ON THE DISTRIBUTION OF PERFECT NUMBERS AND RELATED SEQUENCES VIA THE NOTION OF THE DISC

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Abstract. In this paper we investigate some properties of perfect numbers and associated sequences using the notion of the disc induced by the sum-of-the-divisor function σ. We reveal an important relationship between perfect numbers and abundant numbers.

1. **Introduction**

Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ denotes the sum-of-divisor function, defined as

$$\sigma(N) := \sum_{n|N} n$$

for a fixed $N \in \mathbb{N}$. We say N is a perfect number if and only if $\sigma(N) = 2N$. If N is even then N is called an even perfect number. On the other hand, if N is perfect and is odd then we say it is an odd perfect number. It is still unknown if there exist any odd perfect numbers and the problem for asserting their existence or non-existence still remains an active area of research. Much work has already been done in this area and most subtle and basic properties about odd perfect - if they exist - are now known. The eighteenth century mathematician Leonard Euler was the first to show that if any odd perfect number N exists then it must be of the form

$$N := q^{\beta} \prod_{i=1}^{n} p_i^{\alpha_i}$$

where $q, \beta \equiv 1 \pmod 4$ and $\alpha_i \equiv 0 \pmod 2$ for each $1 \leq i \leq n$. He also showed that all even perfect numbers must be of the form $2^{p-1}(2^p - 1)$ provided $2^p - 1$ is a prime number for a $p \in \mathbb{P}$. It is also know that, if an odd perfect number N exists then it must satisfy the inequality $N > 10^{1500}$ [1]. It is also known that (see [2]) an odd perfect number must not be divisible by 105 and must satisfy the congruence conditions (see [3])

$$N \equiv 1 \pmod{12} \quad \textbf{and} \quad N \equiv 117 \pmod{468} \quad N \equiv 81 \pmod{324}.$$

If there are k of the exponents α_i in the prime factorization of N with $\alpha_i \equiv 0 \pmod 2$, then it is known that the smallest prime factor of N is at most $\frac{k-1}{2}$ [4]. In this case, it has been shown that (see [5])

$$N < 2^{4^{k+1} - 2^{k+1}}$$

Date: October 24, 2022.
2010 *Mathematics Subject Classification.* Primary 11Pxx, 11Bxx; Secondary 11Axx, 11Gxx.
Key words and phrases. disc; degenerative; divisor; perfect; numbers.

and with $q\prod_{i=1}^{k}p_i < 2N^{\frac{17}{26}}$ [6]. The scale of the largest and the second largest prime factor of an odd perfect number - if they exist - has also been studied quite extensively in a series of papers by several authors. It is now known that the largest prime factor of N is greater than 10^{18} (see [7]) and less than $(3N)^{\frac{1}{3}}$ [8]. It has also been shown that the second largest prime factor of an odd perfect number N must be greater than 10^4 and less than $(2N)^{\frac{1}{5}}$ [9]. The third largest prime factor is now known to be greater than 100. All of these result could conceivably be synthesized in a nice way to study the main question of the existence or non-existence of an odd perfect number.
We say a number N is deficient if $\sigma(N) < 2N$ and, respectively, abundant if $\sigma(N) > 2N$. In this paper, by using the notion of the disc induced by arithmetic functions, we expose a subtle connection between perfect numbers and abundant numbers.

2. The notion of the disc induced by arithmetic functions and applications

In this section we introduce and study the notion of the disc induced by arithmetic functions. We find this notion suitable for studying the distribution of perfect numbers and associated sequences.

Definition 2.1. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ and let $a, r \in \mathbb{N}$ be fixed. Then by the disc induced by f with center a and radius r, denoted $\mathcal{D}_f(a, r)$, we mean

$$\mathcal{D}_f(a, r) := |f(m) - a| \leq r$$

for $m \in \mathbb{N}$. We say $s \in \mathcal{D}_f(a, r)$ if and only if $|f(s) - a| \leq r$. We say the disc induced is **degenerative** if there exists some $t \in \mathcal{D}_f(a, 0)$ and we call $\mathcal{D}_f(a, 0)$ the degenerated disc. Otherwise we say the disc induced is non-degenerative. We say the disc induced is **uniformly** degenerative if it is degenerative for all $a \in \mathbb{N}$.

Proposition 2.2. *The following properties hold*

(i) *Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be multiplicative and $s = uv$ with $(u, v) = 1$ with $u, v > 1$. If $s \in \mathcal{D}_g(a, r)$ for a fixed $r, a \in \mathbb{N}$ and*

$$a < \frac{g(u) + g(s)}{2} \quad a < \frac{g(v) + g(s)}{2}$$

then $u \in \mathcal{D}_g(a, r - \epsilon)$ and $v \in \mathcal{D}_g(a, r - \delta)$ for some $\epsilon, \delta > 0$.

(ii) *Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be additive and $s = uv$ with $(u, v) = 1$ with $u, v > 1$. If $s \in \mathcal{D}_g(a, r)$ for a fixed $r, a \in \mathbb{N}$ and*

$$a < \frac{g(u) + g(s)}{2} \quad a < \frac{g(v) + g(s)}{2}$$

then $u \in \mathcal{D}_g(a, r - \epsilon)$ and $v \in \mathcal{D}_g(a, r - \delta)$ for some $\epsilon, \delta > 0$.

Proof. We only prove property (i) since the same approach could be adapted for property (ii). Let $s \in \mathcal{D}_g(a, r)$ and write $s = uv$ such that $(u, v) = 1$ with $u, v > 1$. Then since g is multiplicative and

$$a < \frac{g(u) + g(s)}{2} \quad a < \frac{g(v) + g(s)}{2}$$

we can write

$$|g(u)-a| < |g(s)-a| = |g(u)g(v)-a| \leq r$$

so that there exists some $\epsilon > 0$ such that $|g(s)-a| = |g(u)-a| + \epsilon \leq r$ and it follows that $u \in \mathcal{D}_g(a, r-\epsilon)$. It follows similarly that there exists some $\delta > 0$ such that $v \in \mathcal{D}_g(a, r-\delta)$. □

Remark 2.3. Now we verify an important but yet trivial preparatory observation for asserting the truth of our main result. It conveys the principal notion that no degenerated disc induced by an arithmetic function will ever contain a composite.

Proposition 2.4. *Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be multiplicative (resp. additive). If $s = uv$ with $(u, v) = 1$ and $u, v \geq 3$ such that at least one of the following holds*

$$a < \frac{g(u)+g(s)}{2} \quad a < \frac{g(v)+g(s)}{2}$$

then $s \notin \mathcal{D}_g(a, 0)$.

Proof. Let $s = uv$ such that $(u, v) = 1$ with $u, v \geq 3$ and assume to the contrary that $s \in \mathcal{D}_g(a, 0)$. Since g is multiplicative, let us assume at least one of the following holds

$$a < \frac{g(u)+g(s)}{2} \quad a < \frac{g(v)+g(s)}{2}.$$

Then it follows from Proposition 2.2 that at least one of the following holds

$$u \in \mathcal{D}_g(a, -\epsilon) \quad v \in \mathcal{D}_g(a, -\delta)$$

for some $\epsilon, \delta > 0$. This is impossible since the radius of each of the degenerated disc is negative, thereby ending the proof. □

Theorem 2.5. *If N is a perfect number, then for any $\epsilon > 0$ there exists $l = Nd$ $(d \geq 1)$ with $(N, d) = 1$ such that*

$$l \in \mathcal{D}_\sigma(2l, \epsilon).$$

Otherwise $\sigma(l) > 2l$

Proof. Suppose N is a perfect number and let $l = Nd$ for $d \geq 2$ with $(N, d) = 1$. Assume to the contrary that

$$l \notin \mathcal{D}_\sigma(2l, \epsilon) \quad \textbf{and} \; \sigma(l) - 2l < 0$$

since $l \notin \mathcal{D}_\sigma(2l, \epsilon) \implies \sigma(l) \neq 2l$ and choose $\epsilon > 0$ such that

$$0 < l - \frac{\sigma(l)}{2} \leq \epsilon < 2l - \sigma(l) \tag{2.1}$$

It follows from (2.1) the inequality

$$\frac{\sigma(l)}{2} < l - \frac{\epsilon}{2} \leq \frac{l}{2} + \frac{\sigma(l)}{4}. \tag{2.2}$$

Since N is a perfect number and $l = Nd$ with $(N, d) = 1$ and σ is a multiplicative function, it follows that

$$\frac{\sigma(N)\sigma(d)}{4} < \frac{Nd}{2} \implies \sigma(d) < d$$

which is impossible for all $d \geq 1$. □

Theorem 2.5 tells us something subtle about the distribution of perfect numbers and abundant numbers. It tells us that most abundant numbers can be constructed from knowing a perfect numbers. In other words, it may be possible to construct an abundant numbers by using perfect numbers as a building block. In the worst case scenario, we may also use the constructive regime to produce other perfect numbers of large magnitude.

Corollary 2.6. *If N is a perfect number, then there exists an $l \in \mathbb{N}$ with $l = Nd$ and $(N, d) = 1$ such that $\sigma(l) \geq 2l$.*

Proof. This is an easy consequence of Theorem 2.5. □

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