictates that $g_m = 1$, (32c) implies that $k_m = h_m$, therefore k_m must be a divisor – actually the greatest common divisor – of $[P^1_{(-m)}, 2P^2_{(-m)}]$. Thus, we should have for the first coefficient, $P^1_{(-m)}$,

$$\frac{P^1_{(-m)}}{k_m} = (p^O_m)^{\alpha_m} = l_m, l_m \in \mathbb{N}, l_m \geq 3$$

(35)

with l_m being a positive integer (greater or equal 3). Similarly, the second coefficient, $2P^2_{(-m)}$, should also be divisible by k_m, namely

$$\frac{2P^2_{(-m)}}{k_m} = \frac{2P^2_{(-m)}(p^O_m)^{\alpha_m}}{P^1_{(-m)}} = r_m,\ r_m \in \mathbb{N}, r_m \geq l_m$$

(36)

with r_m being a positive integer which should be greater than (or equal to) $l_m = (p^O_m)^{\alpha_m}$ (see eq. 35) in that the quotient $\frac{2P^2_{(-m)}}{P^1_{(-m)}}$ has to be a positive integer as well (see below).

Clearly, (36) could hold – namely r_m be apositive integer - if and only if at least one term of the product in the numerator – either $2P^2_{(-m)}$ or $(p^O_m)^{\alpha_m}$ - is divisible by $P^1_{(-m)}$ (cf. Courant and Robbins [2], Lemma, p. 47). As a result, we must have that either

$$\frac{2P^2_{(-m)}}{P^1_{(-m)}} = v_m,\ v_m \in \mathbb{N},\ v_m \geq 1$$

(37)

with v_m, being a (positive) integer or,

$$\frac{(p^O_m)^{\alpha_m}}{P^1_{(-m)}} = u_m,\ u_m \in \mathbb{N}, u_m \geq 1$$

(38)

with $u_m$being a (positive) integer - or both - must hold.

While (35) evidently confirms that k_m is a proper divisor of $P^1_{(-m)}$, in that $l_m = (p^O_m)^{\alpha_m}$ is indeed a positive integer, we show, on the contrary, that k_m is not a proper divisor of $2P^2_{(-m)}$, namely neither (37) nor (38) should hold. Thus, $P^1_{(-m)}$ is not a proper divisor of neither $2P^2_{(-m)}$ nor $(p^O_m)^{\alpha_m}$ and thereby eq. (36) is contradicted, namely, r_m is not an integer. We prove both claims in the Appendix; in particular, $P^1_{(-m)}$ not a divisor of $2P^2_{(-m)}$ (cf. **Lemma 2** and **Remark 2**), as well as not a divisor of $(p^O_m)^{\alpha_m}$ (cf. **Lemma 3**).

Since eq. (36) is contradicted (r_m not an integer), it turns out that eq. (32) does not have a solution and thereby eq. (29) also does not have an integer solution. Hence, we obtained a contradiction. By the same token, eqs. (26)-(27) and, by implication, eq. (14) does not hold either, thereby completing our proof that eq. (10) does not have an integer solution. As a result, N^*_O cannot be a perfect odd number, in that inequality (11) should always hold. Thus, to summarise, no twice odd number can be equal to its *Sigma-Function* value and thereby no odd number can be a perfect one.

■

To conclude, we can argue that one of the oldest and most celebrated questions in mathematics has now a definitive answer: odd perfect numbers do not exist.

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Appendix

For Reader's convenience we summarise summation and multiplication rules involving even and odd numbers which are used, explicitly or implicitly, in the main text.

Even and odd numbers, respectively, are denoted below as

$$a(n) = 2n, b(n) = 2n + 1, n \in \mathbb{N}$$

(1A)

(a) Summation Rules for Natural Numbers (⊕ denoting "addition"):

(i) *Even-Number ⊕ Even-Number ≙ Even-Number*

$$a(n_1) + a(n_2) = 2(n_1 + n_2) = a(n_1 + n_2), \qquad n_1, n_2, (n_1 + n_2) \in \mathbb{N}$$

(ii) *Odd-Number ⊕ Odd-Number ≙ Even-Number*

$$b(n_1) + b(n_2) = 2[(n_1 + n_2) + 1] = a(n_1 + n_2 + 1), \qquad n_1, n_2, (n_1 + n_2 + 1) \in \mathbb{N}$$

(iii) *Even-Number ⊕ Odd-Number ≙ Odd-Number*

$$a(n_1) + b(n_2) = 2(n_1 + n_2) + 1 = b(n_1 + n_2), \qquad n_1, n_2, (n_1 + n_2) \in \mathbb{N}$$

(b) Multiplication Rules for Natural Numbers (⊗ denoting "multiplication"):

(iv) *Even-Number ⊗ Even-Number ≙ Even-Number*

$$a(n_1) \cdot a(n_2) = 2 \cdot [2(n_1 \cdot n_2)] = a[a(n_1 \cdot n_2)], n_1, n_2, (n_1 \cdot n_2) \in \mathbb{N}$$

(v) *Odd-Number ⊗ Odd-Number ≙ Odd-Number*

$$b(n_1) \cdot b(n_2) = 2[n_1 + n_2 + 2n_1 n_2] + 1 = b(n_1 + n_2 + 2n_1 n_2),$$
$$n_1, n_2, (n_1 + n_2 + 2n_1 n_2), \in \mathbb{N}$$

(vi) *Even-Number ⊗ Odd-Number ≙ Even-Number*

$$a(n_1) \cdot b(n_2) = 2[n_1 \cdot n_2 + n_1] = a(n_1 \cdot n_2 + n_1), n_1, n_2, (n_1 \cdot n_2 + n_1) \in \mathbb{N}$$

Lemma 2: if $2P^2_{(-m)} > P^1_{(-m)}$, then $P^1_{(-m)}$ not a divisor of $2P^2_{(-m)}$.

Proof: we can assert that

$$P^1_{(-m)} \equiv \prod_{i=1}^{m-1} \left[\left(p_i^O\right)^{\alpha_i + 1} - 1 \right] > P^2_{(-m)} \equiv \prod_{i=1}^{m-1} \left[\left(p_i^O\right)^{\alpha_i + 1} - \left(p_i^O\right)^{\alpha_i} \right]$$

(2A)

Since it is evident that each term of the product in $P^1_{(-m)}$ is greater than its corresponding term in $P^2_{(-m)}$, namely

$$\left(p_i^O\right)^{\alpha_i+1} - 1 > \left(p_i^O\right)^{\alpha_i+1} - \left(p_i^O\right)^{\alpha_i} > 0, \forall\, i = 1, m-1$$

(3A)

recalling that

$$\left(p_i^O\right)^{\alpha_i} > 1\,, p_i^O > 1, \forall\, i = 1, m-1$$

(4A)

then all terms in (3A) is (strictly) positive.

Dividing both side of inequality (2A) by $P^1_{(-m)}$ we get,

$$1 > \frac{P^2_{(-m)}}{P^1_{(-m)}}\,, \qquad P^1_{(-m)} > 0, P^2_{(-m)} > 0,$$

(5A)

and multiplying by 2 both sides of (5A) yields,

$$2 > \frac{2P^2_{(-m)}}{P^1_{(-m)}}$$

(6A)

Since we assumed that $\frac{2P^2_{(-m)}}{P^1_{(-m)}} > 1$, inequality (6A) implies that $P^1_{(-m)}$ is not a divisor of $2P^2_{(-m)}$.

■

Remark 2: it is evident that **Lemma 2** does not encompass the special case in which the quotient $\frac{2P^2_{(-m)}}{P^1_{(-m)}}$ is unity. Hence, to complete the proof that $P^1_{(-m)}$ is never a divisor of $2P^2_{(-m)}$ we need to rule out the case

$$\frac{2P^2_{(-m)}}{P^1_{(-m)}} = 1$$

(7A)

namely,

$$P^1_{(-m)} = 2P^2_{(-m)}$$

(8A)

However, it can be shown that (8A) would contradict eq. (27) (which for convenience is recalled below),

$$P^1_{(-m)} \frac{(p^O_m)^{\alpha_m+1} - 1}{(p^O_m)^{\alpha_m}(p^O_m - 1)} = 2P^2_{(-m)}$$

(9A)

Since the quotient in (9A) is greater than 1

$$\frac{(p^O_m)^{\alpha_m+1} - 1}{(p^O_m)^{\alpha_m}(p^O_m - 1)} = \frac{(p^O_m)^{\alpha_m+1} - 1}{(p^O_m)^{\alpha_m+1} - (p^O_m)^{\alpha_m}} > 1$$

(10A)

as $(p^O_m)^{\alpha_m} > 1$, it must be the case that

$$P^1_{(-m)} \frac{(p^O_m)^{\alpha_m+1} - 1}{(p^O_m)^{\alpha_m}(p^O_m - 1)} > 2P^2_{(-m)}$$

(11A)

if eq. (8A) were to hold. Clearly, (11A) contradicts (9A). But, if eq. (9A) - eq. (27) for that matter - does not hold, then eq. (14) cannot hold either. And if eq. (27) does not hold, our claim that eq. (10) does not have a solution immediately follows and thereby completing our proof that no odd perfect number should exist. However, if eq. (9A) is assumed to hold, inequality (10A) evidently would imply that

$$P^1_{(-m)} < 2P^2_{(-m)}$$

(12A)

If inequality (12A) holds, eq. (27) (or 9A for that matter) again cannot hold, in that $P^1_{(-m)}$ is never a divisor of $2P^2_{(-m)}$. In fact, as a result of (11A) and Lemma 2, we must have,

$$1 < \frac{2P^2_{(-m)}}{P^1_{(-m)}} < 2$$

(13A)

It is worth pointing out that that set of eqs. (7A)-(13A) do not depend upon the value of m, namely the number of (odd) prime factors decomposing N^*_O.

To summarise, assuming that eq. (27) has a solution would lead to a contradiction. Thus eq. (27) cannot hold.

■

Lemma 3: $P^1_{(-m)}$ is not a divisor of $(p^O_m)^{\alpha_m}$.

Proof: We rule out at the outset the trivial case,

$$P^1_{(-m)} > (p^O_m)^{\alpha_m}$$

(14A)

which entails that

$$0 < \frac{(p_m^O)^{\alpha_m}}{P_{(-m)}^1} < 1$$

(15A)

thereby implying that $P_{(-m)}^1$ cannot be a divisor of $(p_m^O)^{\alpha_m}$. More interesting is the exploration of the alternative case,

$$P_{(-m)}^1 \leq (p_m^O)^{\alpha_m}$$

(16A)

We proceed by contradiction in assuming that $(p_m^O)^{\alpha_m}$ is divisible by $P_{(-m)}^1$,

$$\frac{(p_m^O)^{\alpha_m}}{P_{(-m)}^1} = \frac{\overbrace{p_m^O \cdot p_m^O \cdot \ldots \cdot p_m^O}^{\alpha_m}}{\prod_{i=1}^{m-1}\left[\left(p_i^O\right)^{\alpha_i+1} - 1\right],} = z_m, \in \mathbb{N}, z_m \geq 1$$

(17A)

We know that $z_m$can be a (positive) integer if and only if at least one term of the product in the denominator of (17A) should be a divisor of p_m^O (cf., Courant and Robbins, 1948, p. 47, Lemma). If no such term exists our claim is proved.

Let suppose that such a term, say $\left[\left(p_i^O\right)^{\alpha_i+1} - 1\right]$, exists, namely

$$\frac{p_m^O}{\left(p_i^O\right)^{\alpha_i+1} - 1} = q_i, \in \mathbb{N}, q_i \geq 1, i \in \{1,2,\ldots,m-1\}$$

(18A)

Multiplying through by $\left[\left(p_i^O\right)^{\alpha_i+1} - 1\right]$ we get,

$$p_m^O = q_i\left[\left(p_i^O\right)^{\alpha_i+1} - 1\right]$$

(19A)

Dividing by q_i and re-arranging the terms in (19A),

$$\left(p_i^O\right)^{\alpha_i+1} = 1 + \frac{p_m^O}{q_i}$$

(20A)

Clearly, the right-hand-side of (20A) has to be a (positive) integer greater than 2 and therefore the ratio

$$\frac{p_m^O}{q_i}$$

(21A)

should be an integer not less than 2. Recall that p_m^O is a prime and therefore it is divisible only by 1 or p_m^O. Hence, in order to ensure that the fraction (21A) is a (positive) integer we must have, either

$$q_i = 1$$

(22A)

or

$$q_i = p_m^O$$

(23A)

Substituting (22A) and (23A) in (20A) yields

$$\left(p_i^O\right)^{\alpha_i+1} = 1 + p_m^O$$

(24A)

and

$$\left(p_i^O\right)^{\alpha_i+1} = 1 + 1 = 2$$

(25A)

Clearly, (25A) cannot hold in that,

$$\left(p_i^O\right)^{\alpha_i+1} \geq 3$$

(26A)

Hence, (23A) should be discarded.

We are left dealing with (22A) and therefore we proceed by inspecting (24A) whose left-hand-side can be decomposed as follows,

$$\overbrace{p_i^O \cdot p_i^O \cdot \ldots \cdot p_i^O}^{\alpha_i+1} = 1 + p_m^O$$

(27A)

Without loss of generality, we assume that,

$$p_1^O < p_2^O < \cdots < p_{m-1}^O < p_m^O$$

(28A)

Hence, it must be the case that

$$\frac{p_i^O}{1 + p_m^O} < \frac{p_i^O}{p_m^O} < 1, \forall i = 1, m-1$$

(29A)

and therefore $1 + p_m^O$ is not a divisor of any $p_i^O, i = 1. m - 1$. As a result, (27A), and (24A) for that matter, cannot hold and thereby (18A) cannot hold either, namely none of the terms $\left[\left(p_i^O\right)^{\alpha_i+1} - 1\right]$ can be a divisor of p_m^O. Thus (18A) gives rise to a contradiction in that z_m cannot be an integer and thereby ruling out that $(p_m^O)^{\alpha_m}$ is divisible by $P_{(-m)}^1$. Thus, our claim is proved.

■