Proof of the Goldbach Conjecture

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Statement of the Conjecture: << Every even natural integer is the sum of two prime integers>>.

Démonstration:

Let n be an even integer.

Let Pn denote the set of all prime factors less than n defined as follows:

Pn = {1(p_1); 2(p_2); ………; pm} these p_i are listed in ascending order:

$p_1 < p_2 < \ldots\ldots < p_{m-1} < p_m$.

So p_m is the largest prime factor less than n, in other words there is no more prime factor between p_m and n.

We therefore have $\forall\ p_i \in Pn$, $p_i \leq p_m < n$ and $n-p_i < n$.

The contrapositive of the Goldbach conjecture is as follows:

<< There exists an even natural number which is not equal to any sum of two prime numbers >>: Supposition 1 → S1 .

We are therefore going to study this contrapositive for this n.

The meaning of this contrapositive with what was previously defined is:

≪$\forall\ p_i \in Pn,\ n\text{-}p_i \notin Pn$ ≫.

We have $p_1 < p_2 < \ldots\ldots\ldots < p_{m-1} < p_m$ therefore

$n\text{-}p_m < n\text{-}p_{m-1} < \ldots\ldots\ldots < n\text{-}p_2 < n\text{-}p_1$.

Suppose then that $\forall p_i \in Pn,\ p_1 < n\text{-}p_i < p_m$: supposition 2 → S2.

So for $p_i = p_m$ we get $p_1 < n\text{-}p_m < p_m$,

$p_1 < n\text{-}p_m \Rightarrow p_m < n\text{-}p_1$ Contradiction with $n\text{-}p_i < p_m$

This assumption S2 is therefore false (S2 therefore closed).

And then $\exists\ p_{j/n} \in Pn$ such that $n\text{-}\ p_{j/n} \leq p_1$ or $n\text{-}\ p_{j/n} \geq p_m$,

And since $n\text{-}\ p_{j/n} \notin Pn$ then $n\text{-}\ p_{j/n} \neq p_1 = 1$.

We therefore only have the case where $n\text{-}\ p_{j/n} \geq p_m$ more exactly $n\text{-}\ p_{j/n} > p_m$ because $n\text{-}\ p_{j/n}$ is not prime(S1) and p_m is prime.

$n\text{-}\ p_{j/n} > p_m \Rightarrow n\text{-}\ p_m > p_{j/n}$ and therefore

$n\text{-}p_m > p_{j/n} > p_{(j/n)-1} > p_{(j/n)-2} > \ldots\ldots\ldots > p_2 > p_1$.

We will continue this analysis with the largest prime factor pj/n which allows the inequality $n-p_{j/n} > p_m$.

We will then have $n- p_{j/n} > p_m$ and $n- p_{(j/n)+1} < p_m$

With $p_{(j/n)+1}$ the prime factor following the prime factor $p_{j/n}$.

(we can write $p_{(j/n)+1}$ or $p_{(j+1)/n}$; $p_{(j/n)+i}$ or $p_{(j+i)/n}$).

As $p_{j/n} > p_{(j/n)-1} > p_{(j/n)-2} > \ldots\ldots\ldots > p_2 > p_1$ and

$p_{(j/n)+1} < p_{(j/n)+2} < \ldots\ldots\ldots < p_{m-1} < p_m$ we then obtain:

$\rightarrow \forall\ p_k \leq p_{j/n}$, $n- p_k > p_m$ because $n- p_k \geq n- p_{j/n} > p_m$ (which indicates the non-primality of $n- p_k$ for $p_k \leq p_{j/n}$ because there is no prime factor between p_m and n) And

$\rightarrow \forall\ p_k \geq p_{(j+1)/n}$, $n- p_k < p_m$ or $p_1 < n-p_k < p_m$ because $n- p_k < n-p_{(j/n)+1} < p_m$

We will therefore first show the existence of $p_{j/n}$:

$p_{j/n}$ was defined as follows:

$\exists\ p_{j/n} \in Pn$ such that $n- p_{j/n} > p_m$ and $n- p_{(j/n)+1} < p_m$ with $p_{(j/n)+1}$ the prime factor following the prime factor $p_{j/n}$.

Then suppose the opposite: $\forall\ p_j \in Pn$, $n-p_j > p_m$: Assumption 3 $\rightarrow$ S3

So for $p_j=p_m$ we then obtain $n- p_m > p_m \Rightarrow n > 2p_m$

As $n > p_m$ then $p_m < 2p_m < n$.

However, according to Chebychev's theorem, there is always a prime number between q and 2q (with q natural integer >1) and since there is no prime factor between p_m and n then there is no also a prime factor between p_m and $2p_m$ which contradicts the theorem of Tchebychev, we then deduce that $\exists\ p_{j/n} \in Pn$ such that $n- p_{j/n} > p_m$ the assumption S3 is therefore closed.

And at the same time we have just proved that $n- p_m < p_m$.

On the other hand,

→ If $n- p_{m-1} < p_m$ then $n- p_{m-1} < n- p_{j/n}$ because $n- p_{m-1} < p_m < n- p_{j/n} \Rightarrow p_{j/n} < p_{m-1}$ and therefore $p_{j/n}$ is included between p_1 and p_{m-1} and then $p_{(j/n)+1}$ is between p_2 and p_m.

→ If $n- p_{m-1} > p_m$ then we have $n- p_m < p_m$ and $n- p_{m-1} > p_m$ so $p_{j/n}= p_{m-1}$ and $p_{(j/n)+1}=p_m$.

We have therefore just demonstrated the existence of $p_{j/n}$ and $p_{(j/n)+1}$.

We therefore have $\forall\ p_i \in Pn$ such that $p_i \geq p_{(j/n)+1}$

$n- p_{j/n} > p_m > p_i \Rightarrow n- p_{j/n} > p_i \Rightarrow n- p_i > p_{j/n}$

and more particularly $\forall\ p_i \geq p_{(j/n)+1}$, $p_{j/n} < n-p_i < p_m$.

Let us then study the distribution of these $n-p_i$ between $p_{j/n}$ and p_m:

Let p_i be between $p_{j/n}$ and p_m, $\exists!\ p_{i1} > p_{j/n}$ and $\exists!\ p_{i2} > p_{j/n}$ such that p_{i1} and p_{i2} are successive prime factors with $p_{i1} < n- p_i < p_{i2}$, strictly because $n- p_i$ is not prime and p_{i1} and p_{i2} are prime (with $p_1 < n- p_i < p_m$).

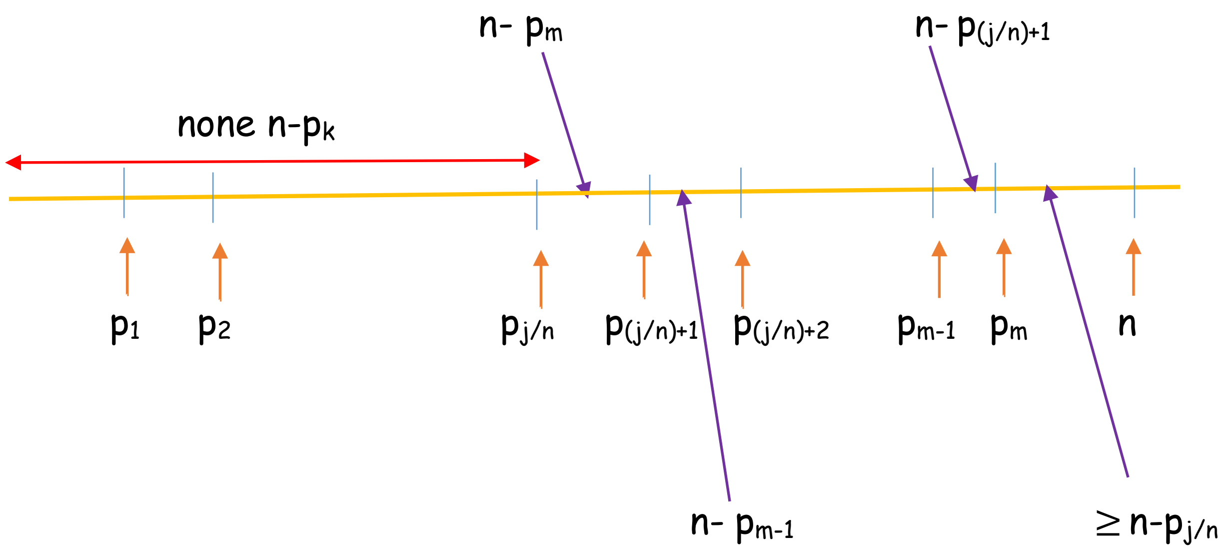
We have $p_{i1} < n- p_i < p_{i2} \Rightarrow p_i < n- p_{i1}$ and $p_i > n- p_{i2} \Rightarrow n- p_{i2} < p_i < n- p_{i1}$

More over for the n- p_k, n- p_{i2} and n- p_{i1} are also successive because

$p_{i1} < p_{i2} < p_{i3} < \ldots \Rightarrow \ldots < n- p_{i3} < n- p_{i2} < n- p_{i1} < \ldots$

this shows two different n- pk cannot belong to the same interval composed by two successive prime factors since there is a prime factor between the two n- p_k ($n- p_{i2} < p_i < n- p_{i1}$)

Let us then schematize this distribution on the following graduated line:



In effect,

$n- p_{j/n} > p_m \Rightarrow n- p_m > p_{j/n}$

and $n- p_{(j+1)/n} < p_m \Rightarrow n- p_m < p_{(j+1)/n}$ whence $p_{j/n} < n- p_m < p_{(j+1)/n}$

the number of p_k between p_m and $p_{(j+1)/n}$ is equal to m-(j+1)+1=m-j.

And the number of n-p_k (with p_k between $p_{(j+1)/n}$ and p_{m-1} because p_m is already used between $p_{j/n}$ and $p_{(j+1)/n}$) is equal to (m-1) -(j+1)+1=m-j-1 which corresponds exactly to the number of intervals between $p_{(j+1)/n}$

and p_m, and since we have $n- p_{m-1} < n- p_{m-2} < \ldots\ldots < n- p_{(j+1)/n}$ then we have exactly the following distribution:

$p_{(j+1)/n} < n-p_{m-1} < p_{(j+2)/n}$; $p_{(j+2)/n} < n- p_{m-2} < n- p_{(j+3)/n}$ .... And $p_{m-1} < n- p_{(j+1)/n} < p_m$.

Because we had demonstrated that between two successive prime factors ($\geq p_{(j+1)/n}$) there is a unique $n-p_k$ ($p_k \geq p_{(j+1)/n}$)

Similarly $n- p_{j/n} > p_m \Rightarrow p_m < n- p_{j/n} < n$,

So all the $n-p_k$ ($p_k \leq p_{j/n}$: $n- p_{j/n} < n-p_k$) are beyond p_m, which confirms the distribution of the $n-p_i$ on the graduated ruler drawn above.

So let's recap all of the above:

→ $\forall$ p_i between $p_{(j+1)/n}$ and p_m we have $p_{j/n} < n- p_i < p_m$

→ Between two successive prime factors greater than $p_{j/n}$, there is a unique $n-p_k$ with $p_{j/n} \leq p_k \leq p_m$.

On the other hand,

We have $\forall$ p_k (between $p_{j/n}$ and p_{m-1}) and $\forall$ p_{k+1} (between $p_{(j+1)/n}$ and p_m), successive prime factors, p_k and p_{k+1} cannot be twin primes ($p_{k+1} - p_k = 2$) because if it was then the only integer that exists between p_k and p_{k+1} is $p_k + 1$ and since $pk < n-p_i < p_{k+1}$ then $n-p_i = p_k + 1$

Which is impossible because $n-p_i$ is odd and $p_k + 1$ is even

From where $\forall\ p_k \geq p_{j/n}$, p_k and p_{k+1} cannot be twin primes.

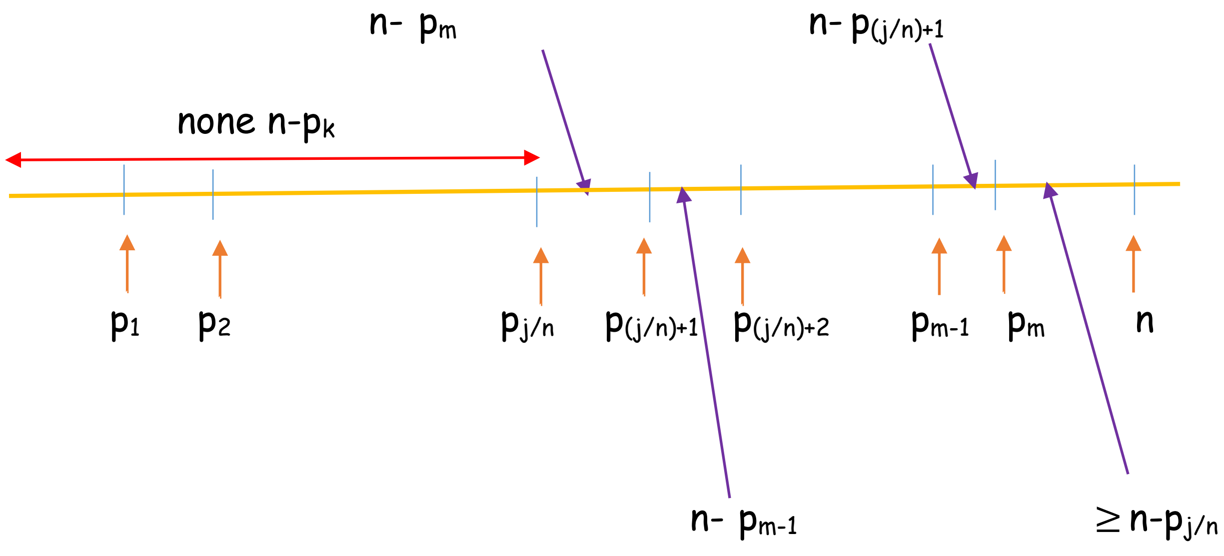
So the only twin primes are those between p_1 and $p_{j/n}$.

Let's recap:

→ $n \in Ep$ (set of even integers), $\forall\ p_i \in Pn$, $n\text{-}p_i \notin Pn$.

→ Between each p_k and p_{k+1} (with k between j/n and m-1) there is a unique $n\text{-}p_i$ (with i between (j+1)/n and m).

Schematized on the following graduated line:



→ There are no twin primes between $p_{j/n}$ and p_m.

→ The only twin primes exist between p_1 and $p_{j/n}$.

We will study later the even numbers between p_m and n,

The first integer (increasing direction) in this case is $p_m + 1$, but it is equal to the sum of two prime numbers, likewise for $p_m + 3$; p_m+5; p_m+7.

We will then reason on the numbers of the form $p_m + 2k+1$ which are the even numbers between strictly p_m and n, with 2k+1 not prime because the numbers of the form $p_m + 2k+1$ with 2k+1 prime meet the criterion: sum of two prime numbers ($\exists$ k1 $\in \mathbb{N}$ such that $n= p_m + 2k1 + 1$).

The numbers of the form $p_m + 2k$ are odd which does not interest us in our case.

The first number such that $p_m + 2k+1$ even and 2k+1 not prime is the number $p_m+ 9$ which we will note n1.

Note that $P_{n1} = P_n$ because there is no longer a prime factor between pm and n ($p_m(n1) = p_m(n)$).

Suppose then that n1 is not equal to any sum of two prime factors of Pn, we will then adopt the same reasoning as for n, where $\exists\ p_{j/n1} \in Pn$ such that $p_{j/n1} < n1 - p_m < p_{(j+1\,)/n1} \Rightarrow p_{j/n1} < p_m + 9 - p_m < p_{(j+1)/n1} \Rightarrow$

$p_{j/n1} < 9 < p_{(j+1)/n1} \Rightarrow$

$p_{j/n1} = 7$ and $p_{(j+1)/n1} = 11$

But we had demonstrated as for n that there are no twin primes between $p_{j/n1}$ and pm whereas in this case there are several twin primes beyond $p_{j/n1} = 7$, contradiction ⇒ $n1 = p_m + 9$ is written as the sum of two prime factors.

Ditto for the second number such that $p_m + 2k+1$ even and 2k+1 not prime, this number is equal to $n2 = p_m + 15$ ($P_{n2} = P_{n1} = Pn$) ⇒ $p_{j/n2} = 13$ and $p_{(j+1)/n2} = 17$ and since there are several twin primes beyond $p_{j/n2} = 13$ then a contradiction and therefore $p_m + 15$ is written as the sum of two prime factors.

Same for $n3 = p_m + 21$ ⇒ $p_{j/n3} = 19$ and $p_{(j+1)/n3} = 23$ and so on….

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