A Recurrence Formula for the Sum of Divisors

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Abstract

In this paper, we describe a simple recurrence formula for the sum of divisors.

0.1 Introduction

The divisor function, $\sigma_k(n)$, for any integer, n, is defined as the sum of the kth powers of the integer divisors of n, i.e. d, and represented as:

$$\sigma_k(n) = \sum_{(d|n)} d^k.$$

When $k = 1$ the divisor function is called the sigma function, sometimes denoted as $\sigma_1(n)$ but conventionally and more often denoted simply as $\sigma(n)$. Here we only consider the case $k = 1$.

The first few values of $\sigma(n)$, where $\sigma(1) = 1$, are (OEIS: A000203):
$1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18, 39, 20, 42, 32, 36, 24, 60, 31, 42, 40...$

It is known that if p is prime and x is any positive integer, then:

$$\sigma(p^x) = (p^{x+1} - 1)/(p - 1). \tag{1}$$

Euler proved the following recurrence:

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) + \cdots$$

$$= \sum_{i \in \mathbb{N}} (-1)^{i+1} \left(\sigma\left(n - \frac{1}{2}\left(3i^2 - i\right)\right) + \sigma\left(n - \frac{1}{2}\left(3i^2 + i\right)\right)\right)$$

where $\sigma(0) = n$ if it occurs and $\sigma(x) = 0$ for $x < 0$, and $\frac{1}{2}\left(3i^2 \mp i\right)$ are consecutive pairs of generalized pentagonal numbers (OEIS: A001318, starting at offset 1). Euler proved this by logarithmic differentiation of the identity in his Pentagonal number theorem.

However, here we suggest a simplified recurrence formula for the divisor function.

0.2 A Simple Recurrence Formula

Here we consider the case for compound n and then for prime n. It is well known that the following recurrence holds for compound n , where $n = st$ (s, t coprime):

$$\sigma(n) = \sigma(s)\sigma(t) \tag{2}$$

$$\Rightarrow \sigma(n) = \sigma(\tfrac{n}{t})\sigma(\tfrac{n}{s}) \tag{3}$$

For example, if $n = 28$, then $s = 4$ and $t = 7$. Since $\sigma(4) = 7$ and $\sigma(7) = 8$, it follows that $\sigma(28) = 7.8 = 56$. NB $s \neq 2, t \neq 14$ since s and t must be coprime.

What is less well-known, though unlikely original, is the case for prime n and powers of primes. We let $n = p$ and generalise the powers such that:

$$\sigma(p^x) = p.\sigma(p^{x-1}) + 1 \tag{4}$$

For example, it is well-known that when $x = 0$ then $\sigma(p) = p + 1$. But if, say, $p^x = 7^3 = 343$, then $p^{x-1} = 7^2 = 49$. Since $\sigma(49) = 57$, it follows that $\sigma(343) = 7.57 + 1 = 400$.