Legendre's conjecture

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This article aims to bring attention to a particular arrangement of odd numbers in which zero represents the central value. In this way, the numbers to the right and left of zero are symmetrical. In this arrangement, there are always two numbers *-1* and *1* which are not occupied. What we want to show is that there is no arrangement of length *2n+1* where the prime numbers $P \leq n$ occupy more positions. This would mean proof of Legendre's conjecture.

Legendre's conjecture states that there is always a prime number between n^2 and $(n+1)^2$.

Since the difference between $(n+1)^2$and n^2 is equal to *2n+1*, Legendre's conjecture tells us there is no arrangement of length *2n+1* where the prime numbers $P \leq n$ occupy all the positions.

The interval of length *2n+1* (n odd), can be represented as follows:

-n –n+1……………-1 0 1 …………………..n-1 n

By removing the even numbers, we get an interval equal to n+1:

-n –n+2……………-1 1 …………………..n-2 n

For example, with *n=7* we have:

-7 -5 -3 -1 1 3 5 7

This proof is based on showing that this way of arranging odd numbers, less than and equal to *n*, is the arrangement that occupies the largest number of positions in an interval of length *n+1*. Consequently, if this hypothesis is true, Legendre's conjecture is correct.

To understand the importance of this arrangement, let's analyze the following situation: we have a prime number *P* and we must position it within an interval of *P+1* consecutive positions trying to occupy the greatest number of positions.

So, for example, if *P=3* the interval has *4* positions:

1 2 3 4

If we put the prime number *3* in position *1*, we also occupy position *4* (the occupied positions are in yellow).

1 2 3 4

If we place the prime number *3* in position *2*, we occupy only position *2*, because the multiple of *3* goes out of range.

1 2 3 4

Same thing if the *3* is placed in position *3*.

1 2 3 4

Therefore, given a prime number *P* and an interval of length *P+1*, the only way to occupy two positions is to place the prime number in the first position. In this way, the first and last positions will be occupied. In all other cases, the multiple of the prime number will go out of range.

If we take *P=11* and use the starting arrangement we get:

-11 -9 -7 -5 -3 -1 1 3 5 7 9 11

For every odd number *D* less than *P* we can define the following intervals of length *D+1*.

-11 -9 -7 -5 -3 -1 1 3 5 7 9 11 *D=3*

-11 -9 -7 -5 -3 -1 1 3 5 7 9 11 *D=5*

-11 -9 -7 -5 -3 -1 1 3 5 7 9 11 *D=7*

-11 -9 -7 -5 -3 -1 1 3 5 7 9 11 *D=9*

-11 -9 -7 -5 -3 -1 1 3 5 7 9 11 *D=11*

We note how in this arrangement the two unoccupied positions (the *-1* and *1*) are positioned centrally for each sub-interval of length *D+1*.

At this point, if we want to occupy any empty position, the odd number *D* that will be moved must leave the position that occupies two places in the length interval *D+1* highlighted in yellow. Consequently, within the yellow range, the free positions will increase.

In conclusion, if we move an odd number *D*, at best we lose a place within the yellow range and earn two places outside the yellow range of length *D+1*.

Example *n=7*:

-7 -5 -3 -1 1 3 5 7

Let us consider the prime number *3*, in yellow the length interval *3+1*.

Position	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
	-7	*-5*	*-3*	*-1*	*1*	*3*	*5*	*7*

In this case, *3* occupies positions *3* and *6,* in this arrangement *3* occupies the largest number of positions in the yellow range but does not occupy the maximum number of positions in the total range. Indeed, if we move the *3* to position *2*, the *3* occupies *3* positions in the total interval (positions *2*, *5* and *8*).

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*
	3			*3*			*3*

However, he lost one position in the yellow range where he now occupies only one position, number *5*.

The two places earned being outside the yellow range are not useful, because they overlap with positions already occupied. Since the two free positions *-1* and *1* are within all the *2D+1* size ranges highlighted in yellow, they are already in an area where the odd numbers *D* are already positioned to occupy the greatest number of positions.

So the only way to occupy all the positions is that an odd number *D* occupies more than two places within an interval of *D+1* consecutive positions, since this is impossible, the *2n+1* positions cannot be occupied by the numbers odd minor and equal to *n*. Consequently, Legendre's conjecture is true.

In this article "The importance of finding the upper bounds for prime gaps in order to solve the twin primes conjecture and the Goldbach conjecture", I show how this technique, with some modifications, can also be used to prove the twin primes conjecture and the Goldbach conjecture.

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