On N-Dimensional Numbers

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Abstract

A number should have a dimension. We can think of a real number as one dimensional number, we can think of a complex number as a two dimensional number. A dimension of a number should be n, where n is any positive integer of our choice. I will introduce algebra of n dimensional numbers for any n positive integer.

1 N-Dimensional Number

Let n be a positive integer. Lets define n dimensional number as:

$$z = \sum_{k=0}^{n-1} a_k i_k = a_0 i_0 + a_1 i_1 + ... + a_{n-1} i_{n-1} \tag{1}$$

where a_k is a real number for every $k = 0, 1, ..., n-1$ and i_k is an imaginary unit of the k-th dimension.

For example, a two dimensional number is then

$$z = \sum_{k=0}^{1} a_k i_k = a_0 i_0 + a_1 i_1 = a_0 + a_1 i_1 \tag{2}$$

a three dimensional number is:

$$z = \sum_{k=0}^{2} a_k i_k = a_0 i_0 + a_1 i_1 + a_2 i_2 = a_0 + a_1 i_1 + a_2 i_2 \tag{3}$$

2 Imaginary Units of the n-th Dimension

Lets define it as i_k for every non-negative integer $k \leq n-1$ such that $i_0 = 1$ and $i_k^2 = -1$ for every positive integer $k \leq n-1$ in a way that the following multiplication rules for imaginary units corresponding to the n-th dimension numbers are used:

$$i_k i_l = (-1)^{\lfloor \frac{k+l-1}{n} \rfloor} i_{k+l \pmod n} \tag{4}$$

for every $0 \leq k < l \leq n-1$, where both k, l are non-negative integers.

For $k = l \geq 0$ we have already mentioned $i_0 = 1$ and $i_k^2 = -1$ for every positive integer k.

For $k \geq l$ we have $i_k i_l = -i_l i_k$, in other words we can say that multiplication of two different imaginary units in the n-th dimension has the anticommucative property, similarly like the in the concept of the Lie brackets.

3 Binary Operations involving Multidimensional Numbers

Using what we have now, we can easily define addition, substraction and multiplication of two n-dimensional numbers $z_1 = \sum_{k=0}^{n-1} a_k i_k$ and $z_2 = \sum_{k=0}^{n-1} b_k i_k$ in this way:

Addition of z_1 and z_2 is defined:

$$z_1 + z_2 = \sum_{k=0}^{n-1} (a_k + b_k) i_k \tag{5}$$

Substraction of z_1 and z_2 is defined:

$$z_1 - z_2 = \sum_{k=0}^{n-1} (a_k - b_k) i_k \tag{6}$$

Multiplication of z_1 and z_2 is defined:

$$z_1 * z_2 = \left(\sum_{k=0}^{n-1} a_k i_k\right)\left(\sum_{k=0}^{n-1} b_k i_k\right) = (a_0 i_0 + a_1 i_1 + ... + a_{n-1} i_{n-1})(b_0 i_0 + b_1 i_1 + ... + b_{n-1} i_{n-1}) \quad (7)$$

To expand this product, i.e. to get rid of the two pairs of brackets, we multiply every term in the first bracket by every term in the second bracket and we use:

$$a_k i_k * b_l i_l = (a_k b_l)(i_k i_l) \quad (8)$$

which we can simplify easily, since $a_k b_l$ is a product of two real numbers, i.e. a real number and a formula for $i_k i_l$ was already stated in this paper.

Multiplication of two n dimensional numbers is commutative for $n = 2$, for $n \geq 3$ it is not commutative in general for any two n dimensional numbers.

Multiplication of an k-th dimensional imaginary unit i_k by a real number c can be done from both left and right with the same result: $c * i_k = i_k * c$.

Multiplication of an n-th dimensional number z by a real number c can be done from both left and right with the same result:

$$c * z = z * c = \sum_{k=0}^{n-1} c a_k i_k = c a_0 i_0 + c a_1 i_1 + ... + c a_{n-1} i_{n-1} \quad (9)$$

Multiplication of an k-th dimensional imaginary unit i_k that is already multiplied by a real number c by a real number d is performed by multiplying the two real numbers c, d and lets create an agreement of treating this product cd as a coefficient of the i_k and thus writing the result of cd on the left according to the i_k like this: $(cd)i_k = h * i_k$ where $h = cd$.

It makes sense to define n-dimensional number conjugate of $z = \sum_{k=0}^{n-1} a_k i_k = a_0 i_0 + a_1 i_1 + ... + a_{n-1} i_{n-1}$ as:

$$\overline{z} = a_0 i_0 - \sum_{k=1}^{n-1} a_k i_k = a_0 i_0 - a_1 i_1 - ... - a_{n-1} i_{n-1} \quad (10)$$

so that $z * \overline{z} = \sum_{k=0}^{n-1} a_k^2 = a_0^2 + a_1^2 + ... + a_{n-1}^2$, however I have only verified that for $n = 2$ and $n = 3$.

4 N-Dimensional Imaginary Units Multiplication Tables

Here are multiplication tables for imaginary units in the n-th dimension for the first few values of $n \geq 3$. The content of these tables is derived using a formula for multiplication $i_k i_l$ for every $0 \leq k < l \leq n-1$, where both k, l are non-negative integers and for $k = l$ on the main diagonal of the following multiplication tables. In each of the following tables in the first column are located imaginary units i_k and in the first row imaginary units i_l. In the intersection of the particular row and and column there is a result of the product $i_k i_l$ in dimension n. Multiplication by i_0 is not considered, because $i_0 = 1$.

$n = 3$

	i_1	i_2
i_1	-1	1
i_2	-1	-1

(11)

$n = 4$

	i_1	i_2	i_3
i_1	-1	i_3	1
i_2	$-i_3$	-1	i_1
i_3	-1	$-i_1$	-1

(12)

$n = 5$

	i_1	i_2	i_3	i_4
i_1	-1	i_3	i_4	1
i_2	$-i_3$	-1	1	$-i_1$
i_3	$-i_4$	-1	-1	$-i_2$
i_4	-1	i_1	i_2	-1

(13)

5 Examples

Here are some examples of using this theory on problems:

example one: Find all 3-dimensional numbers satisfying $x^2 + x + 1 = 0$.

solution: Let $x = a + bi + cj$ (I am using i=i$_1$ and $j = i_2$). We have $(a+bi+cj)^2+(a+bi+cj)+1 = 0$. We expand to get $a^2-b^2-c^2+2abi+2acj+bc-bc+a+bi+cj+1 = 0$. We have $(a^2-b^2-c^2+a+1)+i(2ab+b)+j(2ac+c) = 0$. By comparing real coefficients on both sides we get a system of equations: $a^2-b^2-c^2+a+1 = 0$, $b(2a+1) = 0$ and $c(2a+1) = 0$. There are two cases 1) $a = \frac{-1}{2}$ then $\frac{1}{4} - b^2 - c^2 + \frac{1}{2} = 0$. Hence $b^2 + c^2 = \frac{3}{4}$, which are numbers located on a circle with center being the origin and radius $\sqrt{3}/2$. Second case 2) for a different from minus one half leads to both b, c zero which converts the first equation into $a^2 + a + 1 = 0$ which obviously has no real solutions because of a negative discriminant there. Hence $x = -\frac{1}{2} + bi + cj$ such that $b^2 + c^2 = \frac{3}{4}$ (I repeat $i = i_1$ and $j = i_2$.) are all three dimensional numbers satisfying the given equation.

example two: I was thinking of an extended version of the fundamental theorem of algebra. Given a one variable polynomial of degree m having coefficients n dimensional numbers, what is the number of roots being p dimensional numbers (counting multiple roots multiple times)? The standard version of the fundamental theorem of algebra, proven by Carl Friedrich Gauss (1777 - 1855) claims that for $n = p = 2$ it is exactly m. It seems from the previous example that if $p > n$ then there can be infinitely many solutions however.