Intrinsic Angular Momentum of Classical Electromagnetic Field

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Abstract

We conclude that the expression $\varepsilon_0\mathbf{E}\times\mathbf{A}$ describes the intrinsic angular momentum density of classical electromagnetic field. We also conclude that the mainstream physics community has made a mistake, when failing to recognize this classical quantity.

We study a system that has been defined by a Lagrangian functional

$$\begin{aligned} L(A,\partial_t A) &= -\frac{1}{4\mu_0}\int\limits_{\mathbb{R}^3}\big(\partial_\mu A_\nu(\mathbf{x})-\partial_\nu A_\mu(\mathbf{x})\big)\big(\partial^\mu A^\nu(\mathbf{x})-\partial^\nu A^\mu(\mathbf{x})\big)d^3x \\ &= \frac{1}{2\mu_0}\int\limits_{\mathbb{R}^3}\Big(\Big\|\frac{1}{c}\partial_t\mathbf{A}(\mathbf{x}) \;+\; \nabla_{\mathbf{x}}A^0(\mathbf{x})\Big\|^2 \;-\; \big\|\nabla_{\mathbf{x}}\times\mathbf{A}(\mathbf{x})\big\|^2\Big)d^3x, \end{aligned}$$

where $\mu_0\approx 4\pi\cdot 10^{-7}\mathrm{kg}\cdot\mathrm{m}/\mathrm{C}^2$ is the vacuum magnetic permeability. The functional derivatives of this functional are

$$\frac{\delta L}{\delta A^0(\mathbf{x})} \;=\; -\frac{1}{\mu_0}\nabla_{\mathbf{x}}\cdot\Big(\frac{1}{c}\partial_t\mathbf{A}(\mathbf{x}) \;+\; \nabla_{\mathbf{x}}A^0(\mathbf{x})\Big),$$

$$\frac{\delta L}{\delta A^i(\mathbf{x})} \;=\; -\frac{1}{\mu_0}\big(\partial_i\big(\nabla_{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x})\big) \;+\; \nabla^2_{\mathbf{x}}A_i(\mathbf{x})\big),$$

$$\frac{\delta L}{\delta(\partial_t A^0(\mathbf{x}))} \;=\; 0$$

and

$$\frac{\delta L}{\delta(\partial_t A^i(\mathbf{x}))} \;=\; \frac{1}{\mu_0 c}\Big(-\frac{1}{c}\partial_t A_i(\mathbf{x}) \;+\; \partial_i A^0(\mathbf{x})\Big).$$

Euler-Lagrange equations are

$$\begin{aligned} & D_t\frac{\delta L(A(t,\bullet))}{\delta(\partial_t A^0(\mathbf{x}))} \;=\; \frac{\delta L(A(t,\bullet))}{\delta A^0(\mathbf{x})} \\ \iff\quad & 0 \;=\; \nabla_{\mathbf{x}}\cdot\Big(\frac{1}{c}\partial_t\mathbf{A}(t,\mathbf{x}) \;+\; \nabla_{\mathbf{x}}A^0(t,\mathbf{x})\Big) \end{aligned}$$

and

$$
\begin{aligned}
& D_t \frac{\delta L(A(t,\bullet))}{\delta(\partial_t A^i(\mathbf{x}))} \;=\; \frac{\delta L(A(t,\bullet))}{\delta A^i(\mathbf{x})} \\
\iff \quad & \frac{1}{c}\partial_t\Big(-\frac{1}{c}\partial_t A_i(t,\mathbf{x}) \;+\; \partial_i A^0(t,\mathbf{x})\Big) \;=\; -\partial_i\big(\nabla_{\mathbf{x}}\cdot\mathbf{A}(t,\mathbf{x})\big) \;-\; \nabla_{\mathbf{x}}^2 A_i(t,\mathbf{x}).
\end{aligned}
$$

We can also write the last three equations as

$$
\begin{aligned}
\frac{1}{c}\partial_t\Big(\frac{1}{c}\partial_t\mathbf{A}(t,\mathbf{x}) \;+\; \nabla_{\mathbf{x}}A^0(t,\mathbf{x})\Big) \;&=\; -\nabla_{\mathbf{x}}\big(\nabla_{\mathbf{x}}\cdot\mathbf{A}(t,\mathbf{x})\big) \;+\; \nabla_{\mathbf{x}}^2\mathbf{A}(t,\mathbf{x}) \\
&=\; -\nabla_{\mathbf{x}}\times\big(\nabla_{\mathbf{x}}\times\mathbf{A}(t,\mathbf{x})\big).
\end{aligned}
$$

In this model the electric and magnetic fields can be considered to have been defined by formulas

$$
\begin{aligned}
\mathbf{E}(t,\mathbf{x}) \;&=\; -\partial_t\mathbf{A}(t,\mathbf{x}) \;-\; c\nabla_{\mathbf{x}}A^0(t,\mathbf{x}) \quad \text{and} \\
\mathbf{B}(t,\mathbf{x}) \;&=\; \nabla_{\mathbf{x}}\times\mathbf{A}(t,\mathbf{x}).
\end{aligned}
$$

Euler-Lagrange equations can be written in forms

$$
\nabla_{\mathbf{x}}\cdot\mathbf{E}(t,\mathbf{x}) \;=\; 0
$$

and

$$
\frac{1}{c^2}\partial_t\mathbf{E}(t,\mathbf{x}) \;=\; \nabla_{\mathbf{x}}\times\mathbf{B}(t,\mathbf{x})
$$

that are two of the Maxwell's equations.

The energy of this system is given by the expression

$$
\begin{aligned}
& \int\limits_{\mathbb{R}^3}\frac{\delta L}{\delta(\partial_t A^\mu(\mathbf{x}))}\partial_t A^\mu(\mathbf{x})d^3x \;-\; L \\
&= \frac{1}{2\mu_0}\int\limits_{\mathbb{R}^3}\Big(\Big\|\frac{1}{c}\partial_t\mathbf{A}(\mathbf{x})+\nabla_{\mathbf{x}}A^0(\mathbf{x})\Big\|^2 \;+\; \big\|\nabla_{\mathbf{x}}\times\mathbf{A}(\mathbf{x})\big\|^2\Big)d^3x \\
&\quad + \frac{1}{\mu_0}\int\limits_{\mathbb{R}^3}\Big(\frac{1}{c}\partial_t\mathbf{A}(\mathbf{x})+\nabla_{\mathbf{x}}A^0(\mathbf{x})\Big)\cdot\nabla_{\mathbf{x}}A^0(\mathbf{x})d^3x
\end{aligned}
$$

that also gives the values of Hamiltonian functional. The energy can be written in a form

$$
\begin{aligned}
E(t) \;=\; & \frac{1}{2}\int\limits_{\mathbb{R}^3}\Big(\varepsilon_0\big\|\mathbf{E}(t,\mathbf{x})\big\|^2 \;+\; \frac{1}{\mu_0}\big\|\mathbf{B}(t,\mathbf{x})\big\|^2\Big)d^3x \\
& + \frac{1}{\mu_0 c}\int\limits_{\mathbb{R}^3}\mathbf{E}(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}A^0(t,\mathbf{x})d^3x.
\end{aligned}
$$

Here we used the vacuum permittivity $\varepsilon_0 = \frac{1}{\mu_0 c^2}$. This means that if we want, we can define an energy density

$$\mathcal{E}_{\mathrm{a}}(t,\mathbf{x}) \;=\; \frac{1}{2}\Big(\varepsilon_0\big\|\mathbf{E}(t,\mathbf{x})\big\|^2 \;+\; \frac{1}{\mu_0}\big\|\mathbf{B}(t,\mathbf{x})\big\|^2\Big) \;+\; \frac{1}{\mu_0 c}\mathbf{E}(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}A^0(t,\mathbf{x})$$

and then the total energy can be written in a form

$$E(t) \;=\; \int\limits_{\mathbb{R}^3} \mathcal{E}_{\mathrm{a}}(t,\mathbf{x})d^3x.$$

Mainstream physicists believe that this quantity $\mathcal{E}_{\mathrm{a}}(t,\mathbf{x})$ is not an acceptable energy density, because it depends directly on the vector potential A^μ that doesn't have a physical meaning due to the gauge transformation issue. Then they conclude that we must modify the formula for energy density with the integration by parts technique, which after using $\nabla_{\mathbf{x}}\cdot\mathbf{E}(t,\mathbf{x}) = 0$ gives us another energy density

$$\mathcal{E}_{\mathrm{b}}(t,\mathbf{x}) \;=\; \frac{1}{2}\Big(\varepsilon_0\big\|\mathbf{E}(t,\mathbf{x})\big\|^2 \;+\; \frac{1}{\mu_0}\big\|\mathbf{B}(t,\mathbf{x})\big\|^2\Big).$$

The pointwise values of these $\mathcal{E}_{\mathrm{a}}$ and $\mathcal{E}_{\mathrm{b}}$ are different, but their integrals are the same.

Let's have a look at the momentum of this system. In order to apply Noether's theorem, we must fix some vector $\mathbf{u}$, and then define a translation transformation T_α according to formula

$$\big(T_\alpha(A^\mu)\big)(\mathbf{x}) \;=\; A^\mu(\mathbf{x}-\alpha\mathbf{u}).$$

The derivative of the transformed vector potential at $\alpha = 0$ is

$$\big(D_\alpha T_0(A^\mu)\big)(\mathbf{x}) \;=\; -\mathbf{u}\cdot\nabla_{\mathbf{x}}A^\mu(\mathbf{x}).$$

According to Noether's theorem the momentum of the system in the direction $\mathbf{u}$ is the quantity

$$\begin{aligned}&\int\limits_{\mathbb{R}^3} \big(D_\alpha T_0(A)\big)^\mu(\mathbf{x})\frac{\delta L}{\delta(\partial_t A^\mu(\mathbf{x}))}d^3x \\ &\;=\; -u^j\frac{1}{\mu_0 c}\int\limits_{\mathbb{R}^3}\big(\partial_j A^i(\mathbf{x})\big)\Big(-\frac{1}{c}\partial_t A_i(\mathbf{x}) \;+\; \partial_i A^0(\mathbf{x})\Big)d^3x.\end{aligned}$$

We can remove $\mathbf{u}$ and conclude that the momentum vector is

$$\begin{aligned}\mathbf{P}(t) \;&=\; -\frac{1}{\mu_0 c}\int\limits_{\mathbb{R}^3}\big(\nabla_{\mathbf{x}}A^i(t,\mathbf{x})\big)\Big(-\frac{1}{c}\partial_t A_i(t,\mathbf{x}) \;+\; \partial_i A^0(t,\mathbf{x})\Big)d^3x \\ &=\; \varepsilon_0\int\limits_{\mathbb{R}^3}\sum_{i=1}^{3}\big(\nabla_{\mathbf{x}}A^i(t,\mathbf{x})\big)E^i(t,\mathbf{x})d^3x.\end{aligned}$$

This means that if we want, we can define a momentum density

$$\boldsymbol{\mathcal{P}}_{\rm a}(t,\mathbf{x}) \;=\; \varepsilon_0 \sum_{i=1}^{3} \big(\nabla_{\mathbf{x}} A^i(t,\mathbf{x})\big) E^i(t,\mathbf{x}),$$

and then the total momentum can be written in a form

$$\mathbf{P}(t) \;=\; \int\limits_{\mathbb{R}^3} \boldsymbol{\mathcal{P}}_{\rm a}(t,\mathbf{x}) d^3x.$$

Equation

$$\big(\mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x})\big)^j \;=\; \sum_{i=1}^{3} E^i(t,\mathbf{x})\big(\partial_j A^i(t,\mathbf{x})\big) \;-\; \mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}} A^j(t,\mathbf{x})$$

is true for all $j \in \{1,2,3\}$. This means that the momentum density can alternatively be written as

$$\boldsymbol{\mathcal{P}}_{\rm a}(t,\mathbf{x}) \;=\; \varepsilon_0 \mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}) \;+\; \varepsilon_0 (\mathbf{E}(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}}) \mathbf{A}(t,\mathbf{x}).$$

Similarly as with the energy density, here too mainstream physicists believe that this $\boldsymbol{\mathcal{P}}_{\rm a}$ is not acceptable, because it depends directly on the vector potential. The depence can again be removed with the integration by parts technique, and if we define a new momentum density as

$$\boldsymbol{\mathcal{P}}_{\rm b}(t,\mathbf{x}) \;=\; \varepsilon_0 \mathbf{E}(t,\mathbf{x}) \times \mathbf{B}(t,\mathbf{x}),$$

the equation

$$\mathbf{P}(t) \;=\; \int\limits_{\mathbb{R}^3} \boldsymbol{\mathcal{P}}_{\rm b}(t,\mathbf{x}) d^3x$$

is true again. The pointwise values of $\boldsymbol{\mathcal{P}}_{\rm a}$ and $\boldsymbol{\mathcal{P}}_{\rm b}$ are different, but their integrals are the same.

The integration by parts technique simplifies the formulas for energy and momentum densities, so the mainstream reasoning seems to make sense. However, the reasoning can be criticized too. One way of seeing this thing is that:

There is nothing wrong with the energy and momentum densities depending directly on the vector potential A^μ, because there does not exist a measurement technique for measuring the pointwise values of the densities anyway; only the integrals can be measured by humans.

Doesn't this reasoning seem to make sense too? The quantities $\mathcal{E}_\mathrm{a}$ and $\boldsymbol{\mathcal{P}}_\mathrm{a}$ are the ones that come directly from the generic formulas for Hamiltonian functional and Noether's theorem without extra modifications, so these quantities are simple in that way.

Let's have a look at the angular momentum of this system. In order to apply Noether's theorem, we must fix some vector $\mathbf{u}$, and then define a rotation transformation R_α according to formulas

$$\begin{aligned}\big(R_\alpha(A^0)\big)(\mathbf{x}) &= A^0(e^{-\alpha\mathbf{u}\times}\mathbf{x}) \quad \text{and} \\ \big(R_\alpha(\mathbf{A})\big)(\mathbf{x}) &= e^{\alpha\mathbf{u}\times}\mathbf{A}(e^{-\alpha\mathbf{u}\times}\mathbf{x}).\end{aligned}$$

These formulas actively rotate A^μ around the axis $\mathbf{u}$ with an angle α. The derivative of the transformed vector potential at $\alpha = 0$ is

$$\begin{aligned}\big(D_\alpha R_0(A^0)\big)(\mathbf{x}) &= -\mathbf{u}\cdot(\mathbf{x}\times\nabla_\mathbf{x})A^0(\mathbf{x}) \quad \text{and} \\ \big(D_\alpha R_0(\mathbf{A})\big)(\mathbf{x}) &= \mathbf{u}\times\mathbf{A}(\mathbf{x}) \;-\; \big(\mathbf{u}\cdot(\mathbf{x}\times\nabla_\mathbf{x})\big)\mathbf{A}(\mathbf{x}).\end{aligned}$$

According to Noether's theorem the angular momentum of the system in the direction $\mathbf{u}$ is the quantity

$$\begin{aligned}&\int_{\mathbb{R}^3} \big(D_\alpha R_0(A)\big)^\mu(\mathbf{x})\frac{\delta L}{\delta(\partial_t A^\mu(\mathbf{x}))}d^3x \\ &= -\varepsilon_0\int_{\mathbb{R}^3}\sum_{i=1}^3\Big(\big(\mathbf{u}\times\mathbf{A}(\mathbf{x})\big)^i - \big(\mathbf{u}\cdot(\mathbf{x}\times\nabla_\mathbf{x})\big)A^i(\mathbf{x})\Big)E^i(\mathbf{x})d^3x.\end{aligned}$$

Equation

$$\begin{aligned}\sum_{i=1}^3\big((\mathbf{x}\times\nabla_\mathbf{x})^j A^i(\mathbf{x})\big)E^i(\mathbf{x}) &= \big(\mathbf{x}\times\big(\mathbf{E}(\mathbf{x})\times\mathbf{B}(\mathbf{x})\big)\big)^j \\ &\quad + \big(\mathbf{x}\times\big(\mathbf{E}(\mathbf{x})\cdot\nabla_\mathbf{x}\big)\mathbf{A}(\mathbf{x})\big)^j\end{aligned}$$

is true. This is probably not obvious at a glance, but if one studies the right side carefully, many terms cancel, and what remains is the same as that on the left side. Using this identity the angular momentum in direction $\mathbf{u}$ can be written as

$$\begin{aligned}\mathbf{u}\cdot\mathbf{L}(t) &= \varepsilon_0\mathbf{u}\cdot\int_{\mathbb{R}^3}\Big(\mathbf{E}(t,\mathbf{x})\times\mathbf{A}(t,\mathbf{x}) \\ &\quad + \mathbf{x}\times\Big(\mathbf{E}(t,\mathbf{x})\times\mathbf{B}(t,\mathbf{x}) \;+\; (\mathbf{E}(t,\mathbf{x})\cdot\nabla_\mathbf{x})\mathbf{A}(t,\mathbf{x})\Big)\Big)d^3x\end{aligned}$$

This means that the angular momentum vector is

$$\mathbf{L}(t) = \int_{\mathbb{R}^3}\Big(\varepsilon_0\mathbf{E}(t,\mathbf{x})\times\mathbf{A}(t,\mathbf{x}) \;+\; \mathbf{x}\times\boldsymbol{\mathcal{P}}_\mathrm{a}(t,\mathbf{x})\Big)d^3x.$$

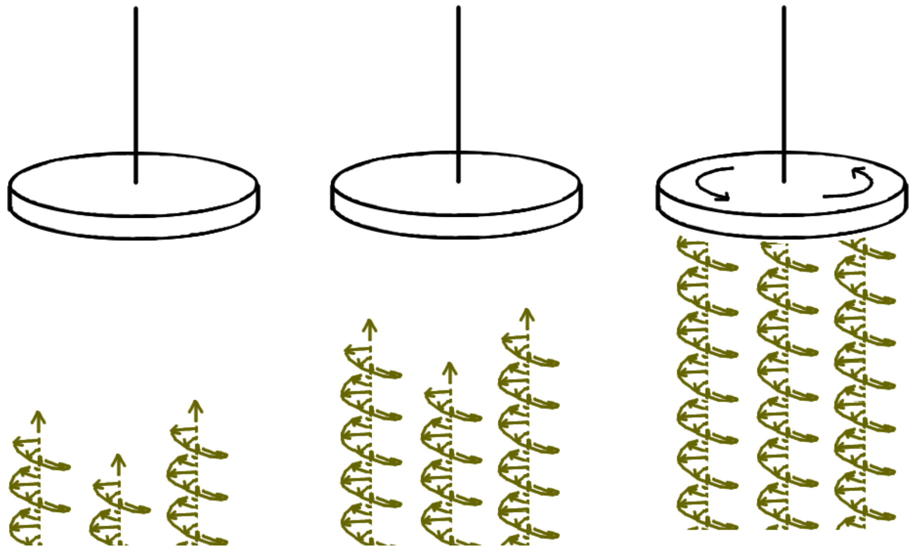

Figure 1: A simplified representation of the R. A. Beth's experiment from 1936. Circularly polarized light hits a solid object, and the solid object turns a little, revealing that the circularly polarized light carries intrinsic angular momentum.

The term $\mathbf{x} \times \boldsymbol{\mathcal{P}}_{\rm a}(t, \mathbf{x})$ describes the orbital angular momentum density, and this term was obviously expected. The term $\varepsilon_0 \mathbf{E}(t, \mathbf{x}) \times \mathbf{A}(t, \mathbf{x})$ may have come as a surprise, but now when we see it, it is natural to interpret it as the intrinsic angular momentum density. If we denote it as

$$\boldsymbol{\mathcal{S}}(t, \mathbf{x}) \;=\; \varepsilon_0 \mathbf{E}(t, \mathbf{x}) \times \mathbf{A}(t, \mathbf{x}),$$

the angular momentum density can then be written as

$$\boldsymbol{\mathcal{L}}(t, \mathbf{x}) \;=\; \boldsymbol{\mathcal{S}}(t, \mathbf{x}) \;+\; \mathbf{x} \times \boldsymbol{\mathcal{P}}_{\rm a}(t, \mathbf{x}),$$

and the angular momentum vector as

$$\mathbf{L}(t) \;=\; \int_{\mathbb{R}^3} \boldsymbol{\mathcal{L}}(t, \mathbf{x}) d^3x.$$

Notice: **If we had been using the mainstream momentum density $\boldsymbol{\mathcal{P}}_{\rm b}(t, \mathbf{x})$, we would not have discovered the intrinsic angular momentum density $\boldsymbol{\mathcal{S}}(t, \mathbf{x})$.** This is why J. D. Jackson does not mention $\boldsymbol{\mathcal{S}}(t, \mathbf{x})$ in any way in the famous book *Classical Electrodynamics*.

Now it makes sense to take a closer look at this expression $\varepsilon_0 \mathbf{E} \times \mathbf{A}$, and check whether it works or not. Let's define an electromagnetic field by formulas

$$\mathbf{E}(t,\mathbf{x}) \;=\; E_0 \begin{pmatrix} \cos\big(k(ct-x^3)\big) \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{B}(t,\mathbf{x}) \;=\; \frac{E_0}{c} \begin{pmatrix} 0 \\ \cos\big(k(ct-x^3)\big) \\ 0 \end{pmatrix},$$

where E_0 and $k > 0$ are some constants. These formulas describe a linearly polarized plane wave that travels in the direction of z-axis. A natural choice for a vector potential that generates this electromagnetic field is

$$A^0(t,\mathbf{x}) \;=\; 0 \qquad \text{and} \qquad \mathbf{A}(t,\mathbf{x}) \;=\; \frac{E_0}{ck} \begin{pmatrix} -\sin\big(k(ct-x^3)\big) \\ 0 \\ 0 \end{pmatrix}.$$

Now

$$\varepsilon_0 \mathbf{E}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x}) \;=\; \mathbf{0},$$

so according to our formula there is no intrinsic angular momentum present. What happens if we instead define an electromagnetic field by formulas

$$\mathbf{E}(t,\mathbf{x}) \;=\; E_0 \begin{pmatrix} \cos\big(k(ct-x^3)\big) \\ \sin\big(k(ct-x^3)\big) \\ 0 \end{pmatrix}$$

and

$$\mathbf{B}(t,\mathbf{x}) \;=\; \frac{E_0}{c} \begin{pmatrix} -\sin\big(k(ct-x^3)\big) \\ \cos\big(k(ct-x^3)\big) \\ 0 \end{pmatrix}?$$

These formulas describe a circularly polarized plane wave that travels in the direction of z-axis. A natural choice for a vector potential that generates this electromagnetic field is

$$A^0(t,\mathbf{x}) \;=\; 0 \qquad \text{and} \qquad \mathbf{A}(t,\mathbf{x}) \;=\; \frac{E_0}{kc} \begin{pmatrix} -\sin\big(k(ct-x^3)\big) \\ \cos\big(k(ct-x^3)\big) \\ 0 \end{pmatrix}.$$

Now

$$\varepsilon_0 \mathbf{E}(t,\mathbf{x}) \times \mathbf{A}(t,\mathbf{x}) \;=\; \frac{\varepsilon_0 E_0^2}{kc} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

so according to our formula there is a non-trivial intrinsic angular momentum present. We see that the expression $\varepsilon_0 \mathbf{E} \times \mathbf{A}$ assigns intrinsic angular momentum correctly to circularly polarized light and not to linearly polarized, and we can conclude that our formula agrees with the Beth's experimental result.

The situation with the integration by parts technique is this: It is possible to remove the direct dependence on vector potential A^μ out of energy, momentum and total angular momentum densities, but it is not possible to remove it out of the intrinsic angular momentum density. So if you believe that the dependence must always be removed with some trick, then the intrinsic angular momentum density becomes a nuisance. This explains why mainstream physicists avoid this quantity. Our conclusion is that the expression $\varepsilon_0 \mathbf{E} \times \mathbf{A}$ describes the intrinsic angular momentum density of the classical electromagnetic field, and that there is nothing wrong with the direct dependence on the vector potential. We should notice that this quantity is not related to quantum mechanics in any special way. We can also conclude that the mainstream physics community has made a mistake, when failing to recognize this classical quantity.