The Maxwell's Equations in Geometric Algebra $Cl_{3,0}$

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Abstract

In this paper, we will obtain the Maxwell's equations using Geometric Algebra $Cl_{3,0}$ (three space dimensions and the time as the trivector of the Algebra). We will obtain the equations with all the possible elements that exist in this Algebra.

We will get as new elements the Electromagnetic Bivector B_{xyz} and the ones related to the angular momentum of the charge -in relation to its rotation or helicoidal trajectory-.

Considering these new elements zero or oscillatory with a zero average, we get the standard Maxwell's Equations (both in its covariant formulism and in Gibbs-Heaviside Algebra.)

Keywords

Geometric Algebra, Covariant formulation of classical Electromagnetism, Maxwell Equations, Electromagnetic trivector

1. Introduction

In this paper, we will obtain the Maxwell's equations using Geometric Algebra $Cl_{3,0}$ (three space dimensions and the time as the trivector of the Algebra).

To arrive to that point, first some hints regarding the position, velocity, current and Electromagnetic Field in Geometric Algebra will be given. Also. a calculation of the Maxwell's equation in standard Algebra will be done so these equations can be compared with the ones that we will obtain in Geometric Algebra $Cl_{3,0}$.

2. The position multivector R

To be able to follow the mathematic framework in this paper, I recommend you read the chapters 2 to 8 of [6] or [7] before continuing. There, you will see how to work in Geometric Algebra $Cl_{3,0}$ considering the time as the 8^{th} degree of freedom (the trivector) of the expanded Geometric Algebra created by the three space vectors.

If you do not know what I am talking about, I strongly recommend you check the masterpiece [1] and the best collection of Geometric Algebra knowledge [3].

If we consider an orthonormal frame in Geometric Algebra $Cl_{3,0}$ composed by the three space vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$.

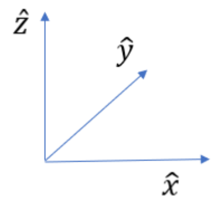

Fig.1 Orthonormal basis vectors in $Cl_{3,0}$

First, as we made in the Annex A1.1 of [2], we will define the position multivector. If we consider a particle or a rigid body as in the Figure 2:

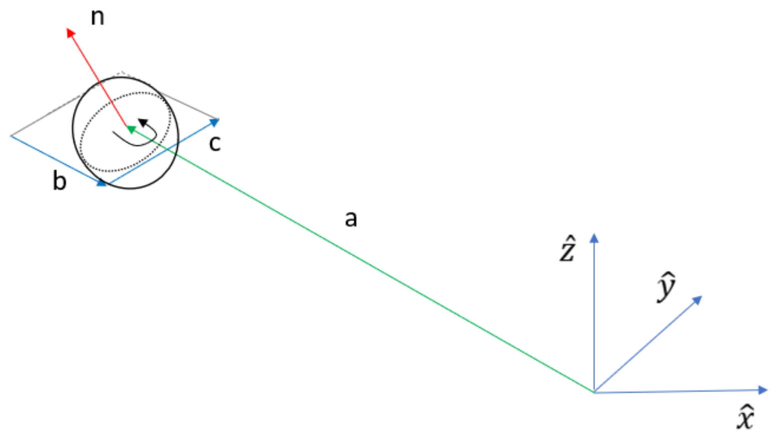

Fig.2 Representation position multivector

This multivector has 8 coordinates (8 degrees of freedom corresponding to the scalar, the three space vectors, the three bivectors and the trivector):

$$R = r_0 + r_x\hat{x} + r_y\hat{y} + r_z\hat{z} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + r_{xyz}\hat{z}\hat{y}\hat{x} \quad (1)$$

If you know something about Geometric Algebra, you will be asking why we have reversed the order of the trivector. You will see this in a minute.

So how this multivector correlated to the Figure 2? First, we see that the vector a corresponds to the linear position of the particle or to the rigid body center of mass:

$$a = r_x\hat{x} + r_y\hat{y} + r_z\hat{z} \quad (2)$$

So, it corresponds to the above elements of the R position multivector. To simplify, we will change the nomenclature of these components to the most classical in literature:

$$r_x = x$$
$$r_y = y$$
$$r_z = z \quad (3)$$

Leading to:

$$a = x\hat{x} + y\hat{y} + z\hat{z} \quad (4)$$

Where the x, y and z without the hat are the spatial coordinates and the $\hat{x}$, $\hat{y}$ and $\hat{z}$ with hat are the space basis vectors.
So, at the moment we are having:

$$R = r_0 + a + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + r_{xyz}\hat{z}\hat{y}\hat{x} \quad (5)$$

$$R = r_0 + x\hat{x} + y\hat{y} + z\hat{z} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + r_{xyz}\hat{z}\hat{y}\hat{x} \quad (6)$$

Now let's go to the bivectors. In Fig.2 you can see that there is a bivector b^c that represents the orientation of a preferred plane in the particle/rigid body. This plane is in general the plane where the rotation will take effect (the plane perpendicular to the rotation axis n) when it happens.

As we are still in a position multivector there is no rotation at this stage, so let's say that this bivector tells us the orientation of the particle/rigid body at a certain moment. If you select a preferred plane solidary to the particle/rigid body, it tells us the orientation of this plane at a certain time. So:

$$b \wedge c = r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} \; (7)$$

Introducing in R:

$$R = r_0 + a + b \wedge c + r_{xyz}\hat{z}\hat{y}\hat{x} \quad (8)$$

$$R = r_0 + x\hat{x} + y\hat{y} + z\hat{z} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + r_{xyz}\hat{z}\hat{y}\hat{x} \quad (6)$$

Regarding the $\hat{z}\hat{y}\hat{x}$, as we commented in chapter 8 of [6] and [7] this corresponds to the $\hat{t}$ vector (being instead the $\hat{x}\hat{y}\hat{z}$ its inverse, the $\hat{t}^{-1}$ vector). So, the element r_{xyz} corresponds to the coordinate of time t. This is:

$$\hat{z}\hat{y}\hat{x} = \hat{t} \quad (9)$$

$$r_{xyz} = t \quad (10)$$

$$R = r_0 + a + b \wedge c + r_{xyz}\hat{z}\hat{y}\hat{x} \quad (8)$$

$$R = r_0 + a + b \wedge c + t\hat{t} \quad (11)$$

$$R = r_0 + x\hat{x} + y\hat{y} + z\hat{z} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + t\hat{t} \quad (12)$$

$$R = r_0 + x\hat{x} + y\hat{y} + z\hat{z} + t\hat{t} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} (13)$$

So, we see that we have the four traditional dimensions (three of space and one of time) included in the position multivector (apart from other elements). This happens even if we have considered only the three space dimensions to start. The time has appeared naturally as the trivector $\hat{z}\hat{y}\hat{x}$.

The only pending element is r_0. The meaning of this element is more obscure. As I have commented in [5[6] the scalars in the multivectors are a kind of scalation factor that affects all the magnitudes that are multiplied by it. So, it could be related to a kind of scalation in the metric appearing in non-Euclidean metrics (kind of local Ricci scalar or trace of the metric tensor).

Another simpler interpretation for r_0, is that the scalars appear when we multiply or divide vectors by themselves. So, it is a necessary degree of freedom to accommodate these results when they appear. For example, we will divide later the time trivector by itself leading to a scalar. But the origin of the value would still remain related to time.

Coming back to the equation of R, we will use the following form (we will use $\hat{z}\hat{y}\hat{x}$ instead of $\hat{t}$) to facilitate the operations:

$$R = r_0 + x\hat{x} + y\hat{y} + z\hat{z} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + t\hat{z}\hat{y}\hat{x} \quad (14)$$

3. The velocity multivector U

To calculate the velocity multivector, we will take the derivative with respect to the proper time τ. The derivative is a division by a differential with its own basis vectors. This means we have to divide by the basis vectors of the time $\hat{t}$ (this is, to postmultiply by the basis vectors of its inverse $\hat{t}^{-1}$).

As τ is the proper time and $\hat{t}$ is the basis vector of the general time coordinate, it is possible that a factor that correlates both should be applied in the vectors, if it is not implicit in the derivation itself. To simplify terms, we will consider a Euclidean metric where this factor is not necessary, and we will just multiply by $\hat{t}^{-1} = \hat{x}\hat{y}\hat{z}$). We will see that this works in Euclidean metric with orthonormal basis, but it could be an issue to be taken into consideration in other cases.

So, taking the derivative of R in (14) with respect to the proper time τ and postmultiplying by $\hat{t}^{-1} = \hat{x}\hat{y}\hat{z}$ we have:

$$U = \frac{dR}{d\tau} = \left(\frac{dr_0}{d\tau} + \frac{dx}{d\tau}\hat{x} + \frac{dy}{d\tau}\hat{y} + \frac{dz}{d\tau}\hat{z} + \frac{dr_{xy}}{d\tau}\hat{x}\hat{y} + \frac{dr_{yz}}{d\tau}\hat{y}\hat{z} + \frac{dr_{zx}}{d\tau}\hat{z}\hat{x} + \frac{dt}{d\tau}\hat{z}\hat{y}\hat{x}\right)\hat{x}\hat{y}\hat{z}$$
$$= \frac{dr_0}{d\tau}\hat{x}\hat{y}\hat{z} + \frac{dx}{d\tau}\hat{y}\hat{z} + \frac{dy}{d\tau}\hat{z}\hat{x} + \frac{dz}{d\tau}\hat{x}\hat{y} - \frac{dr_{xy}}{d\tau}\hat{z} - \frac{dr_{yz}}{d\tau}\hat{x} - \frac{dr_{zx}}{d\tau}\hat{y} + \frac{dt}{d\tau} \quad (15)$$

If you do not understand how the vectors have been multiplied, I recommend you check chapters 2 to 8 of [5] and [6].

You can see that in a Euclidean orthonormal basis, depending on the frame, it could be that $\frac{dt}{d\tau} = 1$. But we will work in a more general case when that derivative could be different than 1 not losing generality.

To be coherent with the nomenclature shown in previous paper [7], we will use the following nomenclature. You can see that some elements will be named-crossed, or sign changed, but this is just a nomenclature. The generality and the correctness are not lost, we are just changing names.

This is the velocity multivector as defined in chapter 11.

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + U_{yz}\hat{x} + U_{zx}\hat{y} + U_{xy}\hat{z} + U_0 \quad (16)$$

This means, the equivalences to be used are:

$$U_{xyz} = \frac{dr_0}{d\tau}$$
$$U_x = \frac{dx}{d\tau}$$
$$U_y = \frac{dy}{d\tau}$$
$$U_z = \frac{dz}{d\tau}$$
$$U_{yz} = -\frac{dr_{yz}}{d\tau}$$
$$U_{zx} = -\frac{dr_{zx}}{d\tau}$$
$$U_{xy} = -\frac{dr_{xy}}{d\tau}$$

$$U_0 = \frac{dt}{d\tau} \qquad (17)$$

Here, we can see what we have commented before regarding the scalar r_0. Now, it is the one multiplying the trivector (it included in $U_{xyz} = \frac{dr_0}{d\tau}$). Does it have anything to do with time now? Is the coordinate of time in the last element $U_0 = \frac{dt}{d\tau}$ not the real one, but the one in U_{xyz}? The important thing is that we will see that the mathematics work. Even if we cannot perfectly understand the philosophy behind the Geometric Algebra $Cl_{3,0}$ with time as the trivector, we will see that it works.

So, summing up the equation of the velocity multivector will have the following form (16). The relation of its coefficients with the ones in the position multivector R can be found in above relations (17).

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + U_{yz}\hat{x} + U_{zx}\hat{y} + U_{xy}\hat{z} + U_0 \qquad (16)$$

As commented in [7], if we compare (16) with standard 4-velocity vector commented in the literature we have 4 elements more marked in bold as following:

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + \boldsymbol{U_{yz}\hat{x}} + \boldsymbol{U_{zx}\hat{y}} + \boldsymbol{U_{xy}\hat{z}} + \boldsymbol{U_0} \qquad (16)$$

This will be of importance when trying to expand already consolidated relations as the Maxwell Equation or the Dirac Equation. Also, as already commented, sometimes it is difficult to discern which was the one existing and which the new one is, between the scalar and the coefficient of the trivector. In this case between $\mathbf{U_0}$ and U_{xyz}. In [7] we considered U_{xyz} as the one existing in the standard 4-velocity vector with successful results.

4. The current multivector J

The electrical current measured in Amperes has the following units:

$$I_{electrical\_current\_units} = A(Ampere) = \frac{C(Coulomb)}{s(second)}$$

Which corresponds to the following basis vectors:

$$I_{electrical\_current\_vectors} = \frac{C}{s} = \frac{scalar}{\hat{t}} = \frac{1}{\hat{t}} = \hat{t}^{-1} = \hat{x}\hat{y}\hat{z}$$

The current density units for the three spatial components of the 4-current [4] are:

$$J_{current\_density\_units} = \frac{I}{Surface} = \frac{\frac{C}{s}}{m^2} = \frac{C}{sm^2}$$

If we consider a current moving in the direction of x axis positive (this is, through a surface in yz plane), we will have:

$$J_{current\_density\_vectors} = \frac{I_{electrical\_current\_vectors}}{Surface\ vectors\ in\ yz} = \frac{\hat{x}\hat{y}\hat{z}}{\hat{y}\hat{z}} = \hat{x}\hat{y}\hat{z}(\hat{y}\hat{z})^{-1} = \hat{x}\hat{y}\hat{z}\hat{z}\hat{y} = \hat{x} \quad (17)$$

So, we can see that the vectors of the density current are the ones in the direction where the current is flowing (as it has been always in standard algebra, but we needed to reconfirm for Geometric Algebra $Cl_{3,0}$).

But, in the standard electrical definition for the current 4-vector J, there is another component that is cρ (see [4]). Where c is the speed of light. As we have had already done in

[5][6] and [7] we will consider the use of units where c=1 and we remove it from the equations.

So, the element staying is ρ, which is the static charge density which units are:

$$\rho_{charge\_density\_units} = \frac{Charge}{Volume} = \frac{C}{m^3}$$

So:

$$\rho_{charge\_density\_vectors} = \frac{C}{m^3} = \frac{scalar}{\hat{z}\hat{y}\hat{x}} = (\hat{z}\hat{y}\hat{x})^{-1} = \hat{x}\hat{y}\hat{z} \qquad (18)$$

Here the definition of the volume can be $\hat{x}\hat{y}\hat{z}$ or $\hat{z}\hat{y}\hat{x}$. Both are ok (they are a sign convention) as far as we are coherent in the subsequent operations. One thing to notice is the parallelism between time and volume. The trivector appears in both cases for the 4-element of a 4-vector indistinctly. Somehow, it is a hint that considering the time as the trivector (in fact, as a kind of volume) is not a so bad idea anyhow. See Annexes.

Coming back to the current multivector, and taking into account (17) and (18), putting all the components together, we will have:

$$J = J_x\hat{x} + J_y\hat{y} + J_z\hat{z} + J_{xyz}\hat{x}\hat{y}\hat{z} \qquad (19)$$

Where J_i is the current density flowing through each of the axes. And J_{xyz} is the charge density ρ commented above. But, as you can guess, we are not stopping here. The goal is to get the most generalized equations possible, so we will introduce all the components of the multivector as:

$$J = J_x\hat{x} + J_y\hat{y} + J_z\hat{z} + J_{xyz}\hat{x}\hat{y}\hat{z} + \boldsymbol{J_{xy}\hat{x}\hat{y}} + \boldsymbol{J_{yz}\hat{y}\hat{z}} + \boldsymbol{J_{zx}\hat{z}\hat{x}} + \boldsymbol{J_0} \qquad (20)$$

Where the elements in bold are the new ones. It is clear that its meaning is obscure, but the idea is to include all the possible components in the equations. If finally, its value is zero, they only have to be deleted.

Anyhow, we could consider the components $\mathbf{J_{ij}}$ as a kind of internal orientation of the charges (if this has any meaning) or a kind of angular momentum during trajectory (helicoidal move). Check [6] and [7] for more details.

The component $\mathbf{J_0}$, as we have commented in similar cases is the scalar. Which function should be a kind of scalation of all the elements that would multiply this magnate or could be a kind of accommodation for future calculations that give scalars as result. Check again [6] and [7] for more comments. Anyhow, we will check later that the component $\mathbf{J_0}$ gets another function.

In fact, there is another simpler way to obtain the J current vector. The J current vector can be defined as [4]:

$$J = \rho_0 U^{\alpha} \qquad (21)$$

Where ρ_0 is the static charge density commented before, so translating to Geometric Algebra:

$$J = (\rho\hat{x}\hat{y}\hat{z})U \qquad (22)$$

Substituting (16) on (22):

$$J = (\rho\hat{x}\hat{y}\hat{z})\left(U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + U_{yz}\hat{x} + U_{zx}\hat{y} + U_{xy}\hat{z} + U_0\right) \qquad (22)$$

$$J = \rho\left(-U_{xyz} - U_x\hat{x} - U_y\hat{y} - U_z\hat{z} + U_{yz}\hat{y}\hat{z} + U_{zx}\hat{z}\hat{x} + U_{xy}\hat{x}\hat{y} + U_0\hat{x}\hat{y}\hat{z}\right) \qquad (23)$$

So, comparing (23) and (20), we will have the following relations:

$$J_x = -\rho U_x$$

$$J_y = -\rho U_y$$
$$J_z = -\rho U_z$$
$$J_{xy} = \rho U_{xy}$$
$$J_{yz} = \rho U_{yz}$$
$$J_{zx} = \rho U_{zx}$$
$$J_0 = -\rho U_{xyz}$$
$$J_{xyz} = \rho U_0 \quad (24)$$

The issue with the change of signs could seem strange but it is a convention anyhow. It is true that it could be related to the index lowering and raising in Minkowski metric. But as I commented in [2] I prefer not to use that Algebra, as when a clear formalism is created for $Cl_{3,0}$, the real necessities of those operations will be defined. At this stage, the idea is to generalize as much as possible all the equations even if in the future different conventions as changes of signs or directions are considered/needed.

5. The Del operator

The del operator (∇) as it is usually defined is (considering c=1):

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} - \frac{\partial}{\partial t}\hat{t} \quad (25)$$

I already commented in Annex A1.5 of [2] that as the Geometric Algebra permits inversion of vectors, it would be a more naturally definition with the vectors dividing (as derivatives are in fact divisions):

$$\nabla = \frac{\partial}{\partial x}\frac{1}{\hat{x}} + \frac{\partial}{\partial y}\frac{1}{\hat{y}} + \frac{\partial}{\partial z}\frac{1}{\hat{z}} + \frac{\partial}{\partial t}\frac{1}{\hat{t}} = \frac{\partial}{\partial x}\hat{x}^{-1} + \frac{\partial}{\partial y}\hat{y}^{-1} + \frac{\partial}{\partial z}\hat{z}^{-1} + \frac{\partial}{\partial t}\hat{t}^{-1} =$$
$$= \frac{\partial}{\partial x}\frac{\hat{x}}{\|\hat{x}\|^2} + \frac{\partial}{\partial y}\frac{\hat{y}}{\|\hat{y}\|^2} + \frac{\partial}{\partial z}\frac{\hat{z}}{\|\hat{z}\|^2} - \frac{\partial}{\partial t}\frac{\hat{t}}{\|\hat{t}\|^2} \quad (26)$$

In an orthonormal basis, all the square norms are equal to 1 so the result would be (25):

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} - \frac{\partial}{\partial t}\hat{t} \quad (25)$$

Now, if we want to generalize the ∇ operator with all the 8 elements of the Geometric Algebra $Cl_{3,0}$, we have to recall all the coordinates of the R position multivector:

$$R = r_0 + x\hat{x} + y\hat{y} + z\hat{z} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + t\hat{z}\hat{y}\hat{x} \quad (14)$$

We see that apart from the x, y, z and t coordinates, we have r_0 and the three r_{ij}.

So, adding them to the del operator:

$$\frac{\partial}{\partial x}\hat{x}^{-1} + \frac{\partial}{\partial y}\hat{y}^{-1} + \frac{\partial}{\partial z}\hat{z}^{-1} + \frac{\partial}{\partial t}\hat{t}^{-1} + \frac{\partial}{\partial r_{xy}}\hat{y}^{-1}\hat{x}^{-1} + \frac{\partial}{\partial r_{yz}}\hat{z}^{-1}\hat{y}^{-1} + \frac{\partial}{\partial r_{zx}}\hat{x}^{-1}\hat{z}^{-1} + \frac{\partial}{\partial r_0}$$

In an orthonormal basis:

$$\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} + \frac{\partial}{\partial t}\hat{x}\hat{y}\hat{z} + \frac{\partial}{\partial r_{xy}}\hat{y}\hat{x} + \frac{\partial}{\partial r_{yz}}\hat{z}\hat{y} + \frac{\partial}{\partial r_{zx}}\hat{x}\hat{z} + \frac{\partial}{\partial r_0}$$

Reordering the bivectors to the standard order of vector products:

$$\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}+\frac{\partial}{\partial t}\hat{x}\hat{y}\hat{z}-\frac{\partial}{\partial r_{xy}}\hat{x}\hat{y}-\frac{\partial}{\partial r_{yz}}\hat{y}\hat{z}-\frac{\partial}{\partial r_{zx}}\hat{z}\hat{x}+\frac{\partial}{\partial r_0}$$

In fact, the trivector of the time (check (14)) should be in reverse order, so it would have a negative sign. But as we have defined to use t^{-1} instead of t, the order is ok with $\hat{x}\hat{y}\hat{z}$ with positive sign.

Anyhow, as we will see later, the above definition of ∇ will not work also. We have to use the following definition with the following change:

$$\nabla=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}+\frac{\partial}{\partial r_0}\hat{x}\hat{y}\hat{z}-\frac{\partial}{\partial r_{xy}}\hat{x}\hat{y}-\frac{\partial}{\partial r_{yz}}\hat{y}\hat{z}-\frac{\partial}{\partial r_{zx}}\hat{z}\hat{x}+\frac{\partial}{\partial t} \quad (26)$$

We have exchanged the r_0 and the t. It is not clear when the time is related to the scalar and when to the trivector. In fact, it already exists an algebra called Algebra of Physical Space (where the time is considered the scalar instead of the trivector [9]). I will comment about in in the Annexes. Some more studies are necessary to define formal processes that can indicate so. In this case, the ∇ works ok with this definition (26) both for Maxwell equations and for Dirac Equation as we will see.

So, putting in bold in (26) the new partial derivatives that appear compared with the standard Del operator (25) we have:

$$\nabla=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}+\frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_0}}\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}-\frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_{xy}}}\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}-\frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_{yz}}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}-\frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_{zx}}}\hat{\boldsymbol{z}}\hat{\boldsymbol{x}}+\frac{\partial}{\partial t} \quad (26)$$

6. The Electromagnetic Field Strength

The work of expanding the Electromagnetic Field Strength in Geometric Algebra $Cl_{3,0}$ was already done in [7] in chapter 11 as:

$$F=E_x\hat{x}+E_y\hat{y}+E_z\hat{z}+B_x\hat{y}\hat{z}+B_y\hat{z}\hat{x}+B_z\hat{x}\hat{y}+\boldsymbol{B_{xyz}}\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}+\boldsymbol{E_0} \quad (27)$$

In bold the new elements. In [6] and [7] its meaning is explained, focusing in the electromagnetic bivector $\mathbf{B_{xyz}}$.

7. The Covariant formulation of the Maxwell's equation in vacuum

In this chapter I will, just follow [4] to obtain the Maxwell's equations in the standard matrix-tensor algebra.

According [4] the Electromagnetic Field Strength is defined as (considering c=1 and ℏ=1):

$$F_{\alpha\beta}=\begin{pmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{pmatrix} \quad (28)$$

Or:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (29)$$

And the Maxwell's equations are defined as:

$$\frac{\partial}{\partial \alpha} F^{\alpha\beta} = \mu_0 J^{\beta} \qquad (30)$$

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}\right) = 0 \quad (31)$$

We will consider the order 4,1,2,3 (t,x,y,z) for rows and columns. If you are a normal not insane person and you use the order 1,2,3,4 you will obtain the same result don't worry (you can check it as an exercise):

$$\frac{\partial}{\partial \alpha} F^{\alpha\beta} = \mu_0 J^{\beta} \qquad (30)$$

Now, we have to sum in repeated dummy indexes and create equations in non repeated indexes (Einstein summation convention [10]):

$$\frac{\partial F^{xx}}{\partial x} + \frac{\partial F^{yx}}{\partial y} + \frac{\partial F^{zx}}{\partial z} + \frac{\partial F^{tx}}{\partial t} = \mu_0 J^x$$

$$\frac{\partial F^{xy}}{\partial x} + \frac{\partial F^{yy}}{\partial y} + \frac{\partial F^{zy}}{\partial z} + \frac{\partial F^{ty}}{\partial t} = \mu_0 J^y$$

$$\frac{\partial F^{xz}}{\partial x} + \frac{\partial F^{yz}}{\partial y} + \frac{\partial F^{zz}}{\partial z} + \frac{\partial F^{tz}}{\partial t} = \mu_0 J^z$$

$$\frac{\partial F^{xt}}{\partial x} + \frac{\partial F^{yt}}{\partial y} + \frac{\partial F^{zt}}{\partial z} + \frac{\partial F^{tt}}{\partial t} = \mu_0 J^t \qquad (32)$$

$$\frac{\partial 0}{\partial x} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial t} = \mu_0 J^x$$

$$-\frac{\partial B_z}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial B_x}{\partial z} - \frac{\partial E_y}{\partial t} = \mu_0 J^y$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} + \frac{\partial 0}{\partial z} - \frac{\partial E_z}{\partial t} = \mu_0 J^z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} + \frac{\partial 0}{\partial t} = \mu_0 J^t \qquad (33)$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{\partial E_x}{\partial t} = \mu_0 J^x$$

$$-\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} - \frac{\partial E_y}{\partial t} = \mu_0 J^y$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{\partial E_z}{\partial t} = \mu_0 J^z$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \mu_0 J^t \quad (34)$$

So, equations (34) are the first set of Maxwell's equations that we should be able to replicate using Geometric Algebra $Cl_{3,0.}$

Let's go to the second equation. Again, we will consider the order 4,1,2,3 (t,x,y,z) for rows and columns. The symbol ε is the Levi-Civita symbol [11]:

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}\right) = 0 \qquad (31)$$

If we make the first case with α=1 (this is, α=x) we have:

$$\frac{1}{2}\frac{\partial}{\partial \alpha}\left(\varepsilon^{1234}F_{34} + \varepsilon^{1243}F_{43} + \varepsilon^{1342}F_{42} + \varepsilon^{1324}F_{24} + \varepsilon^{1423}F_{23} + \varepsilon^{1432}F_{32}\right) = 0$$

$$\frac{1}{2}\frac{\partial}{\partial x}\left(\varepsilon^{1234}F_{zt} + \varepsilon^{1243}F_{tz} + \varepsilon^{1342}F_{ty} + \varepsilon^{1324}F_{yt} + \varepsilon^{1423}F_{yz} + \varepsilon^{1432}F_{zy}\right) = 0$$

$$\frac{1}{2}\frac{\partial}{\partial x}\left(+(-E_z) - (E_z) + E_y - (-E_y) + (-B_x) - (B_x)\right) = 0$$

$$\frac{1}{2}\frac{\partial}{\partial x}\left(-2E_z + 2E_y - 2B_x\right) = 0$$

$$-\frac{\partial E_z}{\partial x} + \frac{\partial E_y}{\partial x} - \frac{\partial B_x}{\partial x} = 0 \quad (35)$$

Making for all the indexes, we get:

$$-\frac{\partial E_z}{\partial x} + \frac{\partial E_y}{\partial x} - \frac{\partial B_x}{\partial x} = 0 \quad (36.1)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_x}{\partial y} - \frac{\partial B_y}{\partial y} = 0$$

$$-\frac{\partial E_y}{\partial z} + \frac{\partial E_x}{\partial z} - \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B_x}{\partial t} + \frac{\partial B_y}{\partial t} + \frac{\partial B_z}{\partial t} = 0 \qquad (36)$$

These equations do not have the classical form obtained in standard algebra. So, I will calculate also using the standard algebra to see if it is easier to make a comparison with Geometric Algebra.

In standard algebra (three dimensions) one of the Maxwell equations [12] has this form (where ∇x is the curl/rotational):

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad (36.2)$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z} = -\frac{\partial B_x}{\partial t}\hat{x} - \frac{\partial B_y}{\partial t}\hat{y} - \frac{\partial B_z}{\partial t}\hat{z}$$

Leading to:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t} = 0$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t} = 0$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t} = 0 \quad (36.3)$$

Also, if we calculate:

$$\nabla \cdot B = 0 \quad (36.4)$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (36.5)$$

So, in this chapter we have calculated different set of equations, both in covariant form (tensors in 4 dimensions) and in standard form (curl/rotational and divergences) in three dimensions. The goal is to reproduce these equations using Geometric Algebra $Cl_{3,0}$ (3 space dimensions) and considering the time as the trivector of this space.

8. The Maxwell Equations in Geometric Algebra $Cl_{3.0}$

We want to reproduce the same results as in chapter 7 (34) and (36) but using Geometric Algebra $Cl_{3,0}$. In [3] (7.14) the Maxwell Equation is reduced to this form in Geometric Algebra:

$$\nabla F = J \quad (37)$$

Now, we can apply the same equation using all the extended definition we have commented for each one of the elements (20)(26)(27) in (37).

I represent in bold, the elements that do not exist in standard algebra (nor in covariant tensor formulation neither in standard Gibbs-Heaviside vector calculations):

$$\nabla F = J \quad (37)$$

$$\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_0}}\boldsymbol{\hat{x}\hat{y}\hat{z}} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_{xy}}}\boldsymbol{\hat{y}\hat{x}} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_{yz}}}\boldsymbol{\hat{z}\hat{y}} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial r_{zx}}}\boldsymbol{\hat{x}\hat{z}} + \frac{\partial}{\partial t}\right)$$

$$\left(E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + \boldsymbol{B_{xyz}\hat{x}\hat{y}\hat{z}} + \boldsymbol{E_0}\right)$$

$$= J_x\hat{x} + J_y\hat{y} + J_z\hat{z} + J_{xyz}\hat{x}\hat{y}\hat{z} + \boldsymbol{J_{xy}\hat{x}\hat{y}} + \boldsymbol{J_{yz}\hat{y}\hat{z}} + \boldsymbol{J_{zx}\hat{z}\hat{x}} + \boldsymbol{J_0} \quad (38)$$

Operating the first product:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial x}\hat{x}\hat{y} - \frac{\partial E_z}{\partial x}\hat{z}\hat{x} + \frac{\partial B_x}{\partial x}\hat{x}\hat{y}\hat{z} - \frac{\partial B_y}{\partial x}\hat{z} + \frac{\partial B_z}{\partial x}\hat{y} + \frac{\partial \boldsymbol{B_{xyz}}}{\partial x}\hat{y}\hat{z} + \frac{\partial \boldsymbol{E_0}}{\partial x}\hat{x} +$$

$$-\frac{\partial E_x}{\partial y}\hat{x}\hat{y} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial y}\hat{y}\hat{z} + \frac{\partial B_x}{\partial y}\hat{z} + \frac{\partial B_y}{\partial y}\hat{x}\hat{y}\hat{z} - \frac{\partial B_z}{\partial y}\hat{x} + \frac{\partial \boldsymbol{B_{xyz}}}{\partial y}\hat{z}\hat{x} + \frac{\partial \boldsymbol{E_0}}{\partial y}\hat{y}$$

$$+\frac{\partial E_x}{\partial z}\hat{z}\hat{x} - \frac{\partial E_y}{\partial z}\hat{y}\hat{z} + \frac{\partial E_z}{\partial z} - \frac{\partial B_x}{\partial z}\hat{y} + \frac{\partial B_y}{\partial z}\hat{x} + \frac{\partial B_z}{\partial z}\hat{x}\hat{y}\hat{z} + \frac{\partial \boldsymbol{B_{xyz}}}{\partial z}\hat{x}\hat{y} + \frac{\partial \boldsymbol{E_0}}{\partial z}\hat{z}$$

$$+\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_0}}\hat{y}\hat{z} + \frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_0}}\hat{z}\hat{x} + \frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_0}}\hat{x}\hat{y} - \frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_0}}\hat{x} - \frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_0}}\hat{y} - \frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_0}}\hat{z} - \frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_0}} + \frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_0}}\hat{x}\hat{y}\hat{z}$$

$$+\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_{xy}}}\hat{y} - \frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_{xy}}}\hat{x} - \frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_{xy}}}\hat{x}\hat{y}\hat{z} + \frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_{xy}}}\hat{z}\hat{x} - \frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_{xy}}}\hat{y}\hat{z} + \frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_{xy}}} + \frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{xy}}}\hat{z} - \frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{xy}}}\hat{x}\hat{y}$$

$$-\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_{yz}}}\hat{x}\hat{y}\hat{z} + \frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_{yz}}}\hat{z} - \frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_{yz}}}\hat{y} + \frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_{yz}}} + \frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_{yz}}}\hat{x}\hat{y} - \frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_{yz}}}\hat{z}\hat{x} + \frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{yz}}}\hat{x} - \frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{yz}}}\hat{y}\hat{z}$$

$$-\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_{zx}}}\hat{z} - \frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_{zx}}}\hat{x}\hat{y}\hat{z} + \frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_{zx}}}\hat{x} - \frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_{zx}}}\hat{x}\hat{y} + \frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_{zx}}} + \frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_{zx}}}\hat{y}\hat{z} + \frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{zx}}}\hat{y} - \frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{zx}}}\hat{z}\hat{x}$$

$$+\frac{\partial E_x}{\partial t}\hat{x}+\frac{\partial E_y}{\partial t}\hat{y}+\frac{\partial E_z}{\partial t}\hat{z}+\frac{\partial B_x}{\partial t}\hat{y}\hat{z}+\frac{\partial B_y}{\partial t}\hat{z}\hat{x}+\frac{\partial B_z}{\partial t}\hat{x}\hat{y}+\frac{\partial B_{xyz}}{\partial t}\hat{x}\hat{y}\hat{z}+\frac{\partial E_0}{\partial t}=$$
$$=J_x\hat{x}+J_y\hat{y}+J_z\hat{z}+J_{xyz}\hat{x}\hat{y}\hat{z}+\boldsymbol{J_{xy}\hat{x}\hat{y}}+\boldsymbol{J_{yz}\hat{y}\hat{z}}+\boldsymbol{J_{zx}\hat{z}\hat{x}}+\boldsymbol{J_0} \quad (39)$$

If we separate the equations by the coefficients that multiply each element in GA (each vector, each bivector, the trivector and the scalar) we obtain:

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}-\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial B_z}}{\boldsymbol{\partial r_{xy}}}+\frac{\boldsymbol{\partial B_x}}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial B_y}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial \boldsymbol{E_0}}{\partial t}=\boldsymbol{J_0}$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial \boldsymbol{B_{xyz}}}{\partial z}+\frac{\partial E_z}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{xy}}}+\frac{\partial B_y}{\boldsymbol{\partial r_{yz}}}-\frac{\partial B_x}{\boldsymbol{\partial r_{zx}}}+\frac{\partial B_z}{\partial t}=\boldsymbol{J_{xy}}$$
$$-\frac{\partial E_z}{\partial x}+\frac{\partial \boldsymbol{B_{xyz}}}{\partial y}+\frac{\partial E_x}{\partial z}+\frac{\boldsymbol{\partial E_y}}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial B_x}}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial B_z}}{\boldsymbol{\partial r_{yz}}}-\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial B_y}{\partial t}=\boldsymbol{J_{zx}}$$
$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}+\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial E_z}}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial E_x}}{\boldsymbol{\partial r_{yz}}}-\frac{\boldsymbol{\partial E_y}}{\boldsymbol{\partial r_{zx}}}+\frac{\boldsymbol{\partial B_{xyz}}}{\partial t}=J_{xyz}$$
$$-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}+\frac{\boldsymbol{\partial E_0}}{\partial z}-\frac{\boldsymbol{\partial B_z}}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{xy}}}+\frac{\boldsymbol{\partial E_y}}{\boldsymbol{\partial r_{yz}}}-\frac{\boldsymbol{\partial E_x}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial E_z}{\partial t}=J_z$$
$$\frac{\partial B_z}{\partial x}+\frac{\partial \boldsymbol{E_0}}{\partial y}-\frac{\partial B_x}{\partial z}-\frac{\boldsymbol{\partial B_y}}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial E_x}}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial E_z}}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial E_y}{\partial t}=J_y$$
$$\frac{\boldsymbol{\partial B_{xyz}}}{\partial x}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\boldsymbol{\partial E_x}}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial B_y}}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial B_z}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial B_x}{\partial t}=\boldsymbol{J_{yz}}$$
$$\frac{\boldsymbol{\partial E_0}}{\partial x}-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}-\frac{\boldsymbol{\partial B_x}}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial E_y}}{\boldsymbol{\partial r_{xy}}}+\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial E_z}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial E_x}{\partial t}=J_x \quad (40)$$

The ones above are the complete set of Maxwell's Equations in Geometric Algebra $Cl_{3,0}$, considering elements not studied at the moment, as the Electromagnetic trivector $\mathbf{B_{xyz}}$ commented in chapter 6 and in [6] and [7]. And also, the bivector electric currents $\mathbf{J_{ij}}$ that could represent a kind of angular momentum of the particles (related to its rotation or orientation) or of its trajectory (helicoidal movement) which mean value should be zero. This means, they can represent an oscillatory movement or rotation that does not affect the average trajectory. And that is the reason, they have not been taken into account in a first place, but they could affect other measurements as spin (see [6] and [7]) for more explanations.

But are equations (40) ok? We can remove the bold elements in the left-hand side to clean the equations and try to compare to the ones in standard algebra.

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=\boldsymbol{J_0} \quad (41)$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}=\boldsymbol{J_{xy}} \quad (42)$$
$$-\frac{\partial E_z}{\partial x}+\frac{\partial E_x}{\partial z}+\frac{\partial B_y}{\partial t}=\boldsymbol{J_{zx}} \quad (43)$$
$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=J_{xyz} \quad (44)$$
$$-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}+\frac{\partial E_z}{\partial t}=J_z \quad (45)$$

$$\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}+\frac{\partial E_y}{\partial t}=J_y \quad (46)$$

$$+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}=\boldsymbol{J_{yz}} \quad (47)$$

$$-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}+\frac{\partial E_x}{\partial t}=J_x \quad (48)$$

If we compare with (34) obtained with covariant formulism (considering a set of units with μ_0=1):

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}-\frac{\partial E_x}{\partial t}=J^x$$

$$-\frac{\partial B_z}{\partial x}+\frac{\partial B_x}{\partial z}-\frac{\partial E_y}{\partial t}=J^y$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}-\frac{\partial E_z}{\partial t}=J^z$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=J^t \quad (34)$$

We see that the equations (41), (45), (46), (48) are the same as above if we consider:

$$J_x=-J^x$$

$$J_y=-J^y$$

$$J_z=-J^z$$

$$J_0=J^t \quad (49)$$

So, again we have an issue with the signs (as we had in (24)), where the current has a different sign convention in both Algebras. But it is coherent within the Geometric algebra itself as the current was defined with opposite direction than velocity (24). So, the effect in the field should the same in both algebras. Anyhow, as commented, the issue of the signs is something that has to be studied in a more formal way to standardize conventions and formulations.

Another issue we see is that J^t is J_0 instead of being J_{xyz}. Again, there is also an issue regarding the consideration of time as the trivector or as the scalar. See annexes for more information.

If we go now to (36) equations:

$$-\frac{\partial E_z}{\partial x}+\frac{\partial E_y}{\partial x}-\frac{\partial B_x}{\partial x}=0 \quad (36.1)$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_x}{\partial y}-\frac{\partial B_y}{\partial y}=0$$

$$-\frac{\partial E_y}{\partial z}+\frac{\partial E_x}{\partial z}-\frac{\partial B_z}{\partial z}=0$$

$$\frac{\partial B_x}{\partial t}+\frac{\partial B_y}{\partial t}+\frac{\partial B_z}{\partial t}=0 \quad (36)$$

We do not have a clear relation. But if we take the first one for example

$$-\frac{\partial E_z}{\partial x}+\frac{\partial E_y}{\partial x}-\frac{\partial B_x}{\partial x}=0 \quad (36.1)$$

We can sum (42) and (43) and subtract (44) to try to reproduce it:

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}=\boldsymbol{J_{xy}} \quad (42)$$

$$-\frac{\partial E_z}{\partial x}+\frac{\partial E_x}{\partial z}+\frac{\partial B_y}{\partial t}=\boldsymbol{J_{zx}} \quad (43)$$

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=J_{xyz} \quad (44)$$

Leading to:

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}-\frac{\partial E_z}{\partial x}+\frac{\partial E_x}{\partial z}+\frac{\partial B_y}{\partial t}-\frac{\partial B_x}{\partial x}-\frac{\partial B_y}{\partial y}-\frac{\partial B_z}{\partial z}=\boldsymbol{J_{xy}}+\boldsymbol{J_{zx}}-J_{xyz}$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_z}{\partial x}-\frac{\partial B_x}{\partial x}=\boldsymbol{J_{xy}}+\boldsymbol{J_{zx}}-J_{xyz}+\frac{\partial E_x}{\partial y}-\frac{\partial B_z}{\partial t}-\frac{\partial E_x}{\partial z}-\frac{\partial B_y}{\partial t}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z} \quad (45)$$

So, we get equation (45) that according to (36.1) should be equal to 0. As we have new elements as the $\mathbf{J_{ij}}$ mixed with standard elements is really difficult if this is correct.

But if we compare with the equations obtained using standard not covariant algebra:

$$\nabla\times E=-\frac{\partial B}{\partial t} \quad (36.2)$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}=0$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{\partial B_y}{\partial t}=0$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}=0 \quad (36.3)$$

We see that equation (42), (43) and (44) are exactly those equations:

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}=\boldsymbol{J_{xy}} \quad (42)$$

$$-\frac{\partial E_z}{\partial x}+\frac{\partial E_x}{\partial z}+\frac{\partial B_y}{\partial t}=\boldsymbol{J_{zx}} \quad (43)$$

$$+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}=\boldsymbol{J_{yz}} \quad (47)$$

Considering that $\mathbf{J_{ij}}$ are zero or have an oscillatory value which mean value is zero.

The only pending equation to be checked is:

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=J_{xyz} \quad (44)$$

If we compare with:

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0 \quad (36.5)$$

We see that they are the same equation if J_{xyz} is equal to zero or is oscillatory with an average value of zero. The meaning of J_{xyz} is obscure. In the beginning it could seem the static density of charge (the electrical current through time) but that role si occupied now by J_0 as can be seen in (34) and (49). We can see it as another degree of freedom regarding the properties of a moving charge that could be taken into account if necessary for explanation of effects to be explored (zitterbewegung, measurements of spin, helicoidal movements etc.). As this property J_{xyz} can affect the fields (and can be affected by them). Something very similar to the Electromagnetic trivector $\mathbf{B_{xyz}}$.

9. Conclusions

In this paper, we have calculated the Maxwell equations Geometric Algebra $Cl_{3,0}$ (three space dimensions and the time being the trivector), leading to:

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}-\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_{xy}}}+\frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_{zx}}}+\frac{\partial \boldsymbol{E_0}}{\partial t}=\boldsymbol{J_0}$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial \boldsymbol{B_{xyz}}}{\partial z}+\frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{xy}}}+\frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_{yz}}}-\frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_{zx}}}+\frac{\partial B_z}{\partial t}=\boldsymbol{J_{xy}}$$
$$-\frac{\partial E_z}{\partial x}+\frac{\partial \boldsymbol{B_{xyz}}}{\partial y}+\frac{\partial E_x}{\partial z}+\frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_{yz}}}-\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial B_y}{\partial t}=\boldsymbol{J_{zx}}$$
$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}+\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_{yz}}}-\frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_{zx}}}+\frac{\partial \boldsymbol{B_{xyz}}}{\partial t}=J_{xyz}$$
$$-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}+\frac{\partial \boldsymbol{E_0}}{\partial z}-\frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{xy}}}+\frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_{yz}}}-\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_{zx}}}+\frac{\partial E_z}{\partial t}=J_z$$
$$\frac{\partial B_z}{\partial x}+\frac{\partial \boldsymbol{E_0}}{\partial y}-\frac{\partial B_x}{\partial z}-\frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_0}}+\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{zx}}}+\frac{\partial E_y}{\partial t}=J_y$$
$$\frac{\partial \boldsymbol{B_{xyz}}}{\partial x}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\boldsymbol{\partial} E_x}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial} B_y}{\boldsymbol{\partial r_{xy}}}-\frac{\boldsymbol{\partial E_0}}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial} B_z}{\boldsymbol{\partial r_{zx}}}+\frac{\partial B_x}{\partial t}=\boldsymbol{J_{yz}}$$
$$\frac{\partial \boldsymbol{E_0}}{\partial x}-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}-\frac{\boldsymbol{\partial} B_x}{\boldsymbol{\partial r_0}}-\frac{\boldsymbol{\partial} E_y}{\boldsymbol{\partial r_{xy}}}+\frac{\boldsymbol{\partial B_{xyz}}}{\boldsymbol{\partial r_{yz}}}+\frac{\boldsymbol{\partial} E_z}{\boldsymbol{\partial r_{zx}}}+\frac{\partial E_x}{\partial t}=J_x \quad (40)$$

Where the elements in bold are new elements as the electromagnetic trivector $\mathbf{B_{xyz}}$ that are not considered in standard algebra or covariant form. Eliminating them form the equations (considering them zero or oscillatory with an average value of zero), we have:

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=\boldsymbol{J_0} \quad (41)$$
$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}=\boldsymbol{J_{xy}} \quad (42)$$
$$-\frac{\partial E_z}{\partial x}+\frac{\partial E_x}{\partial z}+\frac{\partial B_y}{\partial t}=\boldsymbol{J_{zx}} \quad (43)$$
$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=J_{xyz} \quad (44)$$
$$-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}+\frac{\partial E_z}{\partial t}=J_z \quad (45)$$
$$\frac{\partial B_z}{\partial x}-\frac{\partial B_x}{\partial z}+\frac{\partial E_y}{\partial t}=J_y \quad (46)$$
$$+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}=\boldsymbol{J_{yz}} \quad (47)$$

$$-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}+\frac{\partial E_x}{\partial t}=J_x \quad (48)$$

These are the same as these ones obtained with standard algebra (covariant and also Gibbs-Heaviside algebra):

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}-\frac{\partial E_x}{\partial t}=J^x$$

$$-\frac{\partial B_z}{\partial x}+\frac{\partial B_x}{\partial z}-\frac{\partial E_y}{\partial t}=J^y$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}-\frac{\partial E_z}{\partial t}=J^z$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=J^t \quad (34)$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}=0$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{\partial B_y}{\partial t}=0$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}=0 \quad (36.3)$$

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0 \quad (36.5)$$

If we consider:

$$J_x=-J^x$$

$$J_y=-J^y$$

$$J_z=-J^z$$

$$J_0=J^t \quad (49)$$

And the $\mathbf{J}_{ij}$ and $\mathbf{J}_{xyz}$ oscillatory with average value equal to zero or directly zero. The $\mathbf{J}_{ij}$ could be considered as a kind of angular momentum of the moving charge (that could be related to its own rotation or orientation or related to its trajectory (helicoidal)).

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If you consider this helpful, do not hesitate to drop your BTC here:

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A1. Annex A1. Algebra of Physical Space (APS)

In [9] you can find information regarding APS. APS considered the time as the scalar instead as the trivector. The parallelisms of APS with regards as considering the Algebra that considers time as a trivector are clear. In fact, in some steps where a product or a division by the trivector is necessary, the time in fact gets itself imbued in the scalar dimension (we have seen it with this exchange of roles between J_{xyz} and J_0 in the chapter 8 of the paper for example).

I will enumerate which I consider are the advantages of considering time as the trivector (at least in a first step of definitions) instead of the scalar.

- The signature: The signature of time (its square) is almost in all the Algebras opposite to the signature of the space vectors. In a $Cl_{3,0}$ algebra where the square of the space vectors is positive, the square of the time should be negative (as the trivector). The scalars do not fulfil this.
- The possibility of time having a basis vector, “a direction”. This seems more philosophical, but it gets important in practice in the end. If you consider time as the scalar, time is something hanging in the air that nobody can understand exactly its meaning or where it enters to have a role in an equation. Any scalar in the equation does anything to do with time? Considering its basis vector as the trivector, the time is clearly another dimension, and you can stablish for it its own space. You have $\hat{x}\ \hat{y}\ \hat{z}$ and $\hat{t}$. Where $\hat{t} = \hat{z}\hat{y}\hat{x}$ but you do not need even need to know it. You can operate everything with $\hat{t}$ and whenever you consider it necessary, make the transformation to $\hat{z}\hat{y}\hat{x}$. This, you cannot do if the time does not have a vector itself.

- The paravectors. The paravectors are vectors that include the three space vectors and a scalar representing time. This is correct and it works, but due to the points commented before, the time appears as something strange added there with no assigned vector or direction. This makes its meaning obscure which leads to the next point.

The non-written norms of Geometric Algebra:

- The Geometric Algebra has all the elements needed to substitute and eliminate the necessity of imaginary numbers. One of the advantages of Geometric Algebra is that. The imaginary numbers are not imaginary anymore, they are objects (bivectors, trivector) which square is -1. And the imaginary number was a way of filling this hole while these elements were not known, see [6] for example. The paravector issue in APS has led to creating Algebras with imaginary numbers again in Geometric Algebra. This should be avoided.
- Matrices and tensors: The elements of Geometric Algebra have its own way of operating products between them with the geometric product and its properties, so no Matrices or Tensors are needed any more. The Geometric Algebra has come to substitute them.

Another point about time being the trivector, is that sometimes really time and volume are related. We have seen it equation (18) for example, where dividing by a volume you a get as a trivector (time direction) what it is considered the value of a charge through time (the element 4 (called also 0) in a 4-Current vector). It seems like the time is somehow measuring (or related) to the continuous creation of space, see [8] for example.

In [2][5][6][7][8] you have more insights regarding Geometric Algebra.

Anyhow, any effort going towards Geometric Algebra is very positive. It is not my intention to start a war between different Geometric Algebras, just to comment some points.