Bell's theorem and Einstein's worry about quantum mechanics

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Abstract With the use of local dependency of probability density of local hidden variables on the instrument settings, it is demonstrated that Bell's correlation formulation is incomplete. This result concurs with a previous computational violation close to quantum correlation with a computer model based on Einstein locality principles.

Keywords Bell's correlation formula · basic measure theory

1 Introduction

Einstein Podolsky and Rosen started a discussion about the foundation of quantum theory in 1935 [1]. Their work established what later has been called entanglement. Don Howard [2] wrote an interesting history of the discussion that followed from the publication of what we now know as the EPR paradox [1]. Here we will concentrate on Bell's approach to the problem.

In his famous paper, John Bell wrote down [3] a correlation that is based on (local) hidden variables. The experiment where Bell referred to is a spin-spin entanglement experiment. It was based on ideas of David Bohm [4]. Schematically one can formulate it thus

$$[A(\hat{a})] \leftarrow\!\sim \cdots \sim\!\leftarrow\!\sim [S] \sim\!\rightarrow\!\sim \cdots \sim\!\rightarrow [B(\hat{b})] \tag{1}$$

Here, the $[A(\hat{a})]$ and $[B(\hat{b})]$ represent the two distant measuring instruments. The $\hat{a}$ and $\hat{b}$ are the unitary vector setting parameters. The $[S]$ represents the source of an entangled pair of particles.

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Einstein uncovered a correlation between distant measurements. Bell's correlation formula between the setting parameters is presented in equation number (2) of Bell's paper. It is:

$$P(\hat{a},\hat{b}) = \int \rho(\lambda) A(\hat{a},\lambda) B(\hat{b},\lambda) d\lambda \qquad (2)$$

The λ represent hidden variables and $\rho(\lambda)$ represents the propbability density of those variables. The $A(\hat{a},\lambda)$ represents the measurement at $[A(\hat{a})]$ in (1) given the setting $\hat{a}$. For spins, $A(\hat{a},\lambda) = \pm 1$ with 1 a spin-up and -1 a sin-down measurement. From equation (2) a number of inequalities were derived. The CHSH inequality is a very famous inequality and was turned into an experiment by Aspect [5].

2 Thoughts about correlation and locality

Considering the fact that people are awarded Nobelprizes for their work on the inequalities, we will nevertheless argue that the research is incomplete. One cannot conclude from their research that Einstein locality is ruled out in physical "reality". Let us start with noting a work together with Nagata and Nakamura, [6]. Here the mathematics of CHSH is inspected critically and a valid counter example is construed. In [7] a statistical way is construed to locally violate the CHSH with probability nonzero. The criticism on [7] did not touch its conclusion; it is possible locally violate the CHSH with probability nonzero. The following result i.e. a computer program, supports that conclusion.

Then, Geurdes [8] constructed a computer program based on local principles. This program:

- Substantially violates, i.e. ≈ 2.37, the CHSH inequality for 2-dim angular settings at A, $\phi_{\hat{a}_j} \in \{97.39957, 113.48717\}$ and at B, $\phi_{\hat{b}_k} \in \{-82.32930, -26.37997\}$. With, j,k here $1,2$.
- Has results which are close to quantum correlation for all four combinations of $\phi_{\hat{a}_j}$ and $\phi_{\hat{b}_k}$.

If the CHSH really is mathematically solid, i.e. waterproof for local variable models, then the first breach would not be possible. If the local models are in no way able to reproduce quantum correlation, then, the second breach would not have been possible.

Furthermore, we could set up the following analysis. Let us suppose that locality is not violated by allowing that the setting $\hat{a}$ influences a probability density at $[A(\hat{a})]$. Similarly for $\hat{b}$ at $[B(\hat{b})]$. This makes sense in an Einsteinian way when $\hat{a}$ does not influence $[B(\hat{b})]$ and vice versa. Furthermore, in 3 dimensional Euclidian space three orthonormal base vectors are defined by, $\{\hat{e}_k\}_{k=1}^{3}$ with components, $(\hat{e}_k)_n = \delta_{k,n}$. Here $\delta_{k,n} = 1$, when $k = n$ and $\delta_{k,n} = 0$, when

$k \neq n$ and $k, n = 1, 2, 3$. With this definition we are able to write

$$\hat{\omega}(\varphi,\theta) = \sum_{j=1}^{3} \omega_j(\varphi,\theta)\hat{e}_j, \text{ and,} \tag{3}$$

$$\omega_1 = \cos(\varphi)\sin(\theta),\ \omega_2 = \sin(\varphi)\sin(\theta),\ \omega_3 = \cos(\theta)$$

And, $\omega_j = \omega_j(\varphi,\theta)$. The ranges are $\Phi = \{x \in \mathbb{R} : 0 \leq x \leq 2\pi\}$ and $\Theta = \{x \in \mathbb{R} : 0 \leq x \leq \pi\}$. With $||\cdot||$ the Euclidean norm we have $\hat{\omega}^T \cdot \hat{\omega} = ||\hat{\omega}||^2 = 1$ for all $(\varphi,\theta) \in \Phi \times \Theta$. The upper T indicates the transpose of the vector.

Subsequently, with (3), we are able to define $\hat{a} = \hat{\omega}(\varphi_{Aa},\theta_{Aa})$ and $\hat{b} = \hat{\omega}(\varphi_{Bb},\theta_{Bb})$. Both $(\varphi_{Aa},\theta_{Aa})$ and $(\varphi_{Bb},\theta_{Bb})$ in $\Phi \times \Theta$. The $[A(\hat{a})]$ associated hidden variables are denoted by $(\varphi_A,\theta_A) \in \Phi\times\Theta$. The $[B(\hat{b})]$ associated hidden variables are $(\varphi_B,\theta_B) \in \Phi\times\Theta$. If we then, in the language of Pettis integration measure theory [9] write for the A side variables

$$\mu_{\hat{a}}(d\varphi_A d\theta_A) = \delta(\varphi_{Aa} - \varphi_A)\delta(\theta_{Aa} - \theta_A)d\varphi_A d\theta_A \tag{4}$$

While for the B side variables the measure is

$$\mu_{\hat{b}}(d\varphi_B d\theta_B) = \delta(\varphi_{Bb} - \varphi_B)\delta(\theta_{Bb} - \theta_B)d\varphi_B d\theta_B \tag{5}$$

The $\delta(y-x)$ is a Dirac delta function. This is a non-zero "function".

Then, it follows that $\int_{\Phi\times\Theta} \mu_{\hat{a}}(d\varphi_A d\theta_A) = \int_{\Phi\times\Theta} \mu_{\hat{b}}(d\varphi_B d\theta_B) = 1$. Hence, the measures in (4) and (5) are valid short hands for a Bell-form correlation formula. However, the influence of the setting is placed on the density. There is no nonlocality, i.e. $[A(\hat{a})]$ is not influenced by the setting $\hat{b}$ and vice versa, $[B(\hat{b})]$ is not influenced by $\hat{a}$. I.e. the values $(\varphi_{Aa},\theta_{Aa})$ are not influenced by $(\varphi_{Bb},\theta_{Bb})$ and vice versa. The effects are local as one can see from (4) and (5). If people think otherwise they have to come with proof of violation of Einstein locality here. If this proof is not possible -the present author thinks it obviously is not possible- then (4) and (5) are Einstein valid.

Subsequently, let us per pair of entangled particles under investigation -here photons- define a $r_0 \in$ the interval $(0,1)$. The r_0 is randomly selected. Then a measure $\nu_0(dr)$ is defined by

$$\nu_0(dr) = \delta(r_0 - r)dr \tag{6}$$

Here, δ, is again Dirac's delta function and the variable r is in the interval $(0,1)$ as well. Hence, $\nu_0(dr) \geq 0$ and $\int_{-1}^{1} \nu_0(dr) = 1$ and is allowed as density.

Let us then define two functions g_A and g_B with $\Omega_A = (\varphi_A,\theta_A)$, $\Omega_B = (\varphi_B,\theta_B)$ and

$$g_A(\Omega_A,\Omega_B,r_0) = \begin{cases} 1, & 0 < r_0 < \frac{1}{2} \\ \cos\left[\angle\left\{\hat{\omega}\left(\Omega_A\right),\hat{\omega}\left(\Omega_B\right)\right\}\right], & \frac{1}{2} \leq r_0 < 1 \end{cases} \tag{7}$$

The function g_B is defined as follows

$$g_B(\Omega_A,\Omega_B,r_0) = \begin{cases} 1, & \frac{1}{2} \leq r_0 < 1 \\ \cos\left[\angle\left\{\hat{\omega}\left(\Omega_A\right),\hat{\omega}\left(\Omega_B\right)\right\}\right], & 0 < r_0 < \frac{1}{2} \end{cases} \tag{8}$$

The $\angle\{\hat{\omega}(\Omega_A),\hat{\omega}(\Omega_B)\}$ is the angle between unit length vectors $\hat{\omega}(\Omega_A)$ and $\hat{\omega}(\Omega_B)$. Note that $|g_A|\leq 1$ and $|g_B|\leq 1$. Note also that if $\lambda=(\Omega_A,\Omega_B,r)$, the Bell correlation would then be equivalent to

$$P(\hat{a},\hat{b})=\int\rho_{\hat{a}}(\lambda)\rho_{\hat{b}}(\lambda)\rho_{r_0}(\lambda)A(\lambda)B(\lambda)d\lambda \tag{9}$$

We note that the dependence on the settings (which are by definition a local phenomenon) is shifted to the densities. In the next section the integration will be performed in our set of variables and notation.

If we for the moment concentrate on the selection $A(\lambda)=g_A(\Omega_A,\Omega_B,r)$ and $B(\lambda)=g_B(\Omega_A,\Omega_B,r)$, the following integral expression for $P(\hat{a},\hat{b})$, with $d^2\Omega_A=d\varphi_A d\theta_A$ similar B, can be obtained.

$$P(\hat{a},\hat{b})=\int_{\Phi\times\Theta}\mu_{\hat{a}}(d^2\Omega_A)\int_{\Phi\times\Theta}\mu_{\hat{b}}(d^2\Omega_B) \tag{10}$$
$$\times\int_{-1}^{1}\nu_0(dr)g_A(\Omega_A,\Omega_B,r)g_B(\Omega_A,\Omega_B,r)$$

From the definition of $\nu_0(dr)$ it follows

$$P(\hat{a},\hat{b})=\int_{\Phi\times\Theta}\mu_{\hat{a}}(d^2\Omega_A)\int_{\Phi\times\Theta}\mu_{\hat{b}}(d^2\Omega_B)g_A(\Omega_A,\Omega_B,r_0)g_B(\Omega_A,\Omega_B,r_0) \tag{11}$$

and r_0 randomly from interval $(0,1)$ for each pair. Looking at the definition of g_A and g_B in (7) and (8), we arrive from the previous equation at

$$P(\hat{a},\hat{b})=\int_{\Phi\times\Theta}\mu_{\hat{a}}(d^2\Omega_A)\int_{\Phi\times\Theta}\mu_{\hat{b}}(d^2\Omega_B)\cos\left[\angle\{\hat{\omega}(\Omega_A),\hat{\omega}(\Omega_B)\}\right] \tag{12}$$

The subsequent step is to observe that

$$\cos\left[\angle\{\hat{\omega}(\Omega_A),\hat{\omega}(\Omega_B)\}\right]=\hat{\omega}(\Omega_A)^T\cdot\hat{\omega}(\Omega_B)$$

Therefore, the separation in the integration can be performed as

$$P(\hat{a},\hat{b})=\int_{\Phi\times\Theta}\mu_{\hat{a}}(d^2\Omega_A)\int_{\Phi\times\Theta}\mu_{\hat{b}}(d^2\Omega_B)\hat{\omega}(\Omega_A)^T\cdot\hat{\omega}(\Omega_B)= \tag{13}$$
$$\left[\int_{\Phi\times\Theta}\mu_{\hat{a}}(d^2\Omega_A)\hat{\omega}(\Omega_A)\right]^T\cdot\left[\int_{\Phi\times\Theta}\mu_{\hat{b}}(d^2\Omega_B)\hat{\omega}(\Omega_B)\right]$$

Note that, $\Omega_A=(\varphi_A,\theta_A)$ hence by definition of $\mu_{\hat{a}}(d^2\Omega_A)=\mu_{\hat{a}}(d\varphi_A d\theta_A)$ in (4) and of $\mu_{\hat{b}}(d^2\Omega_B)=\mu_{\hat{b}}(d\varphi_B d\theta_B)$ in (5)

$$\int_{\Phi\times\Theta}\mu_{\hat{a}}(d^2\Omega_A)\hat{\omega}(\Omega_A)= \tag{14}$$
$$\int_0^{2\pi}\int_0^{\pi}\delta(\varphi_{Aa}-\varphi_A)\delta(\theta_{Aa}-\theta_A)\hat{\omega}(\varphi_A,\theta_A)\,d\varphi_A d\theta_A=$$
$$\hat{\omega}(\varphi_{Aa},\theta_{Aa})=\hat{a}$$

and similar for $\hat{b}$ on B variables. This implies from (13) that our $P(\hat{a}, \hat{b}) = \hat{a}^T \cdot \hat{b}$. In other words, the quantum correlation has been reproduced from a Bell-type hidden variables model. Why could this not be prevented? The CHSH inequality suggests that this is impossible.

3 An inequality

In this section we will investigate the possibility of a Bell inequality based on our approach. This excercise will teach us something about the usefulness of inequalities and the fact that they can be violated by Bell hidden variables models despite people think otherwise. The equation we base ourselves on is (11) and introduce a slight adaptation. For ease of notation the $\Omega.$ variables will be numbered and we note also that for a third vector $\hat{c}$, we have, $\int_{\Phi\times\Theta} \mu_{\hat{c}}(d^2\Omega_3) = 1$. Therefore

$$P(\hat{a}, \hat{b}) = \int_{\Phi\times\Theta} \mu_{\hat{a}}(d^2\Omega_1) \int_{\Phi\times\Theta} \mu_{\hat{b}}(d^2\Omega_2) \int_{\Phi\times\Theta} \mu_{\hat{c}}(d^2\Omega_3) AB(\Omega_1, \Omega_2) \quad (15)$$

Here, $AB(\Omega_1, \Omega_2)$ is a short-hand for $g_A(\Omega_1, \Omega_2) g_B(\Omega_1, \Omega_2)$, etc. And for completeness, $d^2\Omega_n = d\varphi_n d\theta_n$ with $n = 1, 2, 3$. Take α from the real interval $(-1, 1)$. And so we can write down the triviality

$$\begin{aligned} AB(\Omega_1, \Omega_2) = AB(\Omega_1, \Omega_2)\left[1 + \alpha AB(\Omega_2, \Omega_3)\right] \\ -\alpha AB(\Omega_1, \Omega_2) AB(\Omega_2, \Omega_3) \end{aligned} \quad (16)$$

Then we may note that $1 + \alpha AB(\Omega_2, \Omega_3) \geq 0$. Moreover $\{-AB(\Omega_1, \Omega_2)\} \leq 1$ and $AB(\Omega_2, \Omega_3) \leq 1$, so that $\{-AB(\Omega_1, \Omega_2)\} AB(\Omega_2, \Omega_3) \leq 1$. With an integration procedure like in (15) we then arrive at

$$\begin{aligned} P(\hat{a}, \hat{b}) = \int \mu_{\hat{a}}(d^2\Omega_1) \mu_{\hat{b}}(d^2\Omega_2) \mu_{\hat{c}}(d^2\Omega_3) AB(\Omega_1, \Omega_2)\left[1 + \alpha AB(\Omega_2, \Omega_3)\right] \\ +\alpha \int \mu_{\hat{a}}(d^2\Omega_1) \mu_{\hat{b}}(d^2\Omega_2) \mu_{\hat{c}}(d^2\Omega_3) \{-AB(\Omega_1, \Omega_2)\} AB(\Omega_2, \Omega_3) \end{aligned} \quad (17)$$

Here we have used a somewhat simplified write-up for three integration procedures in (15). I.e.

$$\begin{aligned} \int \mu_{\hat{a}}(d^2\Omega_1) \mu_{\hat{b}}(d^2\Omega_2) \mu_{\hat{c}}(d^2\Omega_3) \\ \equiv \int_{\Phi\times\Theta} \mu_{\hat{a}}(d^2\Omega_1) \int_{\Phi\times\Theta} \mu_{\hat{b}}(d^2\Omega_2) \int_{\Phi\times\Theta} \mu_{\hat{c}}(d^2\Omega_3) \end{aligned}$$

Subsequently,

$$\begin{aligned} P(\hat{a}, \hat{b}) \leq \int \mu_{\hat{a}}(d^2\Omega_1) \mu_{\hat{b}}(d^2\Omega_2) \mu_{\hat{c}}(d^2\Omega_3)\left[1 + \alpha AB(\Omega_2, \Omega_3)\right] \\ +\alpha \int \mu_{\hat{a}}(d^2\Omega_1) \mu_{\hat{b}}(d^2\Omega_2) \mu_{\hat{c}}(d^2\Omega_3) \end{aligned} \quad (18)$$

Because

$$\int \mu_{\hat{a}}(d^2\Omega_1)\mu_{\hat{b}}(d^2\Omega_2)\mu_{\hat{c}}(d^2\Omega_3)AB(\Omega_1,\Omega_2)\left[1+\alpha AB(\Omega_2,\Omega_3)\right] \leq$$
$$\int \mu_{\hat{a}}(d^2\Omega_1)\mu_{\hat{b}}(d^2\Omega_2)\mu_{\hat{c}}(d^2\Omega_3)\left[1+\alpha AB(\Omega_2,\Omega_3)\right]$$

we find that

$$P(\hat{a},\hat{b}) - \alpha P(\hat{b},\hat{c}) \leq 1+\alpha \tag{19}$$

If then we substitue $\alpha = -|\alpha|$ and $\hat{a} = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 0\right)$ and $\hat{b} = \hat{c} = (1,0,0)$, the inequality is

$$\frac{1}{2}\sqrt{2} + |\alpha| \leq 1 - |\alpha|$$

Hence, $|\alpha| \leq \frac{1}{2}\left(1-\frac{1}{2}\sqrt{2}\right) \approx 0.14645$. This implies that if $-1 < \alpha < -0.14645$, the inequality (19) will be violated by the Bell-like expression of (15). Note that $1+\alpha AB(\Omega_2,\Omega_3) = 1-|\alpha|AB(\Omega_2,\Omega_3) \geq 0$ as required.

This result tells us that from (15) a Bell-like inequality can be derived. And that the same expression can violate the inequality and reproduce the quantum correlation. What does this tell us about a big inequality such as CHSH? To be more specific, is an inequality like CHSH sufficient to exclude that Bell's formula reproduces the quantum correlation. For (15) this is not a restriction such as given in (19) for α in the real interval $\left(-1, -\frac{1}{2}\left(1-\frac{1}{2}\sqrt{2}\right)\right)$. Finally, perhaps trivial but when $\hat{a} = \left(0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$, then with the same $\hat{b} = \hat{c} = (1,0,0)$, there is no violation. What is the value of violation vs no violation of an inequality looking at a hidden variables model?

4 Conclusion and discussion

Because of the weight of the matter, one first must acknowledge that our $P(\hat{a},\hat{b})$ is within the concept of what Bell intended with his correlation. To be more specific. Why would a selection of a setting that only affects the density of one associated variable, not be Bell? Secondly, there is no breach of locality as we have already argued in this paper. I.e. selection of $\hat{a}$ does not influence the B variables and vice versa. The settings are Einstein local and settings influence the density of only one variable and $g_a g_B = \cos\left[\angle\left\{\hat{\omega}\left(\Omega_A\right), \hat{\omega}\left(\Omega_B\right)\right\}\right]$ without the necessity to know Ω_{Aa} and Ω_{Bb} and the A integration occurs encapsulated at $[A(\hat{a})]$ and the B integrations encapsulated at $[B(\hat{b})]$. The $\nu_0(dr)$ integration occurs in $[S]$. Note also the possibility of other g_a, g_B with $r_0 \in (0,1)\backslash\left\{\frac{1}{2}\right\}$ random selection. E.g.

$$g_A(\Omega_A,\Omega_B,r_0) = \left(H\left(-\frac{1}{2}+r_0\right) + H\left(\frac{1}{2}-r_0\right)\operatorname{sgn}\left[C_{A,B}\right]\right)\sqrt{|C_{A,B}|}$$
$$g_B(\Omega_A,\Omega_B,r_0) = \left(H\left(\frac{1}{2}-r_0\right) + H\left(-\frac{1}{2}+r_0\right)\operatorname{sgn}\left[C_{A,B}\right]\right)\sqrt{|C_{A,B}|}$$

and $C_{A,B} = \cos\left[\angle\left\{\hat{\omega}\left(\Omega_A\right),\hat{\omega}\left(\Omega_B\right)\right\}\right]$, and sgn the sign function. The $H(x) = 1 \Leftrightarrow x > 0$ and $H(x) = 0 \Leftrightarrow x < 0$. And, $\text{sgn}\left[C_{A,B}\right]\sqrt{|C_{A,B}|}\sqrt{|C_{A,B}|} = C_{A,B}$. Hence, $AB = g_A g_B = C_{A,B}$. This means, the A and B then both simultaneously depend on λ as in Bell's (2). Thirdly, therefore, the use of λ is similar to Bell's. If $\lambda = (\Omega_A, \Omega_B, r) = (\varphi_A, \theta_a, \varphi_B, \theta_B, r)$ are somehow violating locality principles, then, so does Bell's "settings in measurement functions" formulation of the correlation where a product of A and B occur as well. In that case, local hidden variable models would not stand a chance in any experimental test derived from (2). If readers object to the use of $\nu_0(dr)$ then, obviously, the r_0 can be introduced as a $[S]$ viz. (1), parameter without integration procedure. If readers use prejudice to claim that in this case nonlocal hidden variables are employed then they should precisely demonstrate where my locality claim is wrong.

Finally, the result that a quantum correlation reproducing local formulation of Bell's correlation, e.g. (15), violates an associated inequality (19), supports the result where a local computer model violates the CHSH for particular settings [8]. It is justified to claim that the worries of Einstein about the nature of quantum mechanics have not been rightfully addressed in Bell's theorem.

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