Lagrange multipliers and adiabatic limits I

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Abstract

Critical points of a function subject to a constraint can be either detected by restricting the function to the constraint or by looking for critical points of the Lagrange multiplier functional. Although the critical points of the two functionals, namely the restriction and the Lagrange multiplier functional are in natural one-to-one correspondence this does not need to be true for their gradient flow lines. We consider a singular deformation of the metric and show by an adiabatic limit argument that close to the singularity we have a one-to-one correspondence between gradient flow lines connecting critical points of Morse index difference one. We present a general overview of the adiabatic limit technique in the article [FW22b].

The proof of the correspondence is carried out in two parts. The current part I deals with linear methods leading to a singular version of the implicit function theorem. We also discuss possible infinite dimensional generalizations in Rabinowitz-Floer homology. In part II [FW22a] we apply non-linear methods and prove, in particular, a compactness result and uniform exponential decay independent of the deformation parameter.

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1 Introduction

In 1806 it was the observation of Joseph Louis de Lagrange [dL06] that critical points of a function $F(x)$ subject to a constraint $H(x) = 0$ correspond to critical points of the unconstrained function $F_H(x, \tau) = F(x) + \tau H(x)$ which also depends on a Lagrange multiplier τ. More precisely, suppose that M is a finite dimensional manifold, not necessarily symplectic, but equipped with a Riemannian metric G. Let F and H be smooth functions on M such that zero is a regular value of H. Thus $\Sigma := H^{-1}(0)$ is a smooth level hypersurface. Under these assumptions there is a bijection between the critical point sets of the following two functions, namely the **Lagrange multiplier functional**

$$F_H \colon M \times \mathbb{R} \to \mathbb{R}, \quad (u, \tau) \mapsto F(u) + \tau H(u),$$

and the **restriction function** of F to the constraint $\Sigma = H^{-1}(0)$, in symbols

$$f \colon \Sigma \to \mathbb{R}, \quad q \mapsto F(q).$$

The natural bijection is by forgetting the first factor, in symbols

$$\mathrm{Crit}\, F_H \to \mathrm{Crit} f, \quad (x, \tau) \mapsto x. \tag{1.1}$$

The Morse indices differ by 1, namely

$$\mathrm{ind}_{F_H}(x, \tau) = \mathrm{ind}_f(x) + 1.$$

In particular, the difference of the Morse indices at two critical points is independent of the choice of function F_H or f.

Under local properness conditions it was shown in [Fra06] that the Morse homologies of the two functions coincide up to an index shift by 1, namely $\mathrm{HM}_*(F_H) \simeq \mathrm{HM}_{*+1}(f)$. Therefore the Lagrange multiplier function computes the homology of Σ up to a grading shift by 1. The proof of this fact in [Fra06] uses normal deformations of the function F and is hard to generalize to infinite dimensions. Therefore we focus in the present paper on a completely different approach to this homology equivalence which, as well, is much stronger since it gives an isomorphism on chain level and not just on homology level. This approach is based on the adiabatic limit technique developed by Dostoglou and Salamon [DS94] in their proof of a special case of the Atiyah-Floer conjecture. The technique was successfully used and developed further in the context of symplectic vortex equations [Gai99, GS05] and the heat flow [Web99, SW06].

In the context of Lagrange multipliers this adiabatic limit technique works as follows. Pick a parameter $\varepsilon \in (0, 1]$. Then the gradient flow equation of F_H with respect to the product metric $\mathrm{G} \oplus \varepsilon^2$ on $M \times \mathbb{R}$ is given by

$$\partial_s(u, \tau) + \nabla^\varepsilon F_H(u, \tau) = \begin{pmatrix} \partial_s u + \overline{\nabla} F|_u + \tau \overline{\nabla} H|_u \\ \tau' + \varepsilon^{-2} H \circ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.2}$$

for smooth maps $(u, \tau) \colon \mathbb{R} \to M \times \mathbb{R}$ and where ∇^ε is the gradient in the Riemannian manifold $(M \times \mathbb{R}, G \oplus \varepsilon^2)$ and $\overline{\nabla}$ is the gradient in (M, G).

Letting ε formally go to zero one obtains the pair of equations

$$\begin{pmatrix} \partial_s u + \overline{\nabla} F|_u + \tau \overline{\nabla} H|_u \\ H \circ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.3}$$

Equation two tells that u actually takes values in $\Sigma = H^{-1}(0)$. In this case equation one is the downward gradient equation on Σ of the restriction f of F and with respect to the Riemannian metric g given by restricting G (Lemma 3.1).

The main result of part I is the following theorem. Suppose that $x^\mp \in \mathrm{Crit} f$ are critical points of Morse index difference one. Then for each $\varepsilon \in (0, \varepsilon_0]$ we construct a time shift invariant map $\mathcal{T}^\varepsilon \colon \mathcal{M}^0_{x^-,x^+} \to \mathcal{M}^\varepsilon_{x^-,x^+}$ between moduli spaces of gradient flow trajectories $q \colon \mathbb{R} \to \Sigma$ and $(u, \tau) \colon \mathbb{R} \to M \times \mathbb{R}$ which at $\mp\infty$ converge to the critical points $x^\mp$, respectively to $(x^\mp, \tau^\mp) \in \mathrm{Crit} F_H$.

Theorem A. *Assume (f, g) is Morse-Smale. Then there is a constant $\varepsilon_0 \in (0, 1]$, such that for every $\varepsilon \in (0, \varepsilon_0]$ and every pair $x^\mp \in \mathrm{Crit} f$ of index difference one, the map $\mathcal{T}^\varepsilon \colon \mathcal{M}^0_{x^-,x^+} \to \mathcal{M}^\varepsilon_{x^-,x^+}$ is injective.*

To prove the theorem we associate to $q \in \mathcal{M}^0_{x^-,x^+}$ a suitable pair (q, τ) which almost solves the ε-equation (1.2). Then we use the Newton method to find a unique true solution nearby. This is the content of part I (this article).

In part II [FW22a] we shall prove surjectivity by contradiction. If $\mathcal{T}^\varepsilon$ is not surjective for $\varepsilon > 0$ small, there is a sequence of positive reals $\varepsilon_i \to 0$ and a sequence $(u_i, \tau_i) \in \mathcal{M}^{\varepsilon_i}_{x^-,x^+}$ not in the image of $\mathcal{T}^{\varepsilon_i}$. We show that the maps u_i take values near Σ and that they naturally project to maps $\mathfrak{q}_i \colon \mathbb{R} \to \Sigma$ which are almost solutions of the base equation (1.3). We identify true solutions $q_i \colon \mathbb{R} \to \Sigma$ nearby and show that after suitable time shift $\sigma_i \in \mathbb{R}$ we have $(u_i, \tau_i) = \mathcal{T}^{\varepsilon_i}(q_i(\sigma_i + \cdot))$. This contradiction proves surjectivity.

Convention 1.1 (Notation)**.**
a) Tangent and normal bundle of Σ in M are denoted by $T\Sigma \oplus N\Sigma = T_\Sigma M$. Tangent vectors to M based at Σ decompose $X = \xi + \nu = \tan X + \mathrm{nor}\, X$. The dimension of Σ is n, hence $n + 1 = \dim M$.
b) Arguments of maps $H(u)$ are likewise denoted by $H|_u$.
c) For $u \colon \mathbb{R} \to M$, $q \colon \mathbb{R} \to \Sigma$, $\tau \colon \mathbb{R} \to \mathbb{R}$ we often de-parenthesify and write

$$u_s := u(s), \quad q_s := q(s), \quad \tau_s := \tau(s),$$

and

$$\partial_s u := \tfrac{d}{ds} u, \quad \partial_s q := \tfrac{d}{ds} q, \qquad \text{but } \tau' := \tfrac{d}{ds} \tau.$$

d) The symbol $|\cdot|$, applied to real numbers means absolute value, applied to vectors it means vector norm, for example $|\partial_s u| := |\partial_s u|_G$ on (M, G) and $|\partial_s q| := |\partial_s q|_g$ on (Σ, g). Throughout $\|\cdot\|$ denotes L^2-norm.
e) Inner products are denoted by $\langle \cdot, \cdot \rangle$. Depending on context $\langle \cdot, \cdot \rangle$ abbreviates $\langle \cdot, \cdot \rangle_g$ on $T\Sigma$, $\langle \cdot, \cdot \rangle_G$ on TM, $\langle \cdot, \cdot \rangle_2$ on an L^2 space, or other inner products.

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1.1 Outline

Let (M, G) be a Riemannian manifold. Let F and H be smooth functions on M. The **Lagrange multiplier function** is defined by

$$F_H \colon M \times \mathbb{R} \to \mathbb{R}, \quad (x, \tau) \mapsto F(x) + \tau H(x).$$

Hypothesis 1.2. (i) Zero is a regular value of H. (ii) **Local properness:** There exists a constant $\kappa > 0$ such that $\Sigma_\kappa := H^{-1}[-\kappa, \kappa] \subset M$ is compact. (iii) The Riemannian metric G on M is geodesically complete.

By (i) and (ii) the zero level $\Sigma := H^{-1}(0)$ is a smooth compact hypersurface in M, we assume without boundary. By (iii) closed and bounded is equivalent to compact (Theorem of Hopf-Rinow; see e.g. [O'N83, Ch. 5 Thm. 21]). Local properness excludes that H tends to zero at infinity.

Section 2 "Lagrange multiplier function and restriction". The map

$$\iota \colon \Sigma = H^{-1}(0) \hookrightarrow M, \quad q \mapsto q = \iota(q),$$

given by inclusion induces on Σ the Riemannian metric $g := \iota^* G$ and the function $f := \iota^* F$, both given by restriction. Let $\overline{\nabla}$ be the Levi-Civita connection of (M, G) and ∇ the one of (Σ, g). In Section 2.1 we briefly recall some Riemannian hypersurface geometry of (Σ, g, ∇) in $(M, G, \overline{\nabla})$. Since 0 is a regular value of H, along $\Sigma = H^{-1}(0)$ there is an orthogonal decomposition

$$T_\Sigma M = T\Sigma \overset{\perp}{\oplus} \mathbb{R}\overline{\nabla} H, \qquad X = \xi + \nu.$$

Let tan and nor be the corresponding orthogonal projections. The function

$$\chi := -\langle \overline{\nabla} F, V \rangle, \;\; V := \tfrac{\overline{\nabla} H}{|\overline{\nabla} H|^2}, \;\; \text{along } M_{\mathrm{reg}} := \{p \in M \mid dH(p) \neq 0\} \supset \Sigma$$

has the fundamental significance that at each point of Σ the value of χ is the unique real that makes the linear combination

$$\overline{\nabla} F(q) + \chi(q) \overline{\nabla} H(q) \in T_q\Sigma, \quad q \in \Sigma$$

of the two T_qM-valued vectors $\overline{\nabla} F|_q$ and $\overline{\nabla} H|_q$ be tangent to Σ. The function χ plays a crucial role throughout this article, as hinted at by the gradient identities

$$\tan \overline{\nabla} F = \nabla f, \qquad \operatorname{nor} \overline{\nabla} F = -\chi \overline{\nabla} H, \qquad \nabla f = \overline{\nabla} F + \chi \overline{\nabla} H,$$

along Σ. The last identity translates the gradient flow of f on the **base** Σ to the terminology of the **ambience** M. The local flow $\{\varphi_r \colon \Sigma \to M_{\mathrm{reg}}\}$ generated by V near Σ transforms H to the normal form $H(\varphi_r q) = r$ in (2.13). Further important roles play the graph map of χ, called the **canonical embedding**

$$i : \Sigma \to M \times \mathbb{R}, \quad q \mapsto (q, \chi(q)) = (\iota(q), \chi(\iota(q))),$$

and the derivative $I_q\xi := di(q)\xi = (\xi, d\chi(q)\xi)$ for $q \in \Sigma$. We show that the critical point sets $i\,(\mathrm{Crit} f) = \mathrm{Crit} F_H$ are in bijection through the canonical embedding i, the inverse of the forgetful map (1.1). Then we show the Morse index identity $\mathrm{ind}_{F_H}(x,\tau) = \mathrm{ind}_f(x) + 1$ for critical points.

Section 3 "Downward gradient flows". We introduce the downward gradient flow (1.3) on the base (Σ, g), whose solutions q are called **0-solutions**. We introduce the downward gradient flow (1.2) on the product $(M \times \mathbb{R}, G \oplus \varepsilon^2)$ where the metric is deformed by a parameter $\varepsilon > 0$ and whose solutions $z = (u, \tau)$ of (1.2) are called **ε-solutions**.
We define the base energy $E^0(q)$ and the ε-energy $E^\varepsilon(u,\tau)$ for smooth maps $q\colon \mathbb{R} \to \Sigma$ and $(u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ and show the uniform **energy estimates** $E^0(q) = \|\partial_s q\|^2 \le \mathrm{osc} f$ for base flow trajectories q, but for ε-flow trajectories

$$E^\varepsilon(u,\tau) < \infty \quad \Rightarrow \quad E^\varepsilon(u,\tau) = \|\partial_s u\|^2 + \varepsilon^2 \|\tau'\|^2 \le \mathrm{osc} f := \max f - \min f.$$

Two critical points $x^\mp$ of $f\colon \Sigma \to \mathbb{R}$ are called **asymptotic boundary conditions** of a smooth map $q\colon \mathbb{R} \to \Sigma$ if $\lim_{s\to\mp\infty} q(s) = x^\mp$ and of a pair of smooth maps $(u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ if

$$\lim_{s\to\mp\infty} (u(s), \tau(s)) = \left(x^\mp, \chi(x^\mp)\right).$$

Observe that $(x^\mp, \chi(x^\mp)) \in \mathrm{Crit} F_H$. With gradient equations and asymptotic boundary conditions in place there are the usual **energy identities**

$$E^0(q) = f(x^-) - f(x^+), \qquad E^\varepsilon(u,\tau) = f(x^-) - f(x^+) =: c^*,$$

for base flow trajectories q, respectively for ε-flow trajectories (u,τ).

"A priori estimates". The following theorem, proved in part II [FW22a], provides uniform a priori bounds for ε-solutions (u,τ) and all derivatives. The theorem is fundamental for all subsequent sections and it is also rather surprising in view of the factor ε^{-2} in the deformed equations (1.2). The theorem assumes only finite energy of the ε-solutions.

Theorem 1.3 (Uniform a priori bounds for finite energy trajectories)**.** *Assume Hypothesis 1.2 with constant κ. Then there are, a compact subset $K \subset M$, and constants $c_0, c_1, c_2, c_3 > 0$, with the following significance. Assume $(u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ solves the ε-equations (1.2) and is of finite energy $E^\varepsilon(u,\tau) < \infty$.*

(i) *If $\varepsilon \in (0,1]$, then the component u takes values in K and there are bounds*

$$|\tau(s)| \le c_0, \qquad |\partial_s u(s)| + |\tau'(s)| \le c_1, \qquad \left|\overline{\nabla}_s \partial_s u(s)\right| + |\tau''(s)| \le c_2,$$

and $\left|\overline{\nabla}_s \overline{\nabla}_s \partial_s u(s)\right| \le c_3$ at every instant $s \in \mathbb{R}$.

In part II [FW22a] there is actually a part (ii) of the theorem which generalizes the fact that along the compact set Σ the gradient $|\overline{\nabla} H|$ is bounded away from zero to, roughly speaking, neighborhoods of Σ.

Section 4 “Linearized operators”. Fix $x^\mp \in \mathrm{Crit} f$. Let $\mathcal{Q}_{x^-,x^+}$ be the Hilbert manifold of $W^{1,2}$ paths $q\colon \mathbb{R} \to \Sigma$ with asymptotics $x^\mp$. The formula

$$\mathcal{F}^0(q) := \partial_s q + \nabla f|_q = \partial_s q + \overline{\nabla} F(q) + \chi(q) \cdot \overline{\nabla} H(q)$$

defines a section of the Hilbert bundle $\mathcal{L} \to \mathcal{Q}_{x^-,x^+}$ whose fiber $\mathcal{L}_q$ over q consists of the $T\Sigma$-valued L^2 vector fields along q. The zero set $\mathcal{M}^0_{x^-,x^+} := (\mathcal{F}^0)^{-1}(0)$ is called **base moduli space**, the zeroes q **connecting base trajectories**. Linearize $\mathcal{F}^0$ at a zero q to get a linear operator $W^{1,2}(\mathbb{R}, q^*T\Sigma) \to L^2$ given by

$$D^0_q \xi = \nabla_s \xi - \nabla_\xi \nabla f|_q = \overline{\nabla}_s \xi + \overline{\nabla}_\xi \big(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q\big).$$

A pair (f, g) is said **Morse-Smale** if $D^0_q\colon W^{1,2} \to L^2$ is surjective for all $q \in \mathcal{M}^0_{x^-,x^+}$ and $x^\mp \in \mathrm{Crit} f$. The **trivialization** of $\mathcal{F}^0$ at $q \in \mathcal{Q}_{x^-,x^+}$ is the map

$$\mathcal{F}^0_q\colon W^{1,2}(\mathbb{R}, q^*T\Sigma) \to L^2(\mathbb{R}, q^*T\Sigma), \quad \mathcal{F}^0_q(\xi) := \phi(q,\xi)^{-1} \mathcal{F}^0(\exp_q \xi)$$

defined for every ξ of norm smaller than the injectivity radius of (Σ, g), cf. (4.57). Here $\phi = \phi(q,\xi)\colon T_q\Sigma \to T_{\exp_q(\xi)}\Sigma$ is parallel transport, pointwise for $s \in \mathbb{R}$, along the geodesic $r \mapsto \exp_{q(s)}(r\xi(s))$ defined in terms of the exponential map of (Σ, g). The above formula for $D^0_q \xi$ makes sense for general $q \in \mathcal{Q}_{x^-,x^+}$, indeed we shall see that $d\mathcal{F}^0_q(0)\xi = D^0_q \xi$. In the formula for the **formal L^2 adjoint** $(D^0_q)^*$, see (4.43), the term $\nabla_s \xi$ changes sign, as is well known, but it is an interesting little detail that in the ambient formulation a new term II appears twice with the same sign, whereas in D^0_q the two signs were opposite. The operators D^0_q and $(D^0_q)^*$ are bounded, see (4.44). If the asymptotics $x^\mp$ are non-degenerate, then both operators are Fredholm and the Fredholm index is the Morse index difference of the asymptotics, see Proposition 4.4.

Let $\mathcal{Z}_{x^-,x^+}$ be the Hilbert manifold of $W^{1,2}$ paths $z = (u, \tau)\colon \mathbb{R} \to M \times \mathbb{R}$ with asymptotics $z^\mp = (x^\mp, \chi(x^\mp))$. For $\varepsilon > 0$ the formula

$$\mathcal{F}^\varepsilon(u, \tau) \stackrel{(3.30)}{:=} \begin{pmatrix} \partial_s u + \overline{\nabla} F|_u + \tau \overline{\nabla} H|_u \\ \tau' + \varepsilon^{-2} H \circ u \end{pmatrix}$$

defines a section of the Hilbert bundle $\mathcal{L} \to \mathcal{Q}_{x^-,x^+}$ whose fiber $\mathcal{L}_{u,\tau}$ over (u, τ) consists of the L^2 vector fields along (u, τ). The zero set $\mathcal{M}^\varepsilon_{x^-,x^+} := (\mathcal{F}^\varepsilon)^{-1}(0)$ is called **ε-moduli space**, the zeroes (u, τ) **connecting ε-trajectories**. Linearize $\mathcal{F}^\varepsilon$ at a zero to get a linear map $W^{1,2}(\mathbb{R}, u^*TM \oplus \mathbb{R}) \to L^2$ of the form

$$D^\varepsilon_{u,\tau} \begin{pmatrix} X \\ \ell \end{pmatrix} = \begin{pmatrix} \overline{\nabla}_s X + \overline{\nabla}_X \overline{\nabla} F|_u + \tau \overline{\nabla}_X \overline{\nabla} H|_u + \ell \overline{\nabla} H|_u \\ \ell' + \varepsilon^{-2} dH|_u X \end{pmatrix}.$$

For general maps $(u, \tau) \in \mathcal{Z}_{x^-,x^+}$ define $D^\varepsilon_{u,\tau}$ by the right hand side. We use the exponential map Exp of (M, G) to define, about any map $(u, \tau) \in \mathcal{Z}_{x^-,x^+}$, a **trivialization** $\mathcal{F}^\varepsilon_{u,\tau}$, see (4.48), and in (4.49) we show that $d\mathcal{F}^\varepsilon_{u,\tau}(0) = D^\varepsilon_{u,\tau}$.

To get uniform estimates with constants independent of $\varepsilon > 0$ small, we must work with ε-dependent norms suggested on L^2 by the ε-energy identity

$E^\varepsilon(u,\tau)=\|\partial_s u\|^2+\varepsilon^2\|\tau'\|^2$ and on $W^{1,2}$ by the ambient linear estimate below. For $\varepsilon>0$ define

$$\begin{aligned}
\|Z\|_{0,2,\varepsilon}^2 &:= \|X\|^2+\varepsilon^2\|\ell\|^2\\
\|Z\|_{1,2,\varepsilon}^2 &:= \|X\|^2+\varepsilon^2\|\ell\|^2+\varepsilon^2\|\nabla_s X\|^2+\varepsilon^4\|\ell'\|^2\\
\|Z\|_{0,\infty,\varepsilon} &:= \|X\|_\infty+\varepsilon\|\ell\|_\infty\le 3\varepsilon^{-1/2}\|Z\|_{1,2,\varepsilon}
\end{aligned}$$

where $Z=(X,\ell)$; cf. (4.55). The formal adjoint $(D^\varepsilon_{u,\tau})^*$ is defined via the associated $(0,2,\varepsilon)$ inner product and given by formula (4.51). For non-degenerate boundary conditions $x^\mp$ both operators $D^\varepsilon_{u,\tau}$ and $(D^\varepsilon_{u,\tau})^*$ are Fredholm (4.54).

This article, part I, focusses on pairs $(u,\tau)=(q,\chi(q))$ with $q\in\mathcal{M}^0_{x^-,x^+}$. We abbreviate (for the formulas see (6.97) and (5.71))

$$\mathcal{F}^\varepsilon_q:=\mathcal{F}^\varepsilon_{q,\chi(q)},\qquad D^\varepsilon_q:=D^\varepsilon_{q,\chi(q)},\qquad (D^\varepsilon_q)^*:=(D^\varepsilon_{q,\chi(q)})^*.$$

One of two most important linear estimates in adiabatic limit analysis is the **ambient linear estimate**

$$\varepsilon^{-1}\|dH_qX\|+\|\ell\|+\|\overline{\nabla}_sX\|+\varepsilon\|\ell'\|\le C\left(\|D^\varepsilon_qZ\|_{0,2,\varepsilon}+\|X\|\right)$$

for every $Z=(X,\ell)\in W^{1,2}(\mathbb{R},q^*TM\oplus\mathbb{R})$, see (4.60).

Section 5 "Linear estimates". The canonical embedding extends via pointwise evaluation to a map $i\colon\mathcal{Q}_{x^-,x^+}\to\mathcal{Z}_{x^-,x^+}$, $q\mapsto(q,\chi(q))$, between Hilbert manifolds. The linearization $I_q=di(q)\colon T_q\mathcal{Q}_{x^-,x^+}\to T_{i(q)}i(\mathcal{Q}_{x^-,x^+})$ is the map $\xi\mapsto(\xi,d\chi|_q\xi)$. To prepare Section 6, where we view $q\in\mathcal{Q}_{x^-,x^+}$ as an approximate zero $i(q)$ of $\mathcal{F}^\varepsilon$, see (1.4), Section 5 provides estimates for the linear operators *along the image of* i. For pairs $(q,\chi(q))$ we have nice control of the $\tau=\chi(q)$ component, because q takes values in Σ and Σ is compact.

We need to show that if the base flow is Morse-Smale, then so is the ambient ε-flow for all $\varepsilon>0$ small. Let $x^\mp\in\mathrm{Crit}f$ be non-degenerate and $q\in\mathcal{M}^0_{x^-,x^+}$ a connecting base trajectory. Theorem 5.8 provides the key estimates for D^ε_q along the image of $(D^\varepsilon_q)^*$. So the operator

$$R^\varepsilon_q:=(D^\varepsilon_q)^*\left(D^\varepsilon_q(D^\varepsilon_q)^*\right)^{-1}:L^2\stackrel{(\dots)^{-1}}{\longrightarrow}W^{2,2}\stackrel{(D^\varepsilon_q)^*}{\longrightarrow}W^{1,2}$$

is a right inverse of the linearization D^ε_q and uniformly bounded in $\varepsilon>0$ small. Uniformity of the bound is crucial for the Newton iteration to work in Section 6, it triggers the need for weighted Sobolev norms, as mentioned above.

To carry out this program one needs to compare the, by Morse-Smale, surjective base operator D^0_q with the ambient operator D^ε_q. To this end we introduce the orthogonal projection

$$\Pi^\perp_\varepsilon\colon T_{i(q)}\mathcal{Z}_{x^-,x^+}\stackrel{\pi^\perp_\varepsilon}{\longrightarrow}T_q\mathcal{Q}_{x^-,x^+}\stackrel{I_q}{\longrightarrow}T_{i(q)}i(\mathcal{Q}_{x^-,x^+})\subset T_{i(q)}\mathcal{Z}_{x^-,x^+}$$

onto the image of $I_q=di(q)$ and we show that the linear map $\pi^\perp_\varepsilon$ is given by

$$\pi_\varepsilon(X,\ell)=\left(\mathbb{1}+\varepsilon^\alpha\mu^2P\right)^{-1}\left(\tan X+\varepsilon^\beta\ell\nabla\chi|_q\right)$$

with $\alpha = \beta = 2$ and where by definition $P(q(s)) \colon T_{q(s)}\Sigma \to \mathbb{R}\nabla\chi(q(s))$ is the orthogonal projection, at each $s \in \mathbb{R}$, see (5.64). In (5.66) we show that $\|(\mathbb{1} + \varepsilon^\alpha \mu^2 P)^{-1}\| \le 1$. The linearizations are compared in the form $D^0_q \pi_\varepsilon - \pi_\varepsilon D^\varepsilon_q$. The resulting **key estimates** are of the form

$$\begin{aligned}\|Z^*\|_{1,2,\varepsilon} &\le c_1 \left(\varepsilon \|D^\varepsilon_q Z^*\|_{0,2,\varepsilon} + \|\pi_\varepsilon (D^\varepsilon_q Z^*)\|\right)\\ \|dH|_q X^*\| + \varepsilon\|\ell^*\| &\le c_1 \varepsilon \|D^\varepsilon_q Z^*\|_{0,2,\varepsilon}.\end{aligned}$$

for every pair $Z^* := (X^*, \ell^*) \in \operatorname{im}(D^\varepsilon_q)^*|_{W^{2,2}} \subset W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$. In this article the analysis works for $\alpha \in [1,2]$ and $\beta = 2$, so the orthogonal projection works. This is in sharp contrast to the PDE adiabatic limit [SW06, (139)] where the analysis did work for the non-orthogonal projection where $\alpha = 1$ and $\beta = 2$.

In [SW06] there was no analogue of the second of the above key estimates. That second estimate plays a crucial role to prove the uniqueness Theorem 6.2, see estimate after (6.105). We arrived at this new twist in the uniqueness proof by following the philosophy of Arnol′d that mathematics reveals itself through simple non-trivial examples, in our case [FW22b].

Section 6 “Implicit function theorem I – Ambience”. Suppose (f, g) is Morse-Smale and pick a base connecting trajectory $q \in \mathcal{M}^0_{x^-.x^+}$. To find an ε-solution near q we utilize Newton's iteration method which requires a map, say $\mathcal{F}^\varepsilon_q$, defined on a Banach space, so it can be iterated, and whose zeroes are in bijection with the zeroes of $\mathcal{F}^\varepsilon$. Qualitatively, three conditions need to be met. One needs, firstly, a good starting point Z_0 in the sense that its value $\mathcal{F}^\varepsilon_q(Z_0)$ is almost zero, secondly, the derivative $d\mathcal{F}^\varepsilon_q(Z_0)$ must be 'steep enough' in the sense it must admit a right inverse bounded uniformly in ε small and, thirdly, the derivative must not oscillate too wildly near Z_0 which is guaranteed via suitable quadratic estimates.

We are in good shape: The trivialized ambient section $\mathcal{F}^\varepsilon_q$ at the initial point $Z_0 := (0, 0)$ of the Newton iteration has a vanishing first component

$$\mathcal{F}^\varepsilon_q(0,0) = \mathcal{F}^\varepsilon(q, \chi(q)) := \begin{pmatrix} \partial_s q + \overline{\nabla} F(q) + \chi(q)\overline{\nabla} H(q) \\ (\chi(q))' + \varepsilon^{-2} H(q) \end{pmatrix} = \begin{pmatrix} 0 \\ d\chi|_q \partial_s q \end{pmatrix} \tag{1.4}$$

since $-\partial_s q = \nabla f(q) = \overline{\nabla} F(u) + \chi(q)\overline{\nabla} H(u)$. So $\|\mathcal{F}^\varepsilon(q, \chi(q))\|_{0,2,\varepsilon} = \varepsilon \, \|d\chi|_q \partial_s q\|$ is small for ε small. Use the right inverse to define the initial correction term

$$\zeta_0 := -D^{\varepsilon *}_q \left(D^\varepsilon_q D^{\varepsilon *}_q\right)^{-1} \mathcal{F}^\varepsilon_q(0) = -R^\varepsilon_q \mathcal{F}^\varepsilon_q(0).$$

Thus $D^\varepsilon_q \zeta_0 = -\mathcal{F}^\varepsilon_q(0) = (0, -d\chi|_q \partial_s q)$ and so by key estimate one we get

$$\begin{aligned}\|\zeta_0\|_{1,2,\varepsilon} &\le c_1 \left(\varepsilon \|(0, d\chi|_q \partial_s q)\|_{0,2,\varepsilon} + \|(\mathbb{1} + \varepsilon^2\mu^2 P)^{-1}\left(0 + \varepsilon^2 (d\chi|_q \partial_s q)\nabla\chi\right)\|\right)\\ &\le \text{const} \cdot \varepsilon^2.\end{aligned}$$

Now define $Z_1 := Z_0 + \zeta_0$ and add zero in the form $-\mathcal{F}^\varepsilon_q(0) - D^\varepsilon_u \zeta_0$ to get

$$\|\mathcal{F}^\varepsilon_q(Z_1)\|_{0,2,\varepsilon} = \|\mathcal{F}^\varepsilon_q(\zeta_0) - \mathcal{F}^\varepsilon_q(0) - D^\varepsilon_u \zeta\|_{0,2,\varepsilon} \le \text{const} \cdot \varepsilon^{5/2}$$

where the inequality uses the quadratic estimate (6.91). To the next correction term $\zeta_1 := -R_q^\varepsilon \mathcal{F}_q^\varepsilon(Z_1)$ apply the key estimate observing that $D_q^\varepsilon \zeta_1 = -\mathcal{F}_q^\varepsilon(Z_1)$. Iteration provides a Cauchy sequence Z_ν whose limit Z^ε corresponds to a zero of $\mathcal{F}_q^\varepsilon$ and $\|Z^\varepsilon\|_{1,2,\varepsilon} \leq \text{const} \cdot \varepsilon^2$. For the precise statement see the existence Theorem 6.1. The zero is unique in the sense of the uniqueness Theorem 6.2. These two theorems allow to define the map $\mathcal{T}^\varepsilon$ and the short argument in Lemma 6.4 then completes the proof of Theorem A.

1.2 Motivation and general perspective

Let (M, ω) be an exact symplectic manifold where $\omega = d\lambda$. On the free loop space $\mathcal{L}M := C^\infty(\mathbb{S}^1, M)$ consider the negative **area functional** given by

$$\mathcal{A}\colon \mathcal{L}M \to \mathbb{R}, \quad v \mapsto -\int_0^1 v^*\lambda.$$

A smooth function $H\colon M \to \mathbb{R}$, called **Hamiltonian**, induces on the loop space the corresponding **mean value functional**

$$\mathcal{H} = \mathcal{H}_H\colon \mathcal{L}M \to \mathbb{R}, \quad v \mapsto \int_0^1 H \circ v(t)\, dt.$$

On loop space there is the **time reversal involution** defined by

$$\mathcal{T}\colon \mathcal{L}M \to \mathcal{L}M, \quad v \mapsto v^-, \qquad v^-(t) := v(-t).$$

There are the following relations

$$\mathcal{A} \circ \mathcal{T} = -\mathcal{A}, \qquad \mathcal{H} \circ \mathcal{T} = \mathcal{H}. \tag{1.5}$$

The **Rabinowitz action functional** is defined by

$$\mathcal{A}_\mathcal{H}\colon \mathcal{L}M \times \mathbb{R} \to \mathbb{R}, \quad (v, \tau) \mapsto \mathcal{A}(v) + \tau\mathcal{H}(v).$$

The **extended time reversal involution** is defined by

$$\widetilde{\mathcal{T}}\colon \mathcal{L}M \times \mathbb{R} \to \mathcal{L}M \times \mathbb{R}, \quad (v, \tau) \mapsto (v^-, -\tau).$$

From (1.5) it follows the anti-invariance of the Rabinowitz action functional under extended time reversal involution, in symbols

$$\mathcal{A}_\mathcal{H} \circ \widetilde{\mathcal{T}} = -\mathcal{A}_\mathcal{H}.$$

This has the consequence that the extended time reversal involution also acts involutive on the critical point set, in symbols

$$(v, \tau) \in \mathrm{Crit}\mathcal{A}_\mathcal{H} \quad \Leftrightarrow \quad \widetilde{\mathcal{T}}(v, \tau) = (v^-, -\tau) \in \mathrm{Crit}\mathcal{A}_\mathcal{H}.$$

A critical point (v, τ) for τ positive corresponds to a periodic orbit of the Hamiltonian vector field of H of energy zero and period τ. The critical point

$\widetilde{\mathcal{T}}(v,\tau) = (v^-,-\tau)$ corresponds to this orbit traversed *backward* in time. The fixed point set $\operatorname{Fix}\widetilde{\mathcal{T}}|_{\operatorname{Crit}\mathcal{A}_{\mathcal{H}}}$ are pairs $(x,0)$ where x is a point on the energy hypersurface $\Sigma := H^{-1}(0)$ interpreted as a constant loop.

There is no analogue of the time reversal anti-invariance of the Rabinowitz action functional $\mathcal{A}_{\mathcal{H}}$ in symplectic homology or symplectic field theory where periodic orbits are always traversed in *forward* time.

From a physical perspective the time reversal anti-invariance is reminiscent of the Feynman-Stueckelberg interpretation [Stu41, Fey48] of a positron as an electron going backward in time.

From a mathematical perspective the time reversal anti-invariance of the Rabinowitz action functional has strong connections to Tate cohomology, Poincaré-duality, and Frobenius algebras. It led to the discovery by Cieliebak and Oancea [Cie] of the structure of a topological quantum field theory (TQFT) on Rabinowitz-Floer homology. However, the topological quantum field theory structure of Cieliebak and Oancea is not defined on Rabinowitz-Floer homology directly, but on V-shaped symplectic homology. The latter is known to be isomorphic to Rabinowitz-Floer homology as shown by Cieliebak, Frauenfelder, and Oancea [CFO10]. The difficulty to define the TQFT structure directly on Rabinowitz-Floer homology is that, in general, the Rabinowitz action functional does not behave additively with respect to concatenation of loops. For that reason, to our knowledge, nobody defined product structures directly on Rabinowitz-Floer homology. Instead of that, product structures were defined on homologies isomorphic to Rabinowitz-Floer homology, namely, V-shaped symplectic homology by Cieliebak and Oancea [CO18], respectively, on extended phase space by Abbondandolo and Merry [AM18].

For the following reasons we would like to see TQFT structure on Rabinowitz-Floer homology directly.

1. Time reversal anti-invariance for the functional gets lost when going over to V-shaped symplectic homology, respectively, to extended phase space homology. Therefore Poincaré-duality only holds on homology level and not on chain level, as in the case of Rabinowitz action functional.

2. In contrast to symplectic homology the Rabinowitz gradient flow equation is not a PDE but a delay equation. Although the critical points of the Rabinowitz action functional are still solutions of an ODE, the Rabinowitz action functional can easily be generalized to delay equations. In fact, the functional $\mathcal{H}$ not necessarily has to be the mean value of a Hamiltonian on the underlying manifold, but can be a more interesting functional on the free loop space. In particular, in this way one can model interacting particles whose interaction is not necessarily instantaneous, but can happen with some delay [Fra20]. This is in particular of interest in a semi-classical treatment of Helium [CFV21].

As mentioned above the major difficulty to define a TQFT structure on Rabinowitz-Floer homology directly is the complicated behavior of the Rabinowitz action functional on the concatenation of loops. To remedy this situation it was proposed in [Fra22] to take advantage of the following elementary fact. Critical points of a Lagrange multiplier functional are in 1-1 correspondence with critical points of the restriction of the first function to the constraint given by the vanishing of the second function. In the case of the Rabinowitz action functional it means the following. One restricts the negative area functional $\mathcal{A}$ to the constraint $\mathcal{H}^{-1}(0)$, namely the hypersurface in the free loop space consisting of loops whose mean value vanishes. Note that concatenating two loops of mean value zero leads to another loop of mean value zero. Therefore the hypersurface $\mathcal{H}^{-1}(0)$ is invariant under concatenation. Moreover, note that the area functional is additive with respect to concatenation. Therefore the restriction of the area functional to $\mathcal{H}^{-1}(0)$ has the potential of leading to a TQFT for which Poincaré-duality holds on chain level and which should also lead to topological quantum field theories for Hamiltonian delay equations.

In view of the above remarks it is of major interest to understand how the semi-infinite dimensional Morse homology in the sense of Floer of the Rabinowitz action functional $\mathcal{A}_{\mathcal{H}}$ is related to the one of the restriction of the area functional $\mathcal{A}$ to $\mathcal{H}^{-1}(0)$. Motivated by the general perspective we treat in this article the finite dimensional analogue of this question which already has its own interest.

2 Lagrange multiplier function and restriction

Suppose that on a Riemannian manifold (M, G) are given two smooth functions

$$F, H\colon M \to \mathbb{R}$$

such that 0 is a regular value of H, in symbols $H \pitchfork 0$. The function H plays the role of providing a **constraint**, namely the smooth Riemannian hypersurface

$$\Sigma := H^{-1}(0) \overset{\iota}{\hookrightarrow} M, \qquad g := \iota^* G, \qquad f := F|_\Sigma := F \circ \iota\colon \Sigma \to \mathbb{R}, \tag{2.6}$$

equipped with the restriction of F and were ι is the inclusion map. Throughout we assume that Σ is compact and without boundary. We call Σ the **base** of the adiabatic limit construction. Now add to F the constraint function H times a parameter τ to define the **Lagrange multiplier function**

$$F_H\colon M \times \mathbb{R} \to \mathbb{R}, \quad (x, \tau) \mapsto F(x) + \tau H(x). \tag{2.7}$$

The restriction $F_H|_\Sigma = f$ is equal to the restriction of F. The function F_H has the significance that its critical points are in bijection with the critical points x of the restriction f via their so-called Lagrange multipliers $\chi(x)$, see Lemma 2.5.

2.1 Hypersurface geometry

As a preparation we recall relevant facts about the geometry of Riemannian submanifolds following the excellent presentation of O'Neill [O'N83, Chap. 4].

Let (M,G) be a smooth Riemannian manifold and $H\colon M\to\mathbb{R}$ a smooth[1] function with regular value 0. The level set (2.6) endowed with the restriction metric is a smooth Riemannian hypersurface (Σ,g) of (M,G). Let $\mathcal{X}(M)$ be the smooth vector fields along M and $\mathcal{X}(\Sigma)$ those along Σ. Let $\overline{\mathcal{X}}(\Sigma)$ be the restrictions to Σ of vector fields along M, equivalently, the sections of the pull-back bundle $\iota^*TM\to\Sigma$. On (M,G) and (Σ,g), respectively, we denote the Levi-Civita connections by $\overline{\nabla}$ and ∇ and the exponential maps by Exp and exp.

Gradients are orthogonal to level sets. By definition of regular value and codimension 1 the gradient of H is nowhere zero along the hypersurface $\Sigma=H^{-1}(0)$. Thus $\overline{\nabla}H$ generates the normal bundle $N\Sigma=\mathbb{R}\overline{\nabla}H$ of Σ and

$$T_\Sigma M=T\Sigma\overset{\perp}{\oplus}N\Sigma,\qquad X=\xi+\nu,$$

is an orthogonal direct sum along Σ. Hence for any and $X\in T\Sigma M$ there are unique vectors $\xi\in T\Sigma$ and $\nu\in N\Sigma$ such that $X=\xi+\nu$. This defines two orthogonal projections tan and nor, see (2.10) and (2.11).

We denote vectors of TM and vector fields taking values in TM by capital letters such as X,Y and, in contrast, vectors of $T\Sigma$ and vector fields taking values in $T\Sigma$ be greek letters such as ξ,η. By ν we denote elements of $N\Sigma$. See Convention 1.1 for notation of norms and inner products. Here and throughout we silently identify $q\in\Sigma$ with $\iota(q)\in M$ and $\xi\in T\Sigma$ with $T\iota(\xi)\in TM$.

2.1.1 Orthogonal splitting of TM along a neighborhood of Σ

For $p\in M$ the **gradient** $\overline{\nabla}H(p)$ is determined by $dH(p)X=\langle\overline{\nabla}H(p),X\rangle\ \forall X\in T_pM$. An open neighborhood of Σ is provided by the set of regular points

$$\Sigma\subset M_{\rm reg}:=\{p\in M\mid dH(p)\neq 0\}\subset M.$$

Since $\overline{\nabla}H(p)\neq 0$ for $p\in M_{\rm reg}$, there are the **canonical vector fields**

$$U:=\frac{\overline{\nabla}H}{|\overline{\nabla}H|},\quad V:=\frac{\overline{\nabla}H}{|\overline{\nabla}H|^2},\quad \text{along } M_{\rm reg}.$$

The smooth function defined by

$$\chi:=-\tfrac{\langle\overline{\nabla}F,\overline{\nabla}H\rangle}{|\overline{\nabla}H|^2}\quad\text{along } M_{\rm reg}\tag{2.8}$$

provides the coefficient of the orthogonal projection of $\overline{\nabla}F$ onto $-\overline{\nabla}H$; see (2.10). Since $\langle\overline{\nabla}H,\xi\rangle=dH\,\xi=0$ for $\xi\in T\Sigma$, the sum $T_\Sigma M=T\Sigma+\mathbb{R}\cdot\overline{\nabla}H$ is direct and orthogonal. Thus the line bundle $N\Sigma:=\mathbb{R}\overline{\nabla}H$ is the normal bundle of Σ. There are the associated orthogonal projections

$$\tan\colon T_qM\to T_q\Sigma,\quad \mathrm{nor}\colon T_qM\to\mathbb{R}U_q,\quad \tan+\mathrm{nor}=\mathrm{Id}_{T_qM}.\tag{2.9}$$

The vectors of $\mathbb{R}U_q$ are said **normal** to Σ. A vector field $Z\in\overline{\mathcal{X}}(\Sigma)$ is called normal to Σ if each vector $Z(q)$ is. Let $\mathcal{X}(\Sigma)^\perp$ be the vector fields normal to Σ,

[1] throughout smooth means C^∞ smooth

that is the sections of the line bundle $\mathbb{R}\overline{\nabla}H \to \Sigma$. There is the orthogonal vector bundle sum $\overline{\mathcal{X}}(\Sigma) = \mathcal{X}(\Sigma) \oplus \mathcal{X}(\Sigma)^\perp$. The resulting orthogonal projections

$$\begin{aligned} &\mathrm{nor}\colon \overline{\mathcal{X}}(\Sigma) \to \mathcal{X}(\Sigma)^\perp, && \tan\colon \overline{\mathcal{X}}(\Sigma) \to \mathcal{X}(\Sigma), \\ &X \mapsto \langle X, U\rangle\, U = \tfrac{\langle X, \overline{\nabla}H\rangle}{|\overline{\nabla}H|^2}\, \overline{\nabla}H. && X \mapsto X - \mathrm{nor}\, X, \end{aligned} \tag{2.10}$$

are $C^\infty(\Sigma)$-linear and there is the identity $\overline{\mathcal{X}}(\Sigma) \ni X = \tan X + \mathrm{nor}\, X$.

Lemma 2.1 (Gradients and orthogonal decomposition)**.** *It holds that*

$$\begin{aligned} &\tan \overline{\nabla}F = \nabla f && \mathrm{nor}\, \overline{\nabla}F = -\chi \overline{\nabla}H && \overline{\nabla}F = \nabla f - \chi \overline{\nabla}H \\ &\mathrm{nor}\, X = \tfrac{(dH)\, X}{|\overline{\nabla}H|^2} \overline{\nabla}H && |\mathrm{nor}\, X| \le \tfrac{|(dH)\, X|}{m_H} && m_H := \min_\Sigma \left|\overline{\nabla}H\right| > 0 \end{aligned} \tag{2.11}$$

pointwise at $q \in \Sigma$ and for every tangent vector $X \in T_qM$.

Proof. To identify ∇f with the tangential part, pick $\xi \in \mathcal{X}(\Sigma)$. Then

$$\begin{aligned} \langle \nabla f, \xi\rangle_g &= df(\xi) = dF|_\iota\, d\iota(\xi) = \langle \overline{\nabla}F|_\iota, d\iota(\xi)\rangle_G = \langle \overline{\nabla}F|_\iota - \mathrm{nor}\, \overline{\nabla}F|_\iota, d\iota(\xi)\rangle_G \\ &= \langle \tan \overline{\nabla}F|_\iota, \xi\rangle_g. \end{aligned}$$

We subtracted the normal since its inner product with the tangent $d\iota(\xi)$ is zero. As the difference is tangent, we change G to g. Next write $\mathrm{nor}(\overline{\nabla}F) = \alpha \overline{\nabla}H$ for some $\alpha \in C^\infty(\Sigma)$. Then the identity $\overline{\nabla}F = \nabla f + \alpha \overline{\nabla}H$ is the splitting (2.9). Scalar multiply the identity by the normal $\overline{\nabla}H$ to get that

$$\langle \overline{\nabla}F, \overline{\nabla}H\rangle = 0 + \alpha |\overline{\nabla}H|^2.$$

Hence $\alpha = -\chi$ by (2.8). The term $\mathrm{nor}\, X$ is obvious. □

2.1.2 Normal form of H near Σ

Let $\kappa > 0$ be the constant from the local properness Hypothesis 1.2. The vector field $V := \overline{\nabla}H / |\overline{\nabla}H|^2$ along the open neighborhood $M_{\mathrm{reg}} := \{dH \ne 0\}$ of M of Σ in M generates a local flow $\{\varphi_r\}$ on M_{reg}. Since Σ is compact for $\delta \in (0, \kappa)$ small enough the following map is a diffeomorphism onto its image

$$\varphi\colon \Sigma \times (-\delta, \delta) \to U_\Sigma = U_\Sigma(\delta) := \mathrm{im}\, \varphi \subset M, \quad (q.r) \mapsto \varphi_r q.$$

(The map φ provides a retraction $\rho = \rho^2\colon U_\Sigma \to U_\Sigma$.[2]) The identities

$$H(\varphi_0 q) = 0, \qquad \tfrac{d}{dr} H(\varphi_r q) = dH|_{\varphi_r q}\, \tfrac{d}{dr} \varphi_r q = \left\langle \overline{\nabla}H|_{\varphi_r q}, V|_{\varphi_r q}\right\rangle = 1,$$

show that

$$H(\varphi_r q) = r \tag{2.12}$$

for every $(q, r) \in \Sigma \times (-\delta, \delta)$. Thus, for every map $u\colon \mathbb{R} \to M$ that takes values in the image of the flow diffeomorphism φ, there are maps $\mathsf{q}\colon \mathbb{R} \to \Sigma$ and $r\colon \mathbb{R} \to (-\delta, \delta)$, namely $(\mathsf{q}, r) := \varphi^{-1}(u)$ pointwise, such that

$$u = \varphi_r(\mathsf{q}), \qquad r = H(u), \tag{2.13}$$

pointwise at $s \in \mathbb{R}$.

[2] To match the abstract approach [FW22b] define, for each $t \in [0, 1]$, a map $\rho_t\colon U_\Sigma \to U_\Sigma$, $p = \varphi_r q \mapsto \varphi_{-tr} p$. Then $\rho_0 = \mathrm{id}_{U_\Sigma}$, $\rho_1\colon U_\Sigma \to \Sigma$, $\rho_t|_\Sigma = \mathrm{id}_\Sigma\ \forall t \in [0, 1]$. So $\rho := \rho_1 = \rho^2$.

2.1.3 Induced connection

The Levi-Civita connections associated to (M, G) and (Σ, g) are maps

$$\overline{\nabla}\colon \mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M),\qquad \nabla\colon \mathcal{X}(\Sigma)\times\mathcal{X}(\Sigma)\to\mathcal{X}(\Sigma).$$

Via vector field extension from the domain Σ to M the connection $\overline{\nabla}$ gives rise to a map, independent of the chosen extensions $\bar\xi, \overline{X}$, the **induced connection**

$$\overline{\nabla}\colon \mathcal{X}(\Sigma)\times\overline{\mathcal{X}}(\Sigma)\to\overline{\mathcal{X}}(\Sigma),\quad (\xi, X)\mapsto \overline{\nabla}_\xi X := \overline{\nabla}_{\bar\xi}\overline{X},$$

still denoted by the same symbol $\overline{\nabla}$.

Lemma 2.2. *The induced connection satisfies the five axioms that characterize the Levi-Civita connection on the tangent bundle of a Riemannian manifold:*

(i) $C^\infty(\Sigma)$*-linear in* ξ $\qquad \overline{\nabla}_{f\xi}X = f\overline{\nabla}_\xi X$

(ii) $\mathbb{R}$*-linear in* X $\qquad \overline{\nabla}_\xi(\alpha X) = \alpha\overline{\nabla}_\xi X$

(iii) *Leibniz rule* $\qquad \overline{\nabla}_\xi(fX) = (\xi f)X + f\overline{\nabla}_\xi X$

(iv) *torsion free* $\qquad [\xi, \eta] := \xi\eta - \eta\xi = \overline{\nabla}_\xi\eta - \overline{\nabla}_\eta\xi$

(v) *metric* $\qquad \xi\langle X, Y\rangle = \langle\overline{\nabla}_\xi X, Y\rangle + \langle X, \overline{\nabla}_\xi Y\rangle$

for all $\alpha\in\mathbb{R}$*,* $f\in C^\infty(\Sigma)$*,* $\xi,\eta\in\mathcal{X}(\Sigma)$*, and* $X, Y\in\overline{\mathcal{X}}(\Sigma)$*, and where* ξf *is a convenient shorter way to write* $df(\xi)$*.*

Remark 2.3. If both vector fields ξ, η take values in $T\Sigma$, by torsion freeness the difference $\overline{\nabla}_\xi\eta - \overline{\nabla}_\eta\xi$ takes values in $T\Sigma$ as well – the commutator does. This is in general not true for the individual terms. Via the orthogonal projections (2.10) one decomposes the vector field $\overline{\nabla}_\xi\eta\in\overline{\mathcal{X}}(\Sigma)$ into a tangent and a normal part

$$\overline{\nabla}_\xi\eta = \nabla_\xi\eta + \mathrm{II}(\xi, \eta) \tag{2.14}$$

whenever $\xi, \eta\in\mathcal{X}(\Sigma)$ and where

$$\nabla_\xi\eta = \tan\overline{\nabla}_\xi\eta\in\mathcal{X}(\Sigma),\qquad \mathrm{II}(\xi,\eta) := \mathrm{nor}\,\overline{\nabla}_\xi\eta\in\mathcal{X}(\Sigma)^\perp. \tag{2.15}$$

The **second fundamental form tensor** II of the Riemannian submanifold Σ of M is $C^\infty(\Sigma)$-bilinear and symmetric. In our codimension 1 case U generates $\mathcal{X}(\Sigma)^\perp$, so $\mathrm{II}(\xi,\eta)$ is a $C^\infty(\Sigma)$-multiple of U. Multiply (2.14) by U to get

$$\mathrm{II}(\xi,\eta) = \mu(\xi,\eta)\cdot U = \tfrac{\langle\overline{\nabla}_\xi\eta, \overline{\nabla}H\rangle}{|\overline{\nabla}H|^2}\,\overline{\nabla}H,\qquad \mu(\xi,\eta) = \langle\overline{\nabla}_\xi\eta, U\rangle. \tag{2.16}$$

The tensor II appears in the formal adjoint operator $(D^0_q)^*$, see (4.43). The **second fundamental form** B and the **shape operator** S, both associated to the unit normal vector field U, so determined up to sign, are defined by

$$B(\xi,\eta) := \langle S\xi, \eta\rangle \overset{\text{def. } S}{=} \langle\mathrm{II}(\xi,\eta), U\rangle \overset{(2.16)}{=} \langle\overline{\nabla}_\xi\eta, U\rangle$$

for all $\xi,\eta\in\mathcal{X}(\Sigma)$. But $0 = \xi\langle\eta, U\rangle = \langle\overline{\nabla}_\xi\eta, U\rangle + \langle\eta, \overline{\nabla}_\xi U\rangle$. Therefore the shape operator at $q\in\Sigma$ is the symmetric linear map

$$S\colon T_q\Sigma\to T_q\Sigma,\quad \xi\mapsto -\overline{\nabla}_\xi U.$$

Implicitly this tells that $\overline{\nabla}_\xi U$ is tangent to Σ (alternatively hit $\langle U, U\rangle = 1$ by ξ).

2.2 Critical points are in canonical bijection

Critical points of $f = F|_\Sigma$ satisfy $x \in \Sigma$ and

$$0 = \nabla f(x) \overset{(2.11)}{=} \left(\overline{\nabla} F + \chi \overline{\nabla} H\right)(x) \quad \Leftrightarrow \quad \left(dF + \chi dH\right)(x) = 0. \tag{2.17}$$

A point $(p, \tau) \in M \times \mathbb{R}$ is critical for the function $F_H(p, \tau) = F(p) + \tau H(p)$ iff the derivative vanishes

$$dF_H(p, \tau) \begin{pmatrix} X \\ \ell \end{pmatrix} = dF(p)X + \tau \cdot dH(p)X + \ell \cdot H(p) = 0 \tag{2.18}$$

for all $X \in T_pM$ and $\ell \in \mathbb{R}$. Fix $X = 0$ to obtain $H(p) = 0$, that is $p \in \Sigma$. Now fix $\ell = 0$ and set $x := p$ to obtain that (x, τ) is a critical point of F_H iff

$$dF(x) + \tau \cdot dH(x) = 0, \qquad x \in \Sigma.$$

2.2.1 Canonical embedding

Definition 2.4 (Canonical embedding)**.** The graph map of $\chi \colon \Sigma \to \mathbb{R}$, cf. (2.8),

$$i : \Sigma \to M \times \mathbb{R}, \quad q \mapsto (q, \chi(q)) = (\iota(q), \chi(\iota(q))), \tag{2.19}$$

is called the **canonical embedding**. The derivative is denoted and given by

$$I_q := di(q) \colon T_q\Sigma \to T_qM \times \mathbb{R}, \quad \xi \mapsto (\xi, d\chi(q)\xi)\,. \tag{2.20}$$

For simplicity of notation we usually abbreviate $\iota(q)$ by q and $d\iota(q)\xi$ by ξ. Graph maps of smooth functions are embeddings. The Lagrange function $F_H(p, \tau) = F(p) + \tau H(p)$ coincides along the image of i with the restriction $f = F|_\Sigma$ to the zero level Σ of H, in symbols

$$F_H \circ i = f.$$

Lemma 2.5. *The critical points of F_H and f are in bijection, more precisely*

$$\begin{aligned} \mathrm{Crit} F_H &= i(\mathrm{Crit} f) \\ &= \{(x, \chi(x)) \in \Sigma \times \mathbb{R} \mid dF(x) + \chi(x) \cdot dH(x) = 0\}. \end{aligned} \tag{2.21}$$

In particular, along critical points x both functions coincide $f(x) = F_H(x, \chi(x))$.

Proof. Compare (2.17) and (2.21) where $dH(x) \neq 0$ implies $\tau = \chi(x)$. □

2.2.2 Hessians and Morse indices

Suppose $x \in \Sigma$ is a **non-degenerate** critical point of f, that is 0 is not an eigenvalue of the Hessian operator, the covariant derivative of ∇f at x, namely

$$A_x^0 \colon T_x\Sigma \to T_x\Sigma, \quad \xi \mapsto D\nabla f(x)\xi = \nabla_\xi \nabla f(x).$$

This linear map is symmetric; see identity 2 and 3 in (4.40) further below. In local coordinates A^0_x is represented by the Hessian matrix of second derivatives $a^f_x = (\partial_i\partial_j f(x))^n_{i,j=1}$. This matriz is symmetric, hence admits n real eigenvalues, counted with multiplicities. While the Hessian matrix depends on the choice of coordinates, the number of negative eigenvalues does not. The number k of negative eigenvalues, counted with multiplicity, of the Hessian operator A^0_x or, equivalently, of any Hessian matrix a^f_x is called the **Morse index** of x, in symbols $\mathrm{ind}_f(x) = k$.

In the transition from f to F_H, in terms of critical points from $x \in \Sigma$ to $(x, \chi(x)) \in M \times \mathbb{R}$, two new eigenvalues appear, one is positive and the other one is negative. This result is due to the first author [Fra06] where the proof is in local coordinates. It is easy to obtain such coordinates in our scenario: for the submanifold $H^{-1}(0) \hookrightarrow M$ use submanifold coordinates and for the orthogonal complement use the local flow generated by the gradient of H suitably rescaled.

Lemma 2.6 (Morse index increases by 1)**.** *If $x \in \mathrm{Crit}\, f$ is non-degenerate, then so is $(x, \chi(x)) \in \mathrm{Crit}\, F_H$ and the Morse index increases by one, in symbols*

$$\mathrm{ind}_{F_H}(x, \chi(x)) = \mathrm{ind}_f(x) + 1.$$

Remark 2.7. By Lemma 2.5 and 2.6, if f is Morse, so is F_H. Let f be Morse. Since the dimension difference $\dim M - \dim \Sigma = 2$ is two, there always arises together with the negative Hessian eigenvalue exactly one positive eigenvalue. Consequently the Hessian of F_H at a critical point is never negative (positive[3]) definite. Hence critical points of F_H are not minima (maxima), hence not detectable by direct methods using minimization (maximization).

Proof. Given $F, G\colon M \to \mathbb{R}$ with $G \pitchfork 0$, let $\Sigma := H^{-1}(0) \subset M$. Pick a critical point x of $f = F|_\Sigma\colon \Sigma \to \mathbb{R}$. Choose a local coordinate chart between open subsets

$$\phi\colon M \supset V \to U \subset \mathbb{R}^n, \quad p \mapsto \phi(p) = (z_1, \ldots, z_n, r) = (z, r),$$

which takes x to the origin of $\mathbb{R}^n$ and has the following properties:

a) the part of Σ in V corresponds to the part of $\mathbb{R}^n \times 0$ in U;
b) in local coordinates H is given by $(z, r) \mapsto r$. $H(z, r) = r$

Such coordinates exist: By compactness of Σ there is a constant $\delta > 0$ such that the vector field $V = \overline{\nabla} H / |\overline{\nabla} H|^2$ along M_{reg} generates a local flow, notation $\varphi\colon \Sigma \times (-\delta, \delta) \to M$, $(q, r) \mapsto \varphi_r q$. The identities

$$H(\varphi_0 q) = 0, \qquad \tfrac{d}{dr} H(\varphi_r q) = dH|_{\varphi_r q}\, \tfrac{d}{dr}\varphi_r q = \left\langle \overline{\nabla} H|_{\varphi_r q}, \tilde{U}|_{\varphi_r q} \right\rangle = 1,$$

show that $H(\varphi_r(q)) = r$. Compose φ with submanifold coordinates of Σ in M.

[3] Replace f by $-f$.

In the following local coordinate representations of maps are denoted by the same symbols as the maps themselves. For instance, for F in our local coordinates we write $F(z,r)$. In these local coordinates we have

$$\text{(i) } f(z) = F(z,0), \qquad \text{(ii) } F_H(z,r,\tau) = F(z,r) + \tau r.$$

The proof proceeds in two steps. First we consider the special case where $F(z,r) = f(z)$, second we homotop the general case to the special case.

Special case $F(z,r) = f(z)$. The gradient of $F_H(z,r,\tau) = f(z) + \tau r$ is $\nabla F_H(z,r,\tau) = (\nabla f(z), \tau, r)$, so the Hessian at the critical point $(x,0,\chi(x))$ is

$$a_0 := a^{F_H}_{(x,0,\chi(x))} = \begin{bmatrix} a^f_x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since the lower 2×2 diagonal block has eigenvalues $-1, +1$ we are done.

General case $F(z,r)$. The gradient of $F_H(z,r,\tau) = F(z,r) + \tau r$ is given by $\nabla F_H(z,r,\tau) = (\nabla_1 F(z,r), \nabla_2 F(z,r) + \tau, r)$, so the Hessian at $(x,0,\chi(x))$ is the matrix $a_1 = a^{F_H}_{(x,0,\chi(x))}$ given by setting $s=1$ in the interpolating family

$$a_s := \begin{bmatrix} a^f_x & s\nabla_2\nabla_1 F(x,0) & 0 \\ s\nabla_1\nabla_2 F|_{(x,0)} & s\nabla_2\nabla_2 F|_{(x,0)} & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad s\in[0,1].$$

Zero is not an eigenvalue of a_1: Let $(\xi, R, T) \in \ker a_1 \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} \xi \\ R \\ T \end{bmatrix} = \begin{bmatrix} a^f_x\xi + \nabla_2\nabla_1 F(x,0)R \\ \nabla_1\nabla_2 F|_{(x,0)}\xi + \nabla_2\nabla_2 F|_{(x,0)}R + T \\ R \end{bmatrix} = \begin{bmatrix} a^f_x\xi \\ \nabla_1\nabla_2 F|_{(x,0)}\xi + T \\ 0 \end{bmatrix}.$$

Thus $R=0$. Since a^f_x does not have eigenvalue zero, if $a^f_x\xi = 0$, then $\xi = 0$, so $T=0$. For any $s\in[0,1)$ the same argument shows that the matrix a_s does not have eigenvalue 0. But each eigenvalue depends continuously on the matrix a_s, so a_1 and a_0 do have the same number of negative/positive eigenvalues. □

3 Downward gradient flows

3.1 Base flow

The downward gradient equation on the regular hypersurface $(\Sigma, g) = (H^{-1}(0), \iota^* G)$ of the restriction $f = F|_\Sigma \colon \Sigma \to \mathbb{R}$ is given by

$$\begin{aligned} \partial_s q = -\nabla f(q) &\overset{(2.11)}{=} -\big(\overline{\nabla} F + \chi \overline{\nabla} H\big)(q) \\ f &= F\circ\iota\colon \Sigma\to\mathbb{R} \end{aligned} \tag{3.22}$$

for smooth maps $q\colon \mathbb{R}\to\Sigma$ and where χ is defined by (2.8).

Pointwise evaluation at $s \in \mathbb{R}$ extends the canonical embedding (2.19) from points in Σ to smooth maps $q\colon \mathbb{R} \to \Sigma$. The induced embedding, still denoted by

$$i(q) = (\iota \circ q, \chi \circ \iota \circ q) =: (u, \tau), \qquad q\colon \mathbb{R} \to \Sigma, \tag{3.23}$$

is injective consisting of a pair of maps $u = \iota \circ q\colon \mathbb{R} \to M$ and $\tau = \chi \circ \iota \circ q\colon \mathbb{R} \to \mathbb{R}$.

Consider the pair of equations

$$\begin{aligned} \partial_s u + \overline{\nabla} F(u) + \tau \overline{\nabla} H(u) &= 0 \\ H \circ u &= 0 \end{aligned} \tag{3.24}$$

for smooth maps $(u, \tau)\colon \mathbb{R} \to M \times \mathbb{R}$.

Lemma 3.1 (Base equation)**.** *If $q\colon \mathbb{R} \to \Sigma$ solves (3.22), then $(u, \tau) := i(q)$ as defined by (3.23) solves (3.24) and every solution of (3.24) arises this way.*

Proof. Identifying domain and image of $\iota\colon \Sigma \to M$ the first lines of (3.22) and of (3.24) are just the same equation whenever $u = \iota(q)$ and $\tau = \chi(q)$.

Now suppose $(u, \tau)\colon \mathbb{R} \to M \times \mathbb{R}$ solves (3.24). By the second equation u takes values in Σ. This has two consequences. Firstly, we can view u as a map to Σ, notation $q_u\colon \mathbb{R} \to \Sigma$. Secondly, the derivative $\partial_s u$ is tangential to Σ, hence so is $-\partial_s u = \overline{\nabla} F(u) + \tau \overline{\nabla} H(u)$. Take the inner product with the normal field $\overline{\nabla} H(u)$ to get $0 = \langle \overline{\nabla} F(u), \overline{\nabla} H(u) \rangle + \tau |\overline{\nabla} H(u)|^2$. By definition (2.8) this means that $\tau = \chi(u) = \chi(q_u)$. Hence $i(q_u) = (\iota(q_u), \chi(\iota(q_u))) = (u, \tau)$. □

3.1.1 Base energy E^0

Given critical points $x^{\mp}$ of $f\colon \Sigma \to \mathbb{R}$, we impose on a smooth map $q\colon \mathbb{R} \to \Sigma$ the asymptotic boundary conditions

$$\lim_{s \to \mp\infty} q_s = x^{\mp}. \tag{3.25}$$

Definition 3.2. Define the **base energy** of a smooth map $q\colon \mathbb{R} \to \Sigma$ by

$$\begin{aligned} E^0(q) &\overset{\text{def.}}{=} \tfrac{1}{2} \int_{-\infty}^{\infty} |\partial_s q_s|^2 + |\nabla f(q_s)|^2 \, ds \\ &\overset{(3.22)}{=} \tfrac{1}{2} \int_{-\infty}^{\infty} |\partial_s q_s|^2 + \left| \overline{\nabla} F(q_s) + \chi(q_s) \cdot \overline{\nabla} H(q_s) \right|^2 ds \\ &\overset{\text{def.}}{=} E^0(q, \chi(q)). \end{aligned}$$

Lemma 3.3 (Energy identity)**.** *Let $q\colon \mathbb{R} \to \Sigma$ be a smooth solution of (3.22). Then the energy is bounded by the oscillation of f and there is the energy identity*

$$E^0(q) \overset{(3.22)}{=} \|\partial_s q\|^2 \le \operatorname{osc} f := \max f - \min f < \infty \tag{3.26}$$

where $\|\cdot\|$ is the L^2 norm. With asymptotic boundary conditions (3.25) it holds

$$E^0(q) \overset{(3.22)}{=} \|\partial_s q\|^2 \overset{(3.25)}{=} f(x^-) - f(x^+) =: c^*. \tag{3.27}$$

Proof. We see that

$$\begin{aligned}
E^0(q) &\overset{(3.22)}{=} \lim_{T\to\infty} \int_{-T}^{T} |\partial_s q_s|^2 \, ds \\
&\overset{(3.22)}{=} \lim_{T\to\infty} \int_{-T}^{T} -\langle \nabla f(q_s), \partial_s q_s \rangle \, ds \\
&= -\lim_{T\to\infty} \int_{-T}^{T} \tfrac{d}{ds} f(q_s) \, ds \\
&= \lim_{T\to\infty} \left(f(q_{-T}) - f(q_T) \right).
\end{aligned}$$

Now both, (3.26) and (3.27), are obvious. □

3.2 Ambient flow and deformation

Ambient flow. We endow the product $M \times \mathbb{R}$ with the product metric $h^1 := G \oplus 1$ and the associated Levi-Civita connection ∇^1. The downward gradient equation for the function $F_H \colon M \times \mathbb{R} \to \mathbb{R}$ from (2.7), namely $\partial_s z = -\nabla^1 F_H(z)$, is according to (2.18) given by the pair of equations

$$\begin{pmatrix} \partial_s u \\ \tau' \end{pmatrix} = \partial_s z = -\nabla^1 F_H(z) = -\begin{pmatrix} \overline{\nabla} F(u) + \tau \overline{\nabla} H(u) \\ H(u) \end{pmatrix} \tag{3.28}$$

for smooth maps $z = (u, \tau) \colon \mathbb{R} \to M \times \mathbb{R}$. The ambient energy E^1 is $E^{\varepsilon=1}$ in Definition 3.4.

Deformed flow. For $\varepsilon > 0$ consider on $M \times \mathbb{R}$ the rescaled Riemannian metric and associated Levi-Civita connection

$$h^\varepsilon := G \oplus \varepsilon^2, \qquad \nabla^\varepsilon. \tag{3.29}$$

Thus the inner product of elements $Z = (X, \ell)$ and $\tilde{Z} = (\tilde{X}, \tilde{\ell})$ of $T_u M \times \mathbb{R}$ is

$$h^\varepsilon(Z, \tilde{Z}) = \langle X, \tilde{X} \rangle + \varepsilon^2 \ell \tilde{\ell}, \qquad |Z|_\varepsilon^2 := h^\varepsilon(Z, Z) = |X|^2 + \varepsilon^2 \ell^2.$$

By (2.18) the downward ε-gradient equation for the function F_H on $M \times \mathbb{R}$ is

$$\begin{pmatrix} \partial_s u \\ \tau' \end{pmatrix} = \partial_s z = -\nabla^\varepsilon F_H(z) = -\begin{pmatrix} \overline{\nabla} F(u) + \tau \overline{\nabla} H(u) \\ \varepsilon^{-2} H(u) \end{pmatrix} \tag{3.30}$$

for smooth maps $z = (u, \tau) \colon \mathbb{R} \to M \times \mathbb{R}$.

Multiply the second equation by ε^2 and formally set $\varepsilon = 0$ to obtain that $H(u_s) = 0$ $\forall s \in \mathbb{R}$. This suggests that in the limit $\varepsilon \to 0$ the solutions to (3.30) converge to a solution of the base equation (3.24).

3.2.1 Ambient energy E^ε

Given critical points $x^\mp$ of $f\colon \Sigma \to \mathbb{R}$, impose on a smooth map $(u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ the asymptotic boundary conditions

$$\lim_{s\to\mp\infty} (u_s, \tau_s) = \bigl(x^\mp, \chi(x^\mp)\bigr) \stackrel{(2.21)}{\in} \mathrm{Crit} F_H. \tag{3.31}$$

Definition 3.4. The **ε-energy** of a smooth map $z = (u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ is

$$\begin{aligned} E^\varepsilon(u,\tau) :&= \tfrac{1}{2} \int_{-\infty}^{\infty} |\partial_s z_s|_\varepsilon^2 + |\nabla^\varepsilon F_H(z_s)|_\varepsilon^2 ds \\ &= \tfrac{1}{2} \int_{-\infty}^{\infty} |\partial_s u_s|^2 + \varepsilon^2 {\tau_s'}^2 + \left|\overline{\nabla} F(u_s) + \tau_s \overline{\nabla} H(u_s)\right|^2 + \varepsilon^{-2} H(u_s)^2 ds. \end{aligned}$$

Lemma 3.5 (Energy identity)**.** *Given $\varepsilon > 0$, let $(u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ be a solution of (3.30). Then the following is true. a) There is the identity*

$$E^\varepsilon(u,\tau) \stackrel{(3.30)}{=} \|\partial_s u\|^2 + \varepsilon^2 \|\tau'\|^2 \in [0,\infty] \tag{3.32}$$

where $\|\cdot\|$ denotes L^2 norms. b) If, in addition, the energy $E^\varepsilon(u,\tau) < \infty$ is finite, then the energy is bounded by the oscillation of f, in symbols

$$E^\varepsilon(u,\tau) \le \max f - \min f =: \mathrm{osc} f < \infty.$$

c) In case of asymptotic boundary conditions (3.31) there is the energy identity

$$E^\varepsilon(u,\tau) \stackrel{(3.30)}{=} \|\partial_s u\|^2 + \varepsilon^2 \|\tau'\|^2 \stackrel{(3.31)}{=} f(x^-) - f(x^+) =: c^*. \tag{3.33}$$

Proof. Fix $\varepsilon > 0$. We see that

$$\begin{aligned} E^\varepsilon(u,\tau) &\stackrel{(3.30)}{=} \lim_{T\to\infty} \int_{-T}^{T} \left(|\partial_s u_s|^2 + \varepsilon^2 {\tau_s'}^2\right) ds \\ &\stackrel{(3.30)}{=} \lim_{T\to\infty} \int_{-T}^{T} -\left\langle \partial_s u_s, \overline{\nabla} F(u_s) - \tau_s \overline{\nabla} H(u_s)\right\rangle_G - \tau_s' \cdot H(u_s)\, ds \\ &= \; -\lim_{T\to\infty} \int_{-T}^{T} dF|_{u_s} \partial_s u_s + \tau_s\, dH|_{u_s} \partial_s u_s + \tau_s' \cdot H(u_s)\, ds \\ &= \; -\lim_{T\to\infty} \int_{-T}^{T} \frac{d}{ds}\Bigl(F(u_s) + \tau_s H(u_s)\Bigr) ds \\ &= \lim_{T\to\infty} \left(F_H(u_{-T}, \tau_{-T}) - F_H(u_T, \tau_T)\right). \end{aligned}$$

This proves a) and also c) since $F_H(x^-, \chi(x^-)) = F(x^-) + \chi(x^-) H(x^-) = f(x^-) \le \max f$ and similarly at x^+. b) That the right hand side of the displayed formula is $\le \max f - \min f$ will be proved in two steps.

Step 1. *Fix $\varepsilon > 0$. For each $\mu > 0$ there exists a $\delta = \delta(\mu) > 0$ with the following property. At any point $(p,t) \in M \times \mathbb{R}$ where the gradient is δ-small*

$$|\nabla^\varepsilon F_H(p,t)|_\varepsilon^2 = |\overline{\nabla} F(p) + t\overline{\nabla} H(p)|^2 + \varepsilon^{-2} H(p)^2 \leq \delta \tag{3.34}$$

the value of the multiplier function lies in the μ-interval

$$\min f - \mu \leq F_H(p,t) \leq \max f + \mu. \tag{3.35}$$

To prove Step 1 suppose that a point (p,t) satisfies (3.34). Hence $H(p)^2 \leq \delta\varepsilon^2$. Since H is locally proper around zero by Hypothesis 1.2, it follows that for any open neighborhood U of Σ there exists a $\delta_U > 0$ such that the point p lies in U whenever (p,t) satisfies (3.34) for $\delta = \delta_U$. Otherwise, there would exist a sequence $p_\nu \notin U$ with the property that $H(p_\nu) \to 0$, as $\nu \to \infty$. By local properness there is a subsequence p_{ν_k} which converges to a point $p_\infty \in \Sigma = H^{-1}(0)$ contradicting the assumption that none of the p_ν lies in the open neighborhood U of the compact set Σ.
Given $\mu > 0$, we choose $U(\mu)$: Since zero is a regular value of H and $\Sigma = H^{-1}(0)$ is compact there exists an open neighborhood $U(\mu)$ of Σ and constants $c, C > 0$ such that

$$c \leq \inf_U |\overline{\nabla} H|, \quad \sup_U |\overline{\nabla} F| \leq C, \quad \sup_U F \leq \max f + \tfrac{\mu}{2}, \quad \inf_U F \geq \min f - \tfrac{\mu}{2}.$$

We choose $\delta = \delta(\mu)$: Choose $\delta < \min\{\delta_{U(\mu)}, C^2, \frac{\mu^2 c^2}{16\varepsilon^2 C^2}\}$. From (3.34) we deduce firstly that $p \in U(\mu)$ and secondly that, together with

$$\sqrt{\delta} \geq |\overline{\nabla} F(p) + t\overline{\nabla} H(p)| \geq |t\overline{\nabla} H(p)| - |\overline{\nabla} F(p)|,$$

we obtain

$$|t| \leq \tfrac{\sqrt{\delta} + |\overline{\nabla} F(p)|}{|\overline{\nabla} H(p)|} \leq \tfrac{\sqrt{\delta} + C}{c}.$$

From this we get that

$$\begin{aligned} F_H(p,t) = F(p) + tH(p) &\leq \max f + \tfrac{\mu}{2} + \tfrac{\sqrt{\delta}+C}{c} \varepsilon \sqrt{\delta} \\ &\leq \max f + \tfrac{\mu}{2} + \tfrac{2C}{c} \varepsilon \frac{\mu c}{4\varepsilon C} \\ &= \max f + \mu. \end{aligned}$$

This proves the upper bound in (3.35). The lower bound follows similarly.

Step 2. *If (u,τ) is a finite energy solution of the ε-equation. Then*

$$\min f \leq F_H(u_s, \tau_s) \leq \max f.$$

We prove the upper bound in Step 2, the lower bound follows analogously. Assume by contradiction that there exists a time $s_0 \in \mathbb{R}$ such that $F_H(u_{s_0}, \tau_{s_0}) >$

$\max f$. Let $\mu > 0$ be determined by the difference $2\mu := F_H(u_{s_0}, \tau_{s_0}) - \max f$. Let $\delta = \delta(\mu)$ be as in Step 1. Since (u, τ) has finite energy there exists $s_1 \le s_0$ such that $|\nabla^\varepsilon F_H(u_{s_1}, \tau_{s_1})|_\varepsilon^2 \le \delta$. Hence, by (3.35), we have

$$F_H(u_{s_1}, \tau_{s_1}) \le \max f + \mu < \max f + 2\mu = F_H(u_{s_0}, \tau_{s_0}).$$

However, the action is decreasing along the negative gradient flow. This contradiction proves the upper bound. □

4 Linearized operators

4.1 Base Σ

4.1.1 Hilbert manifold $\mathcal{Q}$ and moduli space $\mathcal{M}^0$

Fix two critical points $x^\mp$ of $f\colon \Sigma \to \mathbb{R}$. We denote the Hilbert manifold of all absolutely continuous paths $q\colon \mathbb{R} \to \Sigma$ from x^- to x^+ with square integrable derivative[4] by

$$\mathcal{Q}_{x^-,x^+} := \{q \in W^{1,2}(\mathbb{R}, \Sigma) \mid \lim_{s\to\mp\infty} q(s) = x^\mp\}.$$

We obtain charts for the **Hilbert manifold** $\mathcal{Q}_{x^-,x^+}$ as follows. Let $q_T\colon \mathbb{R} \to \Sigma$ be a smooth map with the property that there is a real $T > 0$ such that $q_T(s) = x^-$ for $s \le -T$ and $q_T(s) = x^+$ for $s \ge T$. Let U_{q_T} be the set of vector fields $\xi \in W^{1,2}(\mathbb{R}, q_T^* T\Sigma)$ such that at each instant of time s the length of $\xi(s)$ is less than the injectivity radius of (Σ, g). The exponential map of (Σ, g) induces a parametrization, still denoted exp, of a neighborhood of q_T in $\mathcal{Q}_{x^-,x^+}$ as follows

$$\exp_{q_T}\colon U_{q_T} \to \mathcal{Q}_{x^-,x^+}, \quad \xi \mapsto \exp_{q_T} \xi, \qquad (\exp_{q_T} \xi)(s) := \exp_{q_T(s)} \xi(s).$$

Consider the tangent bundle of $\mathcal{Q}_{x^-,x^+}$, namely

$$T\mathcal{Q}_{x^-,x^+} \to \mathcal{Q}_{x^-,x^+}, \quad \mathcal{W}_q := T_q\mathcal{Q}_{x^-,x^+} = W^{1,2}(\mathbb{R}, q^*T\Sigma),$$

whose fiber $\mathcal{W}_q := T_q\mathcal{Q}_{x^-,x^+}$ over a path q are the $W^{1,2}$ vector fields along q tangent to Σ. Now consider the vector bundle

$$\mathcal{L} \to \mathcal{Q}_{x^-,x^+}, \quad \mathcal{L}_q := L^2(\mathbb{R}, q^*T\Sigma), \tag{4.36}$$

whose fiber $\mathcal{L}_q$ over a path q consists of the L^2 vector fields along q tangent to Σ. Corresponding inner products are defined by

$$\begin{aligned} \langle \xi, \eta\rangle = \langle \xi, \eta\rangle_2 = \langle \xi, \eta\rangle_{\mathcal{L}_q} &:= \int_{-\infty}^{\infty} \langle \xi(s), \eta(s)\rangle\, ds \\ \langle \xi, \eta\rangle_{1,2} = \langle \xi, \eta\rangle_{\mathcal{W}_q} &:= \int_{-\infty}^{\infty} \langle \xi(s), \eta(s)\rangle + \langle \nabla_s\xi(s), \nabla_s\eta(s)\rangle\, ds \end{aligned}$$

[4] by absolute continuity the derivative, notation $\partial_s q$, exists at almost every instant $s \in \mathbb{R}$

for compactly supported smooth vector fields $\xi, \eta \in C_0^\infty(\mathbb{R}, q^*T\Sigma)$. A section of the vector bundle $\mathcal{L} \to \mathcal{Q}_{x^-,x^+}$, strictly speaking its principal part, is given by

$$\begin{aligned} \mathcal{F}^0 \colon \mathcal{Q}_{x^-,x^+} &\to \mathcal{L}, \\ q &\mapsto \partial_s q + \nabla f(q) \stackrel{(3.22)}{=} \partial_s q + \overline{\nabla} F(q) + \chi(q) \overline{\nabla} H(q). \end{aligned} \tag{4.37}$$

The **base moduli space** is the zero set of the section $\mathcal{F}^0$, in symbols

$$\mathcal{M}^0_{x^-,x^+} = \left\{ q \in \mathcal{Q}_{x^-,x^+} \mid \partial_s q + \overline{\nabla} F(q) + \chi(q) \overline{\nabla} H(q) = 0 \right\}. \tag{4.38}$$

Lemma 4.1 (Regularity and finite energy)**.** *Any element $q \in \mathcal{M}^0_{x^-,x^+}$ is smooth and, by (3.27), of finite energy $E^0(q) = f(x^-) - f(x^+)$.*

Proof. Since by assumption F, H are C^∞ smooth and q is continuous, we see that the derivative $\partial_s q = -\overline{\nabla} F(q) - \chi(q) \cdot \overline{\nabla} H(q)$ is in fact continuous. So $q \in C^1$. But then the right-hand side, hence $\partial_s q$, is C^1, so $q \in C^2$, and so on. □

4.1.2 Linearization of base equation

Linearizing the section $\mathcal{F}^0$ at a zero $q \colon \mathbb{R} \to \Sigma$ we obtain the linear operator

$$D^0_q := d\mathcal{F}^0(q) \colon W^{1,2}(\mathbb{R}, q^*T\Sigma) \to L^2(\mathbb{R}, q^*T\Sigma)$$

which is of the form

$$\begin{aligned} D^0_q \xi &\stackrel{1}{=} \nabla_s \xi + \nabla_\xi \nabla f|_q \stackrel{(2.11)}{=} \nabla_s \xi + \nabla_\xi \left(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q \right) \\ &\stackrel{2}{=} \overline{\nabla}_s \xi + \overline{\nabla}_\xi \left(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q \right) \stackrel{(2.11)}{=} \overline{\nabla}_s \xi + \overline{\nabla}_\xi \nabla f(q) \\ &\stackrel{3}{=} \overline{\nabla}_s \xi + \overline{\nabla}_\xi \overline{\nabla} F|_q + \chi|_q \overline{\nabla}_\xi \overline{\nabla} H|_q + (d\chi|_q \xi) \cdot \overline{\nabla} H|_q. \end{aligned} \tag{4.39}$$

For general elements $q \in \mathcal{M}^0_{x^-,x^+}$ we define D^0_q by (4.39).

Formula 1 arises when linearizing the base formulation of the section, namely $\mathcal{F}^0(q) = \partial_s q + \nabla f(q) = 0$.

Formula 2 arises when linearizing the ambient formulation of the section, namely $\mathcal{F}^0(q) = \partial_s q + \overline{\nabla} F(q) + \chi(q) \cdot \overline{\nabla} H(q) = 0$. Here the second equation in (3.24) imposes the condition that the domain of D^0_q consists of vector fields ξ along q that must be tangent to Σ.
Formula 2 in (4.39) is a sum of vector fields along $q \colon \mathbb{S}^1 \to \Sigma$ each of which a priori takes values in TM. The sum, however, takes values in $T\Sigma$, indeed

$$D^0_q \xi = \overline{\nabla}_s \xi + \overline{\nabla}_\xi \left(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q \right) \stackrel{(3.22)}{=} \overline{\nabla}_s \xi - \overline{\nabla}_\xi \partial_s q = [\partial_s q, \xi],$$

but the commutator of vector fields tangent to Σ is tangent to Σ. The last identity is torsion freeness of the induced connection $\overline{\nabla}$, Lemma 2.2 (iv). The second equation in formula 2 uses the Leibniz rule, Lemma 2.2 (iii).

Symmetry with respect to g of the map $\xi \mapsto \nabla_\xi \nabla f|_q = \nabla_\xi \bigl(\overline{\nabla} F + \chi \overline{\nabla} H\bigr)|_q$,[5] even in the case where $q \in \Sigma$ is a point and $\xi, \eta \in T_q\Sigma$ vectors, is seen as follows

$$
\begin{aligned}
\langle \eta, \overline{\nabla}_\xi \bigl(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q\bigr)\rangle_G &\stackrel{\perp}{=} \langle \eta, \nabla_\xi \bigl(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q\bigr)\rangle_g \\
&\stackrel{(3.22)}{=} \langle \eta, \nabla_\xi \nabla f|_q\rangle_g \\
&\stackrel{3}{=} \xi \langle \eta, \nabla f|_q\rangle_g - \langle \nabla_\xi \eta, \nabla f|_q\rangle_g \\
&\stackrel{4}{=} \bigl(\xi\eta - \nabla_\xi \eta\bigr) f|_q \\
&\stackrel{5}{=} \bigl(\eta\xi - \nabla_\eta \xi\bigr) f|_q .
\end{aligned}
\tag{4.40}
$$

Here step 3 is by metric compatibility of the Levi-Civita connection, step 4 holds since $\langle \eta, \nabla f\rangle = df(\eta) = \eta f$, and step 5 is torsion freeness of ∇.

Alternatively formula 2 arises from formula 1 by substituting both terms $\nabla_s \xi$ and $\nabla_\xi \nabla f(q)$ by differences according to (2.14):

$$
\begin{aligned}
\nabla_s \xi + \nabla_\xi \nabla f|_q &\stackrel{(2.14)}{=} \overline{\nabla}_s \xi - \mathrm{II}(\partial_s q, \xi) \\
&\quad + \overline{\nabla}_\xi \bigl(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q\bigr) - \mathrm{II}\bigl(\xi, \bigl(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q\bigr)\bigr) \\
&\stackrel{(3.22)}{=} \overline{\nabla}_s \xi + \overline{\nabla}_\xi \bigl(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q\bigr).
\end{aligned}
\tag{4.41}
$$

To see the second step substitute $\partial_s q$, then cancel the two II-terms by symmetry. Such cancellation will not happen for the adjoint operator in (4.42) where $\nabla_s \xi$ appears with the opposite sign, but the other term keeps its sign.

Lemma 4.2. *If $\mathcal{F}^0(q) = 0$, then the kernel of D^0_q contains the element $\partial_s q$.*

Proof. Take the covariant derivative ∇_s of the vector field $\partial_s q + \nabla f(q) = 0$. □

4.1.3 Trivialization of base section and derivative

Given a map $q \in W^{\prime 1,2}(\mathbb{R}, \Sigma)$ and a vector field ξ along q, denote (pointwise for $s \in \mathbb{R}$) parallel transport in (Σ, g) along the geodesic $r \mapsto \exp_q(r\xi)$ by

$$
\phi = \phi(q, \xi) \colon T_q\Sigma \to T_{\exp_q(\xi)}\Sigma.
$$

A trivialization of the base section $\mathcal{F}^0$ is given by the map

$$
\mathcal{F}^0_q(\xi) := \phi(q, \xi)^{-1} \mathcal{F}^0(\exp_q \xi) = \phi(q, \xi)^{-1} \left(\partial_s(\exp_q(\xi)) + \nabla f(\exp_q(\xi))\right)
$$

defined on a sufficiently small neighborhood of the origin (so exp is injective) in the Hilbert space scale $h = (h_m)_{m \in \mathbb{N}_0}$ where $h_m = W^{m+1,2}(\mathbb{R}, q^*T\Sigma)$; see [HWZ21] or the introduction [Web22]. The derivative at the origin

$$
d\mathcal{F}^0_q(0)\xi = \tfrac{d}{dr}\Big|_{r=0} \mathcal{F}^0_q(r\xi) = D^0_q \xi
$$

coincides with the linearization (4.39) of the section $\mathcal{F}^0$ at a zero; details are spelled out, e.g., in the proof of Theorem A.3.1 in [Web99].

[5] the map $\xi \mapsto \overline{\nabla}_\xi \bigl(\overline{\nabla} F + \chi \overline{\nabla} H\bigr)$ takes values in TM only, so it cannot be g-symmetric

4.1.4 Formal adjoint

For $q \in W^{1,2}$ the **formal adjoint** $(D_q^0)^* \colon \mathcal{W}_q \to \mathcal{L}_q$ is determined by

$$\langle \eta, D_q^0 \xi \rangle_2 = \langle (D_q^0)^* \eta, \xi \rangle_2, \quad \forall \xi, \eta \in \mathcal{W}_q = W^{1,2}(\mathbb{R}, q^* T\Sigma), \tag{4.42}$$

and consequently given by the first formula in what follows, namely

$$\begin{aligned} (D_q^0)^* \xi &\stackrel{1}{=} -\nabla_s \xi + \nabla_\xi \nabla f|_q \\ &\stackrel{2}{=} -\overline{\nabla}_s \xi + \mathrm{II}(\partial_s q, \xi) \\ &\quad + \overline{\nabla}_\xi \nabla f|_q - \mathrm{II}\big(\xi, \nabla f|_q\big) \\ &\stackrel{3}{=} -\overline{\nabla}_s \xi + \overline{\nabla}_\xi \big(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q\big) + 2\mathrm{II}\big(\xi, \partial_s q\big) \\ &= -\overline{\nabla}_s \xi + \overline{\nabla}_\xi \nabla f|_q + 2\mathrm{II}\big(\xi, \partial_s q\big) \end{aligned} \tag{4.43}$$

for every $\xi \in \mathcal{W}_q$ and where II is defined by (2.16). Step 3 holds for 0-solutions q. To see step 1 it suffices to work in (4.42) with the dense subspace $C_0^\infty(\mathbb{R}, q^* T\Sigma)$. That ∇_s becomes $-\nabla_s$ follows by partial integration and compact support. The map $\xi \mapsto \nabla_\xi \nabla f$ is symmetric by (4.40) and thus it passes from D_q^0 to the adjoint. To obtain step 2 we substituted each of the two terms tangential to Σ, namely $\nabla_s \xi$ and $\nabla_\xi \nabla f(q)$, according to (2.14). In step 3 we replaced ∇f by $\overline{\nabla} F + \chi \overline{\nabla} H$ using (2.11) and in the II-term by $-\partial_s q$ using (3.22) and symmetry of II.

4.1.5 Base linear estimate

Proposition 4.3. *Let $q \in C^1(\mathbb{R}, \Sigma \times \mathbb{R})$ such that $\|\partial_s q\|_\infty < \infty$ is finite. Then there is a constant $c_b = c_b(\|\partial_s q\|_\infty, \|f\|_{C^2(\Sigma)}, \|\mathrm{II}\|_{L^\infty(\Sigma)})$ such that*

$$\|\nabla_s \xi\| + \big\|\overline{\nabla}_s \xi\big\| \le c_b \left(\big\|D_q^0 \xi\big\| + \|\xi\|\right) \tag{4.44}$$

for all vector fields $\xi \in W^{1,2}(\mathbb{R}, q^* TM)$. The estimate also holds for $(D_q^0)^*$.

Proof. Expand the square $\big\|D_q^0 \xi\big\|^2 = \|\nabla_s \xi + \nabla_\xi \nabla f(q)\|^2$ and use Cauchy-Schwarz and Young to get $\|\nabla_s \xi\|^2 \le 2\|D_q^0 \xi\|^2 - 2\|\nabla \nabla f(q)\|_\infty^2 \|\xi\|^2$. By (2.14) $\|\nabla_s \xi\|^2 = \|\overline{\nabla}_s \xi - \mathrm{II}(\partial_s q, \xi)\|^2$, now expand the square. Same for $(D_q^0)^*$. □

4.1.6 Fredholm property

Given a continuously differentiable path $q \colon \mathbb{R} \to \Sigma$, it makes sense to define operators $D_q^0, (D_q^0)^* \colon \mathcal{W}_q \to \mathcal{L}_q$ by the formulae (4.39) and (4.43), respectively.

A continuous linear operator D between Banach spaces is called **Fredholm** if kernel and cokernel are finite dimensional. Finite codimension implies closed image.[6] The difference $\dim \ker D - \dim \operatorname{coker} D$ is called the **Fredholm index**.

[6] Finite codimension of an arbitrary linear subspace Y does not, in general, imply closedness of Y – for an image $Y = \operatorname{im} T$ of a *continuous* operator T it does.

Proposition 4.4. *Let $q \in C^\infty(\mathbb{R}, \Sigma)$ be a path with non-degenerate boundary conditions $\lim_{s\to\mp\infty} q(s) = x^\mp \in \mathrm{Crit} f$. Then the following is true for the operators $D_q^0, (D_q^0)^* \colon \mathcal{W}_q \to \mathcal{L}_q$ defined by (4.39) and (4.43).*

(Exp. decay) *Any kernel element $\xi = \xi(s)$ of D_q^0 or $(D_q^0)^*$ is C^∞ smooth and decays exponentially with all derivatives, as $s \to \mp\infty$. Hence $\|\xi\|, \|\xi\|_\infty < \infty$.*

(Fredholm) *Both operators D_q^0 and $(D_q^0)^*$ are Fredholm and the Fredholm indices are the Morse index differences, namely*

$$\mathrm{index}\, D_q^0 = \mathrm{ind}_f(x^-) - \mathrm{ind}_f(x^+) = -\,\mathrm{index}(D_q^0)^*.$$

Proof of Proposition 4.4. That an operator $\frac{d}{ds} + A(s)$ with invertible asymptotics $A(\mp\infty)$ has exponentially decaying kernel elements, that it is Fredholm, and that the index is the asymptotics' Morse index difference is well known, see e.g. [Sch93]. In suitable trivializations both D_q^0 and $(D_q^0)^*$ are of such form.

That the formal adjoint is Fredholm whenever D_q^0 is (and of the same Fredholm index times -1) follows immediately from the two vector space equalities

$$\ker(D_q^0)^* = \mathrm{coker}\, D_q^0 := \left(\mathrm{im}\, D_q^0\right)^\perp, \qquad \mathrm{coker}\,(D_q^0)^* = \ker D_q^0. \tag{4.45}$$

Vector space equality one. '$\subset$' Pick $\eta \in \ker(D_q^0)^*$. By definition (4.42) of $(D_q^0)^*$ we have $\langle \eta, D_q^0 \xi\rangle = 0$ for every $\xi \in W^{1,2}$. But this means that $\eta \in \left(\mathrm{im}\, D_q^0\right)^\perp$. '$\supset$' Pick $\eta \in \left(\mathrm{im}\, D_q^0\right)^\perp \subset L^2$. Then

$$\begin{aligned} 0 = \langle \eta, D_q^0 \xi\rangle &\overset{(4.39)}{=} \langle \eta, \nabla_s \xi\rangle + \langle \eta, \nabla_\xi \nabla f(q)\rangle \\ &= \quad \langle \eta, \nabla_s \xi\rangle + \langle \nabla_\eta \nabla f(q), \xi\rangle \end{aligned}$$

for every $\xi \in W^{1,2}$. But this is the definition of weak derivative. So η admits a weak derivative, again denoted by $\nabla_s \eta$, and it is given by

$$\nabla_s \eta = \nabla_\eta \nabla f(q) = D\nabla f(q)\, \eta \in L^2.$$

Indeed the last term lies in L^2, because η does and since $D\nabla f(q)$ is of class C^∞ (as f is and by Lemma 4.1) and decays exponentially with all derivatives: indeed $\nabla f(q) = -\partial_s q \in \ker D_q^0$ is a kernel element by Lemma 4.2. Thus $\eta \in W^{1,2}$. Now we can use the defining identity (4.42) of the adjoint to get that

$$0 = \langle (D_q^0)^* \eta, \xi\rangle = \langle (D_q^0)^* \eta, \xi\rangle_g$$

for every $\xi \in W^{1,2}(\mathbb{R}, q^* T\Sigma)$. Thus $(D_q^0)^* \eta = 0$ by nondegeneracy of g.

The proof of vector space equality two is analogous. ☐

4.2 Ambience $M \times \mathbb{R}$

4.2.1 Hilbert manifold $\mathcal{Z}$ and moduli space $\mathcal{M}^\varepsilon$

Fix two critical points $x^\mp$ of $f = F|_\Sigma$. So $(x^\mp, \chi(x^\mp)) \in \mathrm{Crit} F_H$, by Lemma 2.5. We denote the Hilbert manifold of absolutely continuous paths $z = (u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ from z^- to z^+ with square integrable derivative by

$$\mathcal{Z}_{x^-,x^+}, \qquad x^\mp \in \mathrm{Crit} f, \quad \tau^\mp := \chi(x^\mp), \quad z^\mp := (x^\mp, \tau^\mp) \in \mathrm{Crit} F_H.$$

The tangent space at an element $z = (u,\tau)$ are the pairs $Z = (X, \ell)$ consisting of a $W^{1,2}$ vector field X along u and a $W^{1,2}$ function $\ell\colon \mathbb{R} \to \mathbb{R}$, in symbols

$$\mathcal{W}_{u,\tau} := T_{(u,\tau)} \mathcal{Z}_{x^-,x^+} = W^{1,2}(\mathbb{R}, u^*TM \oplus \mathbb{R}).$$

We use the same symbol $\mathcal{L}$ as in (4.36) also for the vector bundle

$$\mathcal{L} \to \mathcal{Z}_{x^-,x^+}, \qquad \mathcal{L}_{u,\tau} := L^2(\mathbb{R}, u^*TM \oplus \mathbb{R})$$

whose fiber $\mathcal{L}_{u,\tau}$ over a path in $M \times \mathbb{R}$ are the L^2 vector fields along (u,τ).

Given a parameter value $\varepsilon > 0$, a section of the vector bundle $\mathcal{L} \to \mathcal{Z}_{x^-,x^+}$ is defined by

$$\mathcal{F}^\varepsilon(u,\tau) := \partial_s(u,\tau) + \nabla^\varepsilon F_H(u,\tau) \overset{(3.30)}{=} \begin{pmatrix} \partial_s u + \overline{\nabla} F|_u + \tau \overline{\nabla} H|_u \\ \tau' + \varepsilon^{-2} H \circ u \end{pmatrix}. \tag{4.46}$$

By definition the zero set is called the **ambient** or **ε-moduli space**, notation

$$\mathcal{M}^\varepsilon_{x^-,x^+} := \{\mathcal{F}^\varepsilon = 0\} \subset \mathcal{Z}_{x^-,x^+}.$$

4.2.2 Linearization of ambient equation

Linearizing the section $\mathcal{F}^\varepsilon$ at a zero $z = (u,\tau)\colon \mathbb{R} \to M \times \mathbb{R}$ provides the operator

$$D^\varepsilon_{u,\tau} := d\mathcal{F}^\varepsilon(u,\tau)\colon \mathcal{W}_{u,\tau} \to \mathcal{L}_{u,\tau}$$

given by $D^\varepsilon_{u,\tau} Z = \nabla^\varepsilon_s Z + \nabla^\varepsilon_Z \nabla^\varepsilon F_H(u,\tau)$ or, equivalently, given by

$$D^\varepsilon_{u,\tau} \begin{pmatrix} X \\ \ell \end{pmatrix} = \begin{pmatrix} \overline{\nabla}_s X + \overline{\nabla}_X \overline{\nabla} F|_u + \tau \overline{\nabla}_X \overline{\nabla} H|_u + \ell \overline{\nabla} H|_u \\ \ell' + \varepsilon^{-2} dH|_u X \end{pmatrix}. \tag{4.47}$$

for $Z = (X, \ell) \in W^{1,2}(\mathbb{R}, u^*TM \oplus \mathbb{R})$. For $(u,\tau) \in \mathcal{Z}_{x^-,x^+}$ define $D^\varepsilon_{u,\tau}$ by (4.47).

4.2.3 Trivialization of ambient section and derivative

Pick a map $(u,\tau) \in W^{,1,2}(\mathbb{R}, M \times \mathbb{R})$ and a vector field (X, ℓ) along it. Denote parallel transport in (M, G) along the geodesic $r \mapsto \mathrm{Exp}_u(rX)$ by

$$\Phi = \Phi(u, X)\colon T_u M \to T_\Gamma M, \qquad \Gamma := \mathrm{Exp}_u(X),$$

pointwise for $s \in \mathbb{R}$. A trivialization of the ambient section $\mathcal{F}^\varepsilon$ is defined by

$$\mathcal{F}^\varepsilon_{u,\tau}(X,\ell) = \begin{pmatrix} \Phi^{-1}\left(\partial_s\Gamma + \overline{\nabla} F|_\Gamma + (\tau+\ell)\overline{\nabla} H|_\Gamma\right) \\ (\tau+\ell)' + \varepsilon^{-2} H|_\Gamma \end{pmatrix} \tag{4.48}$$

for every vector field (X,ℓ) in a sufficiently small (so Exp is injective) ball $\mathcal{O}$ about the origin of $W^{1,2}(\mathbb{R}, u^*TM \oplus \mathbb{R})$.

To calculate the derivative at the origin we utilize the facts about covariant derivation and exponential maps collected in [Web99, appendix A] where the details of essentially the same linearization are spelled out. Abbreviate $\Phi_r := \Phi(u, rX)$ and $\Gamma_r := \mathrm{Exp}_u(rX)$, then $\frac{d}{dr}\big|_0 \Gamma_r = X$ and

$$\begin{aligned}
&d\mathcal{F}^\varepsilon_{u,\tau}(0,0)\begin{pmatrix} X \\ \ell \end{pmatrix} := \tfrac{d}{dr}\big|_0 \mathcal{F}^\varepsilon_{u,\tau}(rX, r\ell) \\
&\stackrel{1}{=} \frac{d}{dr}\Big|_0 \begin{pmatrix} \Phi_r^{-1}\left(\partial_s(\Gamma_r) + \overline{\nabla} F|_{\Gamma_r}\right) + (\tau + r\ell)\Phi_r^{-1}\overline{\nabla} H|_{\Gamma_r} \\ (\tau+r\ell)' + \varepsilon^{-2} H|_{\Gamma_r} \end{pmatrix} \\
&\stackrel{2}{=} \begin{pmatrix} \tfrac{d}{dr}\big|_0 \left(\Phi_r^{-1}\left(\partial_s(\Gamma_r) + \overline{\nabla} F|_{\Gamma_r}\right)\right) + \ell\overline{\nabla} H|_u + \tau\,\tfrac{d}{dr}\big|_0 \left(\Phi_r^{-1}\overline{\nabla} H|_{\Gamma_r}\right) \\ \tfrac{d}{dr}\big|_0 \left((\tau+r\ell)' + \varepsilon^{-2} H|_{\Gamma_r}\right) \end{pmatrix} \\
&\stackrel{3}{=} \begin{pmatrix} \overline{\nabla}_s X + \overline{\nabla}_X \overline{\nabla} F|_u + \tau \overline{\nabla}_X \overline{\nabla} H|_u + \ell \overline{\nabla} H|_u \\ \ell' + \varepsilon^{-2} dH|_u X \end{pmatrix} \stackrel{(4.47)}{=} D^\varepsilon_{u,\tau}\begin{pmatrix} X \\ \ell \end{pmatrix}.
\end{aligned} \tag{4.49}$$

Step 1 is by definition of $\mathcal{F}^\varepsilon_{u,\tau}$ and linearity of parallel transport. Step 2 uses the Levi-Civita connection $\overline{\nabla}$ of (M,G) and the Leibniz rule. Step 3 holds by Theorem A.3.1 in [Web99], more precisely by terms 1 and 3 in the proof.

4.2.4 Formal adjoint and Fredholm property

The **formal adjoint** $(D^\varepsilon_{u,\tau})^* \colon \mathcal{W}_{u,\tau} \to \mathcal{L}_{u,\tau}$ with respect to the $(0,2,\varepsilon)$ inner product associated to the $(0,2,\varepsilon)$ norm, defined in (4.55) below, is determined by

$$\left\langle \tilde{Z}, D^\varepsilon_{u,\tau} Z \right\rangle_{0,2,\varepsilon} = \left\langle (D^\varepsilon_{u,\tau})^* \tilde{Z}, Z \right\rangle_{0,2,\varepsilon}, \quad \forall Z, \tilde{Z} \in \mathcal{W}_{u,\tau}. \tag{4.50}$$

The formal $(0,2,\varepsilon)$ adjoint is then given by the formula

$$\begin{aligned}
(D^\varepsilon_{u,\tau})^* \begin{pmatrix} X \\ \ell \end{pmatrix} &= (D^\varepsilon_z)^* Z \\
&\stackrel{2}{=} -\nabla^\varepsilon_s Z + \nabla^\varepsilon_Z \nabla^\varepsilon F_H|_z \\
&\stackrel{3}{=} \begin{pmatrix} -\overline{\nabla}_s X + \overline{\nabla}_X \overline{\nabla} F|_u + \tau \overline{\nabla}_X \overline{\nabla} H|_u + \underline{\ell \overline{\nabla} H|_u} \\ -\ell' + \underline{\varepsilon^{-2} dH|_u X} \end{pmatrix}
\end{aligned} \tag{4.51}$$

for every $Z = (X,\ell) \in \mathcal{W}_{u,\tau} = W^{1,2}(\mathbb{R}, u^*TM \oplus \mathbb{R})$. Concerning identity 2, an s-derivative turns, by partial integration, into minus an s-derivative and the operator $Z \mapsto \nabla^\varepsilon_Z \nabla^\varepsilon F_H$ is symmetric by an argument analoguous to (4.40). Alternatively, analyze (4.50) term by term. Apart from the two arguments we just gave, the two underlined terms in (4.51) satisfy the identity

$$\langle \tilde{X}, \underline{\ell \overline{\nabla} H|_u} \rangle + \varepsilon^2 \langle \tilde{\ell}, \underline{\varepsilon^{-2} dH|_u X} \rangle = \langle \tilde{\ell} \overline{\nabla} H|_u, X \rangle + \varepsilon^2 \langle \varepsilon^{-2} dH|_u \tilde{X}, \ell \rangle. \tag{4.52}$$

Mind the tildes. To see the equality write out the inner products as integrals.

Proposition 4.5 (Fredholm property). *For a path $z = (u,\tau) \in \mathcal{Z}^{\varepsilon}_{x^-,x^+}$ with non-degenerate boundary conditions $x^{\mp} \in \mathrm{Crit} f$ the following is true. Both operators $D^{\varepsilon}_{u,\tau}, (D^{\varepsilon}_{u,\tau})^* \colon \mathcal{W}_{u,\tau} \to \mathcal{L}_{u,\tau}$ are Fredholm and*

$$\ker(D^{\varepsilon}_{u,\tau})^* = \mathrm{coker}\, D^{\varepsilon}_{u,\tau} := \left(\mathrm{im}\, D^{\varepsilon}_{u,\tau}\right)^{\perp}, \qquad \mathrm{coker}\,(D^{\varepsilon}_{u,\tau})^* = \ker D^{\varepsilon}_{u,\tau}. \tag{4.53}$$

The Fredholm and Morse indices are related by

$$\mathrm{index}\, D^{\varepsilon}_{u,\tau} = \mathrm{ind}_f(x^-) - \mathrm{ind}_f(x^+) = -\,\mathrm{index}(D^{\varepsilon}_{u,\tau})^*. \tag{4.54}$$

Proof. Analogous to Proposition 4.4; use in addition Lemma 2.6. □

4.2.5 Suitable ε-dependent norms

To obtain uniform estimates for the right inverse with constants independent of $\varepsilon > 0$ small, we must work with ε-dependent norms which are suggested on L^2 by the energy identity (3.33) and on $W^{1,2}$ by the fundamental estimate (4.60). For compactly supported smooth vector fields $Z = (X,\ell)$ along (u,τ) define

$$\begin{aligned}
\|Z\|_{0,2,\varepsilon} &:= \left(\|X\|^2 + \varepsilon^2\|\ell\|^2\right)^{1/2} \\
&\le \|X\| + \varepsilon\|\ell\| \\
\|Z\|_{0,\infty,\varepsilon} &:= \|X\|_\infty + \varepsilon\|\ell\|_\infty \\
\|Z\|_{1,2,\varepsilon} &:= \left(\|X\|^2 + \varepsilon^2\|\ell\|^2 + \varepsilon^2\|\nabla_s X\|^2 + \varepsilon^4\|\ell'\|^2\right)^{1/2} \\
&\le \|X\| + \varepsilon\|\ell\| + \varepsilon\|\overline{\nabla}_s X\| + \varepsilon^2\|\ell'\| \stackrel{(4.59)}{\le} 2^{\frac{3}{2}}\|Z\|_{1,2,\varepsilon}.
\end{aligned} \tag{4.55}$$

Lemma 4.6. *Let $(u,\tau) \in W^{1,2}(\mathbb{R}, M\times\mathbb{R})$ and $\varepsilon > 0$. Then there is the estimate*

$$\varepsilon^{1/2}\|Z\|_{0,\infty,\varepsilon} \le 3\|Z\|_{1,2,\varepsilon} \tag{4.56}$$

for every $Z = (X,\ell) \in W^{1,2}(\mathbb{R}, u^*TM \oplus \mathbb{R})$.

Proof. For $v \colon \mathbb{R} \to \mathbb{R}$ of class C^1 and compactly supported it holds that

$$|v(s)| \cdot v(s) = \int_{-\infty}^{s} \underbrace{\tfrac{d}{d\sigma}\left(|v(\sigma)| \cdot v(\sigma)\right)}_{=2|v(\sigma)|v'(\sigma)} d\sigma = 2\langle |v|, v'\rangle_{L^2} \le 2\|v\|\cdot\|v'\| \le \|v\|_{1,2}$$

where the last step is by Young $ab \le a^2/2 + b^2/2$ and $\|v\|_{1,2}^2 := \|v\|^2 + \|v'\|^2$. So

$$\|v\|_\infty \le \|v\|_{1,2}. \tag{4.57}$$

Use that C^1_0 is dense in $W^{1,2}$ on the domain $\mathbb{R}$, then (4.57) provides the Cauchy property of the approximating sequence, so (4.57) remains true for $v \in W^{1,2}$.

Now we rescale. For $\beta\in\mathbb{R}$ and $\varepsilon>0$ define $v_\beta\colon \mathbb{R}\to\mathbb{R}$ by $v_\beta(s):=v(\varepsilon^{2\beta}s)$. Note that $\|v_\beta\|_\infty=\|v\|_\infty$, but the L^2-norms behave as follows

$$\|v_\beta\|^2=\int_{-\infty}^\infty v(\underbrace{\varepsilon^{2\beta}s}_{\sigma(s)})^2\,ds=\varepsilon^{-2\beta}\int_{-\infty}^\infty v(\sigma)^2\,d\sigma=\varepsilon^{-2\beta}\|v\|^2,$$

$$\|v_\beta'\|^2=\int_{-\infty}^\infty (v'(\underbrace{\varepsilon^{2\beta}s}_{\sigma(s)})\varepsilon^{2\beta})^2\,ds=\varepsilon^{-2\beta}\varepsilon^{4\beta}\int_{-\infty}^\infty (v'(\sigma))^2\,d\sigma=\varepsilon^{2\beta}\|v'\|^2.$$

Now square (4.57) to v_β to see that

$$\begin{aligned}\|v\|_\infty^2=\|v_\beta\|_\infty^2 &\stackrel{(4.57)}{\le} \|v_\beta\|^2+\|v_\beta'\|^2\le\left(\varepsilon^{-\beta}\|v\|\right)^2+\left(\varepsilon^{\beta}\|v'\|\right)^2\\ &\le\left(\varepsilon^{-\beta}\|v\|+\varepsilon^{\beta}\|v'\|\right)^2\end{aligned}$$

whenever $\beta\in\mathbb{R}$ and $\varepsilon>0$. Take the square root, then multiply by ε^β to obtain

$$\varepsilon^\beta\|v\|_\infty\le\|v\|+\varepsilon^{2\beta}\|v'\|. \tag{4.58}$$

With $\beta=\frac{1}{2}$ apply (4.58) for $v(s)=|X(s)|=|X(s)|_G$ and $v(s)=\ell(s)$ to obtain

$$\sqrt{\varepsilon}\|Z\|_{0,\infty,\varepsilon}\stackrel{(4.55)}{=}\sqrt{\varepsilon}\|X\|_\infty+\sqrt{\varepsilon}\varepsilon\|\ell\|_\infty\stackrel{(4.58)}{\le}\|X\|+\varepsilon\|X'\|+\varepsilon\|\ell\|+\varepsilon^2\|\ell'\|.$$

Now the square root of the inequality for non-negative reals

$$(a_1+\dots+a_k)^2\le 2^{k-1}(a_1^2+\dots+a_k^2) \tag{4.59}$$

in case $k=4$ completes the proof of Lemma 4.6. □

4.2.6 Ambient linear estimate along maps $i(q)$

The most important uniform linear estimates in an adiabatic limit are the fundamental estimate, in our case the ambient linear estimate, Theorem 4.7 below,[7] and the key estimate, Theorem 5.8.

In the following we consider maps q that take values in the compact regular hypersurface Σ. Thus we can work directly with the (positive) minimal length $m_H:=\min_\Sigma|\overline{\nabla}H|>0$ along Σ, instead of invoking part (ii) of Theorem 1.3 which only works for small $\varepsilon>0$. In fact, Section 4.2.6 can be generalized to maps $(u,\tau)\in C^1(\mathbb{R},M\times\mathbb{R})$ with $\|\partial_s u\|_\infty+\|\tau\|_\infty<c_w$ and for $\varepsilon>0$ small.

Theorem 4.7. *Let $q\in C^1(\mathbb{R},\Sigma)$. Let $\|\partial_s q\|_\infty<c_w$ be bounded by a constant. Then there is a constant $c_a=c_a(m_H,c_w,\|H\|_{C^2(\Sigma)},\|F\|_{C^2(\Sigma)})>0$ such that*

$$\varepsilon^{-1}\|dH|_q X\|+\|\ell\|+\|\overline{\nabla}_s X\|+\varepsilon\|\ell'\|\le c_a\left(\|D_q^\varepsilon Z\|_{0,2,\varepsilon}+\|X\|\right) \tag{4.60}$$

for all $\varepsilon>0$ and $Z=(X,\ell)\in W^{1,2}(\mathbb{R},q^*TM\oplus\mathbb{R})$. The estimate continues to hold for $(D_q^\varepsilon)^*$. The constants c_a is invariant under s-shifts of q.

[7] In PDE cases, such as [SW06], the ambient linear estimate is often much weaker than in our ODE case, so it must be improved to what we refer to as the fundamental estimate.

Proof. Fix $Z=(X,\ell)$ in the dense subset $C_0^\infty(\mathbb{R},q^*TM\oplus\mathbb{R})$. Take the square

$$\begin{aligned}\|D_q^\varepsilon Z\|_{0,2,\varepsilon}^2 &= \|\overline{\nabla}_s X + \overline{\nabla}_X\left(\overline{\nabla}F|_q + \chi|_q\overline{\nabla}H|_q\right) + \ell\overline{\nabla}H|_q\|_{L^2_q}^2\\ &\quad + \varepsilon^2\|\ell' + \varepsilon^{-2}dH|_qX\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Consider the first term in the sum. Expand the square to get

$$\begin{aligned}&\|\overline{\nabla}_s X + \overline{\nabla}_X\left(\overline{\nabla}F|_q + \tau\overline{\nabla}H|_q\right) + \ell\overline{\nabla}H|_q\|_{L^2_q}^2\\
&= \|\overline{\nabla}_s X\|_{L^2_q}^2 + \|\overline{\nabla}_X\left(\overline{\nabla}F|_q + \tau\overline{\nabla}H|_q\right)\|_{L^2_q}^2 + \|\ell\overline{\nabla}H|_q\|_{L^2_q}^2\\
&\quad + 2\left\langle \sqrt{2}\overline{\nabla}_X\left(\overline{\nabla}F|_q + \tau\overline{\nabla}H|_q\right), \tfrac{1}{\sqrt{2}}\ell\overline{\nabla}H|_q\right\rangle_{L^2_q}\\
&\quad + 2\left\langle \tfrac{1}{\sqrt{2}}\overline{\nabla}_s X, \sqrt{2}\overline{\nabla}_X\left(\overline{\nabla}F|_q + \tau\overline{\nabla}H|_q\right)\right\rangle_{L^2_q} + 2\left\langle \overline{\nabla}_s X, \ell\overline{\nabla}H|_q\right\rangle_{L^2_q}\\
&\geq \tfrac{1}{2}\|\overline{\nabla}_s X\|_{L^2_q}^2 + \tfrac{1}{2}\|\ell\overline{\nabla}H|_q\|_{L^2_q}^2 - 3\|\overline{\nabla}_X\left(\overline{\nabla}F|_q + \tau\overline{\nabla}H|_q\right)\|_{L^2_q}^2 + 2\left\langle \overline{\nabla}_s X, \ell\overline{\nabla}H|_q\right\rangle_{L^2_q}\\
&\geq \tfrac{1}{2}\|\overline{\nabla}_s X\|_{L^2_q}^2 + \tfrac{{m_H}^2}{2}\|\ell\|_{L^2(\mathbb{R})}^2 - 3\left(\|F\|_{C^2(\Sigma)} + \|\tau\|_\infty\|H\|_{C^2(\Sigma)}\right)\|X\|_{L^2_q}^2\\
&\quad + \underline{2\left\langle \overline{\nabla}_s X, \ell\overline{\nabla}H|_q\right\rangle_{L^2_q}}.\end{aligned}$$

Here we also used Cauchy-Schwarz followed by Young's inequality, then we pulled out the L^∞ norms. Next consider the second term in the sum. Expand the square and integrate by parts to get

$$\begin{aligned}&\varepsilon^2\|\ell' + \varepsilon^{-2}dH|_qX\|_{L^2(\mathbb{R})}^2\\
&= \varepsilon^2\|\ell'\|_{L^2(\mathbb{R})}^2 + \varepsilon^{-2}\|dH|_qX\|_{L^2(\mathbb{R})}^2 + 2\left\langle \ell', \langle\overline{\nabla}H|_q, X\rangle_G\right\rangle_{L^2(\mathbb{R})}\\
&= \varepsilon^2\|\ell'\|_{L^2(\mathbb{R})}^2 + \varepsilon^{-2}\|dH|_qX\|_{L^2(\mathbb{R})}^2\\
&\quad - \left\langle \tfrac{m_H}{\sqrt{2}}\ell, 2\tfrac{\sqrt{2}}{m_H}\langle\overline{\nabla}_s\overline{\nabla}H|_q, X\rangle_G\right\rangle_{L^2(\mathbb{R})} - 2\left\langle \ell, \langle\overline{\nabla}H|_q, \overline{\nabla}_s X\rangle_G\right\rangle_{L^2(\mathbb{R})}\\
&\geq \varepsilon^2\|\ell'\|_{L^2(\mathbb{R})}^2 + \varepsilon^{-2}\|dH|_qX\|_{L^2(\mathbb{R})}^2 - \tfrac{{m_H}^2}{4}\|\ell\|_{L^2(\mathbb{R})}^2\\
&\quad - \tfrac{4\|\partial_s u\|_\infty^2\|H\|_{C^2(\Sigma)}^2}{{m_H}^2}\|X\|_{L^2_q}^2 - \underline{2\left\langle \ell, \langle\overline{\nabla}H|_q, \overline{\nabla}_s X\rangle_G\right\rangle_{L^2(\mathbb{R})}}.\end{aligned}$$

To obtain the inequality we used Cauchy-Schwarz followed by Young's inequality, then we pulled out the L^∞ norms. Adding the two estimates the underlined terms cancel and we obtain the estimate (4.60).

The estimate for the formal adjoint follows exactly the same way. Here the derivative terms show up with a minus sign. The underlined terms now show up with a factor -1 and so they still cancel. □

Remark 4.8. Under the hypotheses of Theorem 4.7 there are $C,\varepsilon_0>0$ with

$$\varepsilon^{-1}\|dH|_qX\| + \|\ell\| + \|\overline{\nabla}_s X\| + \varepsilon\|\ell'\| \leq C\left(\|D_q^\varepsilon Z\|_{0,2,\varepsilon} + \|\tan X\|\right)$$

for every $Z=(X,\ell)\in W^{1,2}(\mathbb{R},q^*TM\oplus\mathbb{R})$ and whenever $\varepsilon\in(0,\varepsilon_0]$. Similarly for $(D_q^\varepsilon)^*$ and the constants C,ε_0 are invariant under s-shifts of q.

To see this decompose $X = \tan X + \operatorname{nor} X$ on the right of (4.60) to obtain

$$\|X\| \le \|\tan X\| + \|\operatorname{nor} X\| \overset{(2.11)}{\le} \|\tan X\| + \tfrac{\varepsilon}{m_H} \varepsilon^{-1} \left\| dH|_q X \right\| .$$

Incorporate the last summand into the left-hand side of (4.60) for small ε.

Corollary 4.9. *Let $q \in C^1(\mathbb{R}, \Sigma)$. Let $\|\partial_s q\|_\infty < c_w$ be bounded by a constant. Let ε_0 be the constant in Remark 4.8. Then there is a constant $C_a > 0$ with*

$$\tfrac{1}{3}\varepsilon^{1/2} \|Z\|_{0,\infty,\varepsilon} \le \|Z\|_{1,2,\varepsilon} \le \varepsilon C_a \|D_q^\varepsilon Z\|_{0,2,\varepsilon} + \|\tan X\| \tag{4.61}$$

for all $\varepsilon \in (0, \varepsilon_0]$ and $Z = (X, \ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$. The estimate also holds for $(D_q^\varepsilon)^*$. The constants C_a, ε_0 are invariant under s-shifts of q.

Proof. By definition (4.55) of the $(1, 2, \varepsilon)$-norm, by writing $X = \tan X + \operatorname{nor} X$, and since $\|\operatorname{nor} X\| \le \frac{1}{m_H} \left\| dH|_q X \right\|$ by (2.11), we get that

$$\begin{aligned} \|Z\|_{1,2,\varepsilon} &\le \|\tan X\| + \|\operatorname{nor} X\| + \varepsilon\|\ell\| + \varepsilon\|\overline{\nabla}_s X\| + \varepsilon^2\|\ell'\| \\ &\le \|\tan X\| + \varepsilon \cdot \tfrac{\max\{m_H, 1\}}{m_H} \left(\varepsilon^{-1}\|dH|_q X\| + \|\ell\| + \|\overline{\nabla}_s X\| + \varepsilon\|\ell'\|\right) . \end{aligned}$$

Now apply Remark 4.8. Inequality (4.56) concludes the proof. □

5 Linear estimates

Throughout Section 5 we study linearized operators along maps q which take values in the compact hypersurface Σ. Thus we can work with the constant

$$m_H := \min_\Sigma \left|\overline{\nabla} H\right| > 0,$$

see (2.11), which does not impose restrictions on the values of $\varepsilon > 0$, in sharp contrast to the constant c_κ that appears in part (ii), see [FW22a], of the a priori Theorem 1.3 requiring a small parameter interval $(0, \varepsilon_\kappa]$.

5.1 Canonical embedding and orthogonal projection

The elements q of the Hilbert manifold $\mathcal{Q}_{x^-,x^+}$ are paths that take values in the regular level set $\Sigma = H^{-1}(0)$ along which the map χ defined by (2.8) is well defined. By (2.19) and (3.23) there is the **canonical embedding**

$$i \colon \mathcal{Q}_{x^-,x^+} \to \mathcal{Z}_{x^-,x^+}, \quad q \mapsto (q, \chi(q)),$$

which is useful to compare the base solutions q and the ε-solutions (u, τ). At a path $q \in \mathcal{Q}_{x^-,x^+}$ the linearization of the natural embedding is given by

$$\begin{aligned} & T_q\mathcal{Q}_{x^-,x^+} \to T_{i(q)} i(\mathcal{Q}_{x^-,x^+}) \qquad \subset T_{i(q)}\mathcal{Z}_{x^-,x^+} \\ I_q := di|_q \colon & W^{1,2}(\mathbb{R}, q^*T\Sigma) \to W^{1,2}(\mathbb{R}, q^*T\Sigma \oplus \mathbb{R}) \subset W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R}) \\ & \xi \mapsto (\xi, d\chi|_q \xi) . \end{aligned}$$

Definition 5.1 (Orthogonal projection)**.** At $q \in \mathcal{Q}_{x^-,x^+}$ the $(0,2,\varepsilon)$-orthogonal projection onto the image of the linearized embedding I_q is the composition

$$\Pi_\varepsilon^\perp = I_q \pi_\varepsilon^\perp \colon T_{i(q)}\mathcal{Z}_{x^-,x^+} = W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R}) \to W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$$

whose value on $Z = (X, \ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$ is determined by

$$\left\langle Z - I_q \pi_\varepsilon^\perp Z, I_q \xi \right\rangle_{0,2,\varepsilon} = 0 \tag{5.62}$$

for every vector field $\xi \in T_q\mathcal{Q}_{x^-,x^+} = W^{1,2}(\mathbb{R}, q^*T\Sigma)$.

Lemma 5.2. *a) The linear map $\pi_\varepsilon^\perp \colon T_{i(q)}\mathcal{Z}_{x^-,x^+} \to T_q\mathcal{Q}_{x^-,x^+}$ is given by*

$$\pi_\varepsilon^\perp(X,\ell) = \left(\mathbb{1} + \varepsilon^2\mu^2\, P\right)^{-1} \left(\tan X + \varepsilon^2 \ell \nabla\chi|_q\right), \qquad \mu := \left|\nabla\chi(q)\right|, \tag{5.63}$$

for every pair $Z = (X, \ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$. Here $\nabla\chi$ is the gradient in (Σ, g) and P is the pointwise orthogonal projection[8]

$$\begin{aligned} P = P_q \colon T_q\Sigma &\to V_q := \mathbb{R}\nabla\chi|_q \subset T_q\Sigma \\ \xi &\mapsto \frac{\langle \nabla\chi|_q, \xi\rangle}{\mu^2}\, \nabla\chi|_q, \end{aligned} \tag{5.64}$$

where q actually abbreviates $q(s)$ for $s \in \mathbb{R}$. By compactness of Σ the constant $\mu_\infty := \max\{1, \|\nabla\chi\|_{L^\infty(\Sigma)}\}$ is finite. b) It holds that $\pi_\varepsilon^\perp I_q = \mathbb{1}$, so $(\Pi_\varepsilon^\perp)^2 = \Pi_\varepsilon^\perp$.

Proof. a) Let $\xi_0 := \pi_\varepsilon^\perp(X,\ell)$. By (5.62) the vector field ξ_0 lives in $T\Sigma$ and

$$\begin{aligned} 0 &= \langle X - \xi_0, \xi\rangle_G + \varepsilon^2 \left(\ell - d\chi|_q \xi_0\right) d\chi|_q \xi \\ &= \left\langle \tan X - \xi_0 + \varepsilon^2 \left(\ell - \langle \nabla\chi, \xi_0\rangle\right) \nabla\chi, \xi \right\rangle \end{aligned}$$

pointwise at $s \in \mathbb{R}$ and for every $\xi \in T_q\mathcal{Q}_{x^-,x^+}$. We wrote $X = \tan X + \operatorname{nor} X$, we used that $\xi \perp \operatorname{nor} X$, and we replaced the metric G by g. By non-degeneracy

$$\tan X + \varepsilon^2 \ell \nabla\chi = \xi_0 + \varepsilon^2 \langle \nabla\chi, \xi_0\rangle \nabla\chi = \xi_0 + \varepsilon^2\mu^2 P\xi_0$$

and so $\pi_\varepsilon^\perp(X,\ell) = \xi_0 = \left(\mathbb{1} + \varepsilon^2 \langle\nabla\chi, \mathbb{1}\rangle \nabla\chi\right)^{-1} \left(\tan X + \varepsilon^2 \ell \nabla\chi\right)$.

b) Apply the isomorphism in (5.67) to the desired identity $\xi = \pi_\varepsilon^\perp I_q \xi$ to get equivalently $\xi + \varepsilon^2\mu^2 P\xi = \xi + \varepsilon^2 (d\chi|_q \xi)\nabla\chi$ which is true by definition of P. □

5.1.1 Ansatz for a suitable projection

In previous adiabatic limits [DS94, Gai99, Web99, GS05, SW06] – where the spatial part involves differential equations, so the flow equation is a PDE and not just an ODE as in the present article – it was crucial for the functioning of the Newton iteration not to choose the operator $\pi_\varepsilon^\perp$ associated to the orthogonal projection $\Pi_\varepsilon^\perp = I_q \pi_\varepsilon^\perp$. There the natural orthogonal choice did produce an

[8] if $\nabla\chi(q(s) = 0$ vanishes at some s, then $\mu_{q(s)}^2 P_{q(s)} = 0$ is the zero map at that s

abundance of powers of ε in one component, but a lack in the other one. To balance this out one can introduce parameters $\alpha, \beta > 0$ and make the Ansatz

$$\pi_\varepsilon(X, \ell) := \left(1\!\!1 + \varepsilon^\alpha \mu^2 P\right)^{-1} \left(\tan X + \varepsilon^\beta \ell \nabla\chi|_q\right). \tag{5.65}$$

It seems a common principle that the epsilon power $\beta = 2$ that shows up in the *orthogonal* projection (5.63) and also in the ε-equation (3.30), is the right value of β. Usually the value $\beta = 2$ is suggested, too, when comparing the linear operators D_q^0 and D_q^ε, see the proof of Proposition 5.5. In the present article the choice $\beta = 2$ also optimizes the Uniqueness Theorem 6.2, see (6.105). For $\alpha = 1$ the operator comparison estimate (5.72) has a nicely equilibrated right hand side, but the orthogonal choice $\alpha = 2$ works as well.

Lemma 5.3 (Le. 4.1.5)**.** *Let $q \in W^{1,2}(\mathbb{R}, \Sigma)$ and $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned}
\left\|(1\!\!1 + \varepsilon^\alpha \mu^2 P)^{-1}\xi\right\| &\le \|\xi\| \\
\left\|(1\!\!1 + \varepsilon^\alpha \mu^2 P)^{-1}P\xi\right\| &\le \|\xi\| & (1\!\!1 + \varepsilon^\alpha \mu^2 P)^{-1}P &= \tfrac{P}{1+\varepsilon^\alpha\mu^2} \\
\left\|(1\!\!1 + \varepsilon^\alpha \mu^2 P)^{-1}\varepsilon^{\alpha/2}\mu P\xi\right\| &\le \tfrac{1}{2}\|\xi\| & \tfrac{\varepsilon^{\alpha/2}}{1+\varepsilon^\alpha\mu(s)^2} &\le \tfrac{1}{2\mu(s)} \\
\left\|(1\!\!1 + \varepsilon^\alpha \mu^2 P)^{-1}\varepsilon^{\alpha}\mu^2 P\xi\right\| &\le \|\xi\| & \tfrac{\varepsilon^{\alpha}}{1+\varepsilon^\alpha\mu(s)^2} &\le \tfrac{1}{\mu(s)^2}
\end{aligned} \tag{5.66}$$

for all constants $\varepsilon > 0$, vector fields $\xi \in W^{1,2}(\mathbb{R}, q^*T\Sigma)$, and reals $s \in \mathbb{R}$.

Recall that $P^2 = P$, pointwise at $q(s)$, is a projection, an orthogonal one, hence of norm 1. So estimate one with ξ replaced by $P\xi$ implies estimate two. Note that estimate two in the lemma allows for removing the square root μP, at cost $\varepsilon^{\alpha/2}$, of the operator $(\mu P)^2 = \mu^2 P$ that appears in $(1\!\!1 + \varepsilon^\alpha \mu^2 P)^{-1}$, whereas removing $(\mu P)^2 = \mu^2 P$ itself has cost $\varepsilon^\alpha \mu^2$. These facts are somewhat hidden since $P^2 = P$. As it turns out only estimates one and two in Lemma 5.3 are of significance in the present ODE adiabatic limit. In sharp contrast, the refined estimates three and four were foundational in the PDE adiabatic limit [SW06] where $P = \nabla_t$ is one spatial derivative. At present the finer estimate three in the lemma can still be used for cosmetics, for example to get constant 2 in estimate three in (5.68), as opposed to a factor involving μ_∞, see (5.70).

Proof. Let $\varepsilon > 0$ and $\xi \in W^{1,2}(\mathbb{R}, q^*T\Sigma)$. Pick $s \in \mathbb{R}$. The operator

$$B(s) := 1\!\!1 + \varepsilon^\alpha \mu_{q(s)}^2 P_{q(s)} \colon T_{q(s)}\Sigma \to T_{q(s)}\Sigma \tag{5.67}$$

is symmetric since the projection is orthogonal, thus the eigenvalues are real. The eigenvalues of $B(s)$ are positive: The projection $P_{q(s)}$, defined by (5.64), has eigenvalue 0 on $V_{q(s)}^\perp$ and 1 on the line $V_{q(s)} = \mathbb{R}\nabla\chi|_{q(s)}$. Thus the operator $B(s)$ has eigenvalue 1 on $V_{q(s)}^\perp$ and $1 + \varepsilon^\alpha \mu_{q(s)}^2$ on $V_{q(s)}$. Hence $B(s)$ is invertible. The inverse $B(s)^{-1}$ has spectrum $\{1, (1 + \varepsilon^\alpha \mu_{q(s)}^2)^{-1}\}$, thus norm 1. Hence

$$\left\|(1\!\!1 + \varepsilon^\alpha \mu^2 P)^{-1}\xi\right\| = \left\|B(s)^{-1}\xi\right\| \le \|\xi\|.$$

This proves estimate one. For estimate two replace ξ by $P\xi$ and use that by orthogonality $|P_{q(s)}\xi(s)| \le |\xi(s)|$ at any $s \in \mathbb{R}$. The symmetric operator

$$(\mathbb{1} + \varepsilon^\alpha \mu_{q(s)}^2 P_{q(s)})^{-1} P_{q(s)} \colon T_{q(s)}\Sigma \to T_{q(s)}\Sigma$$

has eigenvalue 0 on $V_{q(s)}^\perp$ and $1/(1+\varepsilon^\alpha \mu_{q(s)}^2)$ on $V_{q(s)} = \operatorname{im} P_{q(s)} = \mathbb{R}\nabla\chi(q(s))$. This proves in (5.66) the identity in line two. By Young $1 \cdot \varepsilon^{\alpha/2}\mu \le (1^2 + (\varepsilon^{\alpha/2}\mu)^2)/2$, hence $\varepsilon^{\alpha/2}\mu/(1+\varepsilon^\alpha\mu^2) \le 1/2$ and this implies estimate three. Clearly $\varepsilon^\alpha\mu^2/(1+\varepsilon^\alpha\mu^2) \le 1$ and this implies estimate four. □

5.1.2 Component estimates

As discussed prior to Lemma 5.3 we already choose $\beta = 2$.

Lemma 5.4. *Let $q \in W^{1,2}(\mathbb{R},\Sigma)$. In π_ε let $\alpha \in [1,2]$ and $\beta = 2$. Then*

$$\begin{aligned}
\|X - \pi_\varepsilon Z\| &\le \tfrac{1}{m_H} \|dH|_q X\| + \varepsilon^\alpha \mu_\infty^2 \|P \tan X\| + \varepsilon^2 \mu_\infty \|\ell\| \\
\|\ell - d\chi|_q \pi_\varepsilon Z\| &\le \mu_\infty \|P \tan X\| + 2\|\ell\| \\
\|Z - I_q \pi_\varepsilon Z\|_{0,2,\varepsilon} &\le \tfrac{1}{m_H} \|dH|_q X\| + 2\mu_\infty^2 \varepsilon \|P\tan X\| + 4\mu_\infty \varepsilon \|\ell\| \\
\|\pi_\varepsilon Z\| &\le \|I_q \pi_\varepsilon Z\|_{0,2,\varepsilon} \le 2\|Z\|_{0,2,\varepsilon}
\end{aligned} \tag{5.68}$$

for all constants $\varepsilon \in (0,1]$ and pairs $Z = (X,\ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$ where

$$m_H := \min_\Sigma \left|\overline{\nabla} H\right| > 0, \qquad \mu_\infty := \max\{1, \|\nabla\chi\|_{L^\infty(\Sigma)}\} \in [1,\infty). \tag{5.69}$$

Proof. Given q and $Z = (X,\ell)$, we denote

$$\xi_0 := \pi_\varepsilon Z = (\mathbb{1} + \varepsilon^\alpha \mu^2 P)^{-1} \left(\tan X + \varepsilon^2 \ell \nabla\chi\right).$$

Write $X = \operatorname{nor} X + B^{-1}(B \tan X)$, with B given by (5.67), in order to obtain

$$X_1 := X - \xi_0 = \operatorname{nor} X + (\mathbb{1} + \varepsilon^\alpha\mu^2 P)^{-1} \left(\varepsilon^\alpha \mu^2 P \tan X - \varepsilon^2 \ell \nabla\chi\right)$$

pointwise at $s \in \mathbb{R}$. By (2.11) and Lemma 5.3, we get

$$\|X_1\| \le \tfrac{1}{m_H} \|dH|_q X\| + \varepsilon^\alpha \mu_\infty^2 \|P \tan X\| + \varepsilon^2 \mu_\infty \|\ell\|$$

Similarly, we get

$$\begin{aligned}
\ell_1 :&= \ell - d\chi|_q \xi_0 \\
&= \ell - d\chi|_q \left(\mathbb{1} + \varepsilon^\alpha \mu^2 P\right)^{-1} \left(\tan X + \varepsilon^2 \ell \nabla\chi\right) \\
&= \ell - \left\langle \nabla\chi, \left(\mathbb{1} + \varepsilon^\alpha\mu^2 P\right)^{-1} \left(P \tan X + (\mathbb{1} - P)\tan X + \varepsilon^2 \ell \nabla\chi\right)\right\rangle \\
&\stackrel{4}{=} \ell - \tfrac{\langle \nabla\chi, P\tan X\rangle}{1+\varepsilon^\alpha\mu^2} - 0 - \tfrac{\varepsilon^2\mu^2}{1+\varepsilon^\alpha\mu^2} \ell
\end{aligned}$$

By Lemma 5.3 we get

$$\|\ell_1\| \le \mu_\infty \|P \tan X\| + 2\|\ell\|.$$

For later use in (5.70), note that by equality 4 above

$$d\chi(q)\xi_0 = \tfrac{\langle \nabla\chi, P\tan X\rangle}{1+\varepsilon^\alpha\mu^2} + \tfrac{\varepsilon^\alpha\mu^2}{1+\varepsilon^\alpha\mu^2}\varepsilon^{2-\alpha}\ell.$$

Take the sum of the estimates for X_1 and ℓ_1 to obtain

$$\begin{aligned}
\left\|Z - I_q\pi_\varepsilon Z\right\|_{0,2,\varepsilon} &\le \left\|X_1\right\| + \varepsilon\left\|\ell_1\right\| \\
&\le \tfrac{1}{m_H}\left\|dH|_q X\right\| + \mu_\infty^2\varepsilon(1+\varepsilon^{\alpha-1})\left\|P\tan X\right\| \\
&\quad + 2\mu_\infty\varepsilon(1+\varepsilon)\left\|\ell\right\|.
\end{aligned}$$

Now use the hypotheses $\alpha \ge 1$ and $\varepsilon \le 1$. By Lemma 5.3, also using the finer third estimate, applied to the earlier identity for ξ_0, and for $d\chi(q)\xi_0$, we get

$$\begin{aligned}
\left\|\xi_0\right\| &\le \left\|\tan X\right\| + \tfrac{1}{2}\varepsilon^{2-\frac{\alpha}{2}}\left\|\ell\right\|, \\
\left\|d\chi|_q\xi_0\right\| &\le \tfrac{1}{2}\varepsilon^{-\frac{\alpha}{2}}\left\|\tan X\right\| + \varepsilon^{2-\alpha}\left\|\ell\right\|.
\end{aligned} \tag{5.70}$$

Square these two inequalities and take the sum to obtain

$$\begin{aligned}
\left\|I_q\pi_\varepsilon Z\right\|_{0,2,\varepsilon}^2 &= \left\|\xi_0\right\|^2 + \varepsilon^2\left\|d\chi|_q\xi_0\right\|^2 \\
&\le 2(1+\tfrac{1}{4}\varepsilon^{2-\alpha})\left\|\tan X\right\|^2 + 2\varepsilon^{2-\alpha}(\tfrac{1}{4}+\varepsilon^{2-\alpha})\varepsilon^2\left\|\ell\right\|^2 \\
&\le 3\left(\left\|\tan X\right\|^2 + \varepsilon^2\left\|\ell\right\|^2\right).
\end{aligned}$$

Note that $\|\tan X\| \le \|X\|$ since tan is an orthogonal projection. The proof of Lemma 5.4 is complete. □

5.2 Comparing the base and ambient linear operators

We keep focusing on the special class of the ambient linear operators, see (4.47), along the canonical embedding $i\colon q \mapsto (q, \chi(q))$. The aim of this section is to control, downstairs in q-space, the difference between the base linear operator along q and the ambient linear operator along $i(q)$.

For $q \in C^1(\mathbb{R}, \Sigma)$ denote the ambient linear operators along the graph of χ over q by $D_q^\varepsilon := D_{q,\chi(q)}^\varepsilon$ and $(D_q^\varepsilon)^* := (D_{q,\chi(q)}^\varepsilon)^*$. These operators have the form

$$\begin{aligned}
D_q^\varepsilon \begin{pmatrix} X \\ \ell \end{pmatrix} &\overset{(4.47)}{=} \begin{pmatrix} \overline{\nabla}_s X + \overline{\nabla}_X \overline{\nabla} F|_q + \chi(q)\overline{\nabla}_X\overline{\nabla}H|_q + \ell\overline{\nabla}H|_q \\ \ell' + \varepsilon^{-2} dH|_q X \end{pmatrix} \\
(D_q^\varepsilon)^* \begin{pmatrix} X \\ \ell \end{pmatrix} &\overset{(4.51)}{=} \begin{pmatrix} -\overline{\nabla}_s X + \overline{\nabla}_X \overline{\nabla} F|_q + \chi(q)\overline{\nabla}_X\overline{\nabla}H|_q + \ell\overline{\nabla}H|_q \\ -\ell' + \varepsilon^{-2} dH|_q X \end{pmatrix}
\end{aligned} \tag{5.71}$$

for every $Z = (X, \ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$.

Proposition 5.5. *In π_ε let $\alpha > 0$ and $\beta = 2$. Let $q \in C^1(\mathbb{R}, \Sigma)$ be a map with bounded derivative $\partial_s q$. Then there is a constant $c_d > 0$ such that*

$$\left\|(D_q^0)^*\pi_\varepsilon Z - \pi_\varepsilon (D_q^\varepsilon)^* Z\right\| \le \varepsilon c_d \left(\tfrac{1}{\varepsilon}\left\|dH|_q X\right\| + \varepsilon^{\alpha-1}\left\|\tan X\right\| + \varepsilon\left\|\ell\right\|\right) \tag{5.72}$$

for every $Z = (X, \ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$ whenever $\varepsilon \in (0, 1]$. The same is true for $D_q^0\pi_\varepsilon - \pi_\varepsilon D_q^\varepsilon$. The constant c_d is invariant under s-shifts of q.

Note that for $\alpha = 1$ all three terms on the right hand side of (5.72) are of the same quality in terms of powers of ε as in the ambient linear estimate (4.60).

5.2.1 Commutators along Σ

The proof of Proposition 5.5 below suggests the value $\beta = 2$. For better reading we set $\beta = 2$ already now. Let $\alpha \in \mathbb{R}$.

A commutator with the inverse operator $\left(\mathbb{1} + \varepsilon^\alpha \mu^2 P\right)^{-1}$ should be rewritten in terms of a commutator with the operator itself. The reason is that commutators are additive and the first summand of $\mathbb{1} + \varepsilon^\alpha \mu^2 P$ commutes with anybody, thus disappears, and the second summand then brings in the precious factor ε^α.

Here is an example of this technique, further below in (5.75) there will be another one. In preparation to prove Proposition 5.5 note that along Σ it holds

$$\begin{aligned}[\nabla_s, \left(\mathbb{1} + \varepsilon^\alpha \mu^2 P\right)^{-1}] &= \left(\mathbb{1} + \varepsilon^\alpha \mu^2 P\right)^{-1} [\mathbb{1} + \varepsilon^\alpha \mu^2 P, \nabla_s] \left(\mathbb{1} + \varepsilon^\alpha \mu^2 P\right)^{-1} \\ &= \varepsilon^\alpha \left(\mathbb{1} + \varepsilon^\alpha \mu^2 P\right)^{-1} [\mu^2 P, \nabla_s] \left(\mathbb{1} + \varepsilon^\alpha \mu^2 P\right)^{-1}\end{aligned}$$

where, by definition (5.64) of P, the last commutator has the form

$$[\mu^2 P, \nabla_s]\xi = -\langle \nabla_s \nabla\chi, \xi\rangle \nabla\chi - \langle \nabla\chi, \xi\rangle \nabla_s \nabla\chi$$

for every $\xi \in W^{1,2}(\mathbb{R}, q^*T\Sigma)$. Thus, abbreviating $B \stackrel{(5.67)}{:=} \mathbb{1} + \varepsilon^\alpha \mu^2 P$, we get

$$[\nabla_s, B^{-1}]\cdot = -\varepsilon^\alpha B^{-1} \Big(\left\langle \nabla_s \nabla\chi, B^{-1}\cdot \right\rangle \nabla\chi + \left\langle \nabla\chi, B^{-1}\cdot \right\rangle \nabla_s \nabla\chi \Big). \tag{5.73}$$

Proof of Proposition 5.5. Let $Z = (X, \ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$. We abbreviate $\xi_0 := \pi_\varepsilon Z$ and write the operator π_ε in the general form

$$\xi_0 := \pi_\varepsilon Z = B^{-1} \left(\tan X + \varepsilon^\beta \ell \nabla\chi\right), \qquad B \stackrel{(5.67)}{:=} \mathbb{1} + \varepsilon^\alpha \mu^2 P,$$

in order to identify how the natural choice $\beta = 2$ arises. For simplicity of reading we mainly omit arguments q and $q(s)$. By (4.43) the adjoint of D_q^0 is given by

$$\begin{aligned}(D_q^0)^* \pi_\varepsilon Z &\stackrel{(4.43)}{=} -\nabla_s \xi_0 + \nabla_{\xi_0} \nabla f \\ &\stackrel{\xi_0}{=} -B^{-1} \nabla_s \left(\tan X + \varepsilon^\beta \ell \nabla\chi\right) - [\nabla_s, B^{-1}] \left(\tan X + \varepsilon^\beta \ell \nabla\chi\right) \\ &\quad + \nabla_{B^{-1}(\tan X + \varepsilon^\beta \ell \nabla\chi)} \nabla f \\ &\stackrel{(5.73)}{=} -B^{-1} \Big(\underline{\nabla_s \tan X} + \underline{\varepsilon^\beta \ell' \nabla\chi} + \varepsilon^\beta \ell \nabla_s \nabla\chi \Big) \\ &\quad + \varepsilon^\alpha B^{-1} \Big(\langle \nabla_s \nabla\chi, \xi_0\rangle \nabla\chi + \langle \nabla\chi, \xi_0\rangle \nabla_s \nabla\chi \Big) \\ &\quad + \nabla_{B^{-1} \tan X} \nabla f + \varepsilon^\beta \ell \nabla_{B^{-1} \nabla\chi} \nabla f.\end{aligned}$$

The underlined terms annihilate their twins below when we take the difference.

We write $(D_q^\varepsilon)^* Z =: (X^*, \ell^*)$, where $(D_q^\varepsilon)^*$ is given by (5.71), then

$$
\begin{aligned}
\pi_\varepsilon (D_q^\varepsilon)^* Z &= \pi_\varepsilon (X^*, \ell^*) \\
&= B^{-1} \left(\tan X^* + \varepsilon^\beta \ell^* \nabla \chi \right) \\
&\stackrel{3}{=} B^{-1} \tan \Big(-\overline{\nabla}_s X + \overline{\nabla}_X \left(\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q \right) - (d\chi|_q X) \overline{\nabla} H + \ell \overline{\nabla} H \Big) \\
&\quad + B^{-1} \left(-\varepsilon^\beta \ell' + \varepsilon^{\beta-2} dH|_q X \right) \nabla \chi \\
&\stackrel{4}{=} -B^{-1} \left(\underline{\nabla_s \tan X} + \tan \overline{\nabla}_s \mathrm{nor}\, X - \overline{\nabla}_{\tan X} \nabla f - \tan \overline{\nabla}_{\mathrm{nor}\, X} \nabla f \right) \\
&\quad - B^{-1} \left(\underline{\varepsilon^\beta \ell' \nabla \chi} - \varepsilon^{\beta-2} (dH|_q X) \nabla \chi \right) .
\end{aligned}
$$

In identity 3 we pulled out the term $\overline{\nabla}_X$ from the sum of two terms whereby the extra term $-(d\chi|_q X)\overline{\nabla} H$ arises. Identity 4 substitutes $\overline{\nabla} F|_q + \chi|_q \overline{\nabla} H|_q$ for ∇f, by (2.11), and uses that $\tan \overline{\nabla} H = 0 = \tan \mathrm{II}$ and that $\tan \overline{\nabla} \chi = \nabla \chi$, by (2.11). We wrote $\overline{\nabla}_s X = \overline{\nabla}_s (\tan X + \mathrm{nor}\, X)$ and $\overline{\nabla}_X \nabla f = \overline{\nabla}_{\tan X} \nabla f + \overline{\nabla}_{\mathrm{nor}\, X} \nabla f$, then we used (2.14) and that normal parts II vanish under tangential projection.

Take the difference, so the s-derivatives (underlined) disappear, and utilize (2.14), to obtain (the lower signs are for $D_q^0 \pi_\varepsilon - \pi_\varepsilon D_q^\varepsilon$)

$$
\begin{aligned}
&(D_q^0)^* \pi_\varepsilon Z - \pi_\varepsilon (D_q^\varepsilon)^* Z \\
&= -\varepsilon^{\beta-2} (dH|_q X) B^{-1} \nabla \chi \mp \varepsilon^\beta \ell \left(B^{-1} \nabla_s \nabla \chi - \tfrac{1}{1+\varepsilon^\alpha \mu^2} \nabla_{\nabla \chi} \nabla f \right) \\
&\quad \pm \varepsilon^\alpha B^{-1} \big(\langle \nabla_s \nabla \chi, \xi_0 \rangle \nabla \chi + \langle \nabla \chi, \xi_0 \rangle \nabla_s \nabla \chi \big) \\
&\quad + \nabla_{B^{-1} \tan X} \nabla f - B^{-1} \nabla_{\tan X} \nabla f \\
&\quad \pm B^{-1} \tan \overline{\nabla}_s \mathrm{nor}\, X \mp B^{-1} \tan \overline{\nabla}_{\mathrm{nor}\, X} \nabla f.
\end{aligned}
\tag{5.74}
$$

To finish the proof it remains to inspect line by line the L^2 norm of these four lines, denoted by $L_1, \ldots, L_4$. The coefficient $\varepsilon^{\beta-2}$ suggests to choose $\beta \geq 2$. In view of line four, see analysis below, choosing $\beta > 2$ does not improve the overall estimate for the term $dH|_q X$. So the value $\beta = 2$ that appears in the orthogonal projection will be just fine.[9]

To estimate line one L_1 we use that $\left\| B^{-1} \right\| \leq 1$, by (5.66), to obtain

$$
\|L_1\| \leq \mu_\infty \varepsilon^{\beta-2} \|dH|_q X\| + c_a \varepsilon^\beta \|\ell\|
$$

where c_a depends on $\|\partial_s q\|_\infty$, the $C^2(\Sigma)$-norms of χ and f, and on μ_∞.

Concerning line two L_2, by definition of ξ_0 and since $\left\| B^{-1} \right\| \leq 1$, we obtain

$$
\|L_2\| \leq C \varepsilon^\alpha \|\xi_0\| , \qquad \|\xi_0\| \leq \|\tan X\| + \mu_\infty \varepsilon^\beta \|\ell\| ,
$$

where C depends on $\|\partial_s q\|_\infty$, the $C^2(\Sigma)$-norm of χ, and μ_∞.

[9] We do not see here the phenomenon that the two most unpleasant terms, here $dH|_q X$, appear with opposite signs, one with ε^0 and one with $\varepsilon^{\beta-2}$ thereby *enforcing* the choice $\beta = 2$, as opposed to [SW06, p. 1132, formula for $\pi_\varepsilon \mathcal{D}_u^\varepsilon \zeta$, unpleasant terms $\nabla_t \eta$ already cancelled].

Line three L_3 in (5.74) is of the form

$$[\Phi, B^{-1}] = B^{-1}[B, \Phi]B^{-1} = B^{-1}[\mathbb{1} + \varepsilon^\alpha \mu^2 P, \Phi]B^{-1} = \varepsilon^\alpha \mu^2 B^{-1}[P, \Phi]B^{-1}$$

where $\Phi\colon W^{1,2}(\mathbb{R}, q^*TM) \to W^{1,2}(\mathbb{R}, q^*TM)$ is given by $\Phi\xi = \nabla_\xi \nabla f$. Thus

$$\begin{aligned} \|L_3\| &= \left\|[\Phi, B^{-1}] \tan X\right\| \\ &= \left\|\varepsilon^\alpha \mu^2 B^{-1} \left(P\nabla_{B^{-1}\tan X}\nabla f - \nabla_{PB^{-1}\tan X}\nabla f\right)\right\| \\ &\le \varepsilon^\alpha \mu_\infty^2 \|f\|_{C^2(\Sigma)} \|\tan X\| \end{aligned} \tag{5.75}$$

since $\left\|B^{-1}\right\| \le 1$, by (5.66), and since orthogonal projection have $\|P\| = 1$.

Line four L_4 in (5.74): For the first summand, by (2.10) and Leibniz, we get

$$\overline{\nabla}_s \mathrm{nor}\, X = \left(\tfrac{\langle \overline{\nabla} H, X\rangle}{|\overline{\nabla} H|^2}\right)' \overline{\nabla} H + \tfrac{\langle \overline{\nabla} H, X\rangle}{|\overline{\nabla} H|^2} \overline{\nabla}_s \overline{\nabla} H.$$

Now use orthogonality $\overline{\nabla} H \perp \tan X$ and write $X = \tan X + \mathrm{nor}\, X$ to obtain

$$\tan \overline{\nabla}_s \mathrm{nor}\, X = \tfrac{\langle \overline{\nabla} H, \mathrm{nor}\, X\rangle}{|\overline{\nabla} H|^2} \tan \overline{\nabla}_s \overline{\nabla} H$$

where the right-hand side is linear in $\mathrm{nor}\, X$. Use this formula to get the estimate

$$\left\|\tan \overline{\nabla}_s \mathrm{nor}\, X\right\| \le \left\|\tfrac{\tan \overline{\nabla}_s \overline{\nabla} H}{|\overline{\nabla} H|}\right\|_\infty \|\mathrm{nor}\, X\| \overset{(2.11)}{\le} \tfrac{\|\overline{\nabla}_\cdot \overline{\nabla} H\|_\infty \|\partial_s q\|_\infty}{m_H^2} \|dH|_q X\| \tag{5.76}$$

where $\|\overline{\nabla}_\cdot \overline{\nabla} H\|_\infty$ is over the compact Σ. For the second summand of L_4 we get

$$\left\|\tan \overline{\nabla}_{\mathrm{nor}\, X} \nabla f\right\| \le \left\|\overline{\nabla}_{\mathrm{nor}\, X} \nabla f\right\| \le \left\|\overline{\nabla}_\cdot \nabla f\right\|_\infty \|\mathrm{nor}\, X\| \overset{(2.11)}{\le} \tfrac{\left\|\overline{\nabla}_\cdot \nabla f\right\|_\infty}{m_H} \|dH|_q X\|.$$

For $\alpha > 0$, $\beta = 2$, and $\varepsilon > 0$ the estimates together prove the L^2 bound (5.72). All estimates are invariant under s-shifts of q, because all constants depend on the L^∞ norm of $\partial_s q$. The proof of Proposition 5.5 is complete. □

5.3 Right inverse – key estimate

In this section we show that if the base flow is Morse-Smale, then so is the ambient ε-flow for all $\varepsilon > 0$ small, see Theorem 5.8.

Definition of right inverse

Suppose that $q \in \mathcal{M}^0_{x^-, x^+}$. By Morse-Smale the linear operator

$$D^0_q \colon W^{1,2}(\mathbb{R}, \Sigma) \to L^2(\mathbb{R}, \Sigma)$$

is surjective. By (4.45) this is equivalent to injectivity of the adjoint $(D^0_q)^*$. Here the Fredholm operator property of D^0_q and $(D^0_q)^*$ enters which holds true, see Proposition 4.4, since Morse-Smale implies Morse.

The main result of this section, Theorem 5.8, tells that surjectivity of D_q^0 implies, for $\varepsilon > 0$ small, surjectivity of D_q^ε, equivalently injectivity of $(D_q^\varepsilon)^*$. As $\ker D_q^\varepsilon = \operatorname{im}(D_q^\varepsilon)^*$, by analogy to (4.45), the composition $D_q^\varepsilon D_q^{\varepsilon *}\colon W^{2,2} \to L^2$ is a bijection and, as a composition of bounded operators, it is bounded. So $D_q^\varepsilon D_q^{\varepsilon *}$ has a bounded inverse by the open mapping theorem. Then the operator

$$R_q^\varepsilon := (D_q^\varepsilon)^* \left(D_q^\varepsilon (D_q^\varepsilon)^*\right)^{-1} : L^2 \xrightarrow{(\dots)^{-1}} W^{2,2} \xrightarrow{(D_q^\varepsilon)^*} W^{1,2} \tag{5.77}$$

is bounded and a right inverse of the operator D_q^ε given by (5.71).

Boundedness of R_q^ε is not enough to get a bijection $\mathcal{T}^\varepsilon\colon \mathcal{M}^0_{x^-,x^+} \to \mathcal{M}^\varepsilon_{x^-,x^+}$ between base and ambient moduli spaces for every parameter value $\varepsilon > 0$ small. To achieve this via the Newton method, what we need is a *uniform* bound that works for every $\varepsilon > 0$ small. Uniform boundedness of the right inverse amounts to establishing uniform estimates for D_q^ε along the image of the formal adjoint. This is also part of Theorem 5.8. To have a chance to obtain uniform bounds in ε one works with Sobolev norms $\|\cdot\|_{0,2,\varepsilon}$ and $\|\cdot\|_{1,2,\varepsilon}$ weighted by suitable powers of ε, see (4.55). The weights are suggested by, respectively, the ε-energy identity and the ambient linear estimate.

5.3.1 The Fredholm operator interchange estimate

In adiabatic limit analysis when one proves the key estimates for the linearized operator along the image of the adjoint (in the present article Theorem 5.8) one needs to interchange the base and ambient operators at some point. For future reference we include the proof of an abstract version of [SW06, Le. D.7] for Fredholm operators D and D'. In practice D' is the formal adjoint of D, so the isomorphism hypothesis on the maps A and B is satisfied automatically.

Lemma 5.6. *Let $D, D'\colon W \to E$ be Fredholm operators between Banach spaces such that W is contained and dense in E and such that the maps defined by*

$$A\colon \ker D \overset{\simeq}{\to} coker D' := \tfrac{E}{\operatorname{im} D'}, \qquad B\colon \ker D' \overset{\simeq}{\to} \operatorname{coker} D := \tfrac{E}{\operatorname{im} D},$$
$$\xi \mapsto \xi + \operatorname{im} D' \qquad\qquad\qquad \eta \mapsto \eta + \operatorname{im} D$$

are isomorphisms. Let D be surjective. Then there is a constant c such that

$$\begin{aligned} \|\eta\|_W &\le c\|D'\eta\|_E \\ \|\xi\|_W &\le c\left(\|\xi - D'\eta\|_E + \|D\xi\|_E\right) \end{aligned} \tag{5.78}$$

for all $\xi, \eta \in W$.

Proof of Lemma 5.6. Since D is surjective D' is injective as the isomorphism B shows. Hence estimate one in (5.78) follows from the open mapping theorem; see e.g. [Rud91, Thm. 4.13]

The linear map $P\colon E \to E/\operatorname{im} D'$, defined by $\xi \mapsto \xi + \operatorname{im} D'$, is continuous since the target space is of finite dimension. The operator

$$T\colon W \to E \oplus \tfrac{E}{\operatorname{im} D'}, \quad \xi \mapsto (D\xi, P\xi)\,,$$

is an injective Fredholm operator: Linearity is clear and continuity holds by continuity of D and of P. Note that $\ker T \subset \ker D$. For injectivity let $\xi \in \ker T$, then $D\xi = 0$ and $0 = P\xi = A\xi$. But then $\xi = 0$ since A is an isomorphism. The image of T is closed, since so is the image of D and since the dimension of $\ker D$ is finite. The image of T has finite codimension, since so has D and since $\frac{E}{\operatorname{im} D'}$ is of finite dimension.

By injectivity and closed range the operator T, as a map $W \to \operatorname{im} T$, is a bijection between Banach spaces. Thus by the open mapping theorem, see e.g. [Rud91, Cor. 2.12 (c)], there is a constant $c > 0$ such that

$$\|\xi\|_W \le c\|T\xi\| = c\left(\|D\xi\|_E + \|P\xi\|_{E/\operatorname{im} D'}\right)$$

for every $\xi \in W$. Given $\eta \in W$, then $D'\eta \in \operatorname{im} D' = \ker P$. Thus, by continuity of P with constant C, we get $\|P\xi\| = \|P(\xi - D'\eta)\| \le C\|\xi - D'\eta\|_E$. $\square$

5.3.2 Weak injectivity estimate of $(D_q^\varepsilon)^*$

To show injectivity of $(D_q^\varepsilon)^* \colon W^{1,2} \to L^2$ amounts to prove the last estimate in (5.79) with the $(1,2,\varepsilon)$-norm on the left-hand side. In this section we aim for the weaker $(0,2,\varepsilon)$-norm and this is why we use the term weak injectivity.

Proposition 5.7 (Weak injectivity of adjoint $(D_q^\varepsilon)^*$)**.** *In π_ε let $\alpha \in [1,2]$ and $\beta = 2$. Let $x^\mp \in \operatorname{Crit} f$ be non-degenerate and $q \in \mathcal{M}^0_{x^-,x^+}$ a connecting base trajectory such that $D_q^0 \colon W^{1,2} \to L^2$ is surjective. Then there are constants $c > 0$ and $\varepsilon_0 \in (0,1]$ such that for any parameter value $\varepsilon \in (0,\varepsilon_0]$ it holds that*

$$\begin{aligned}
\|X\| &\le c\left(\varepsilon\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + \|\pi_\varepsilon (D_q^\varepsilon)^* Z\|\right)\\
\|dH(u)X\| + \varepsilon\|\ell\| &\le c\left(\varepsilon\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + \varepsilon\|\pi_\varepsilon (D_q^\varepsilon)^* Z\|\right)\\
\|Z\|_{0,2,\varepsilon} &\le c\left(\varepsilon\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + \|\pi_\varepsilon (D_q^\varepsilon)^* Z\|\right)\\
\|Z\|_{0,2,\varepsilon} &\le c\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} \quad \text{(weak injectivity estimate)}
\end{aligned} \tag{5.79}$$

for every $Z = (X,\ell) \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$.

Proof. Let $\varepsilon \in (0,1]$. A base connecting trajectory $q \in \mathcal{M}^0_{x^-,x^+}$ is smooth, by Lemma 4.1, and $\|\partial_s q\| \le \operatorname{osc} f$ is finite, by the energy identity (3.26). So the difference Proposition 5.5 applies. By Lemma 5.6, which applies due to the Fredholm Proposition 4.4, there is a constant $c_0 > 0$ such that

$$\|\xi\| \le c_0\|(D_q^0)^*\xi\|$$

for every $\xi \in W^{1,2}(\mathbb{R}, q^*T\Sigma)$. The inequality for $\xi = \pi_\varepsilon Z$ is used in step 2 of what follows. In step 1 and 3 add zero and use the triangle inequality to get

$$\begin{aligned}
\|X\| &\le \|X-\pi_\varepsilon Z\| + \|\pi_\varepsilon Z\| \\
&\overset{\substack{\text{comps.}\\(5.68)}}{\le} \tfrac{1}{m_H}\left\|dH|_q X\right\| + \varepsilon^\alpha \mu_\infty^2 \left\|P\tan X\right\| + \varepsilon^2\mu_\infty \left\|\ell\right\| + c_0\|(D_q^0)^*\pi_\varepsilon Z\| \\
&\le \tfrac{1}{m_H}\left\|dH|_q X\right\| + \varepsilon^\alpha \mu_\infty^2 \left\|P\tan X\right\| + \varepsilon^2\mu_\infty \left\|\ell\right\| + c_0\|\pi_\varepsilon (D_q^\varepsilon)^* Z\| \\
&\quad + c_0\|(D_q^0)^*\pi_\varepsilon Z - \pi_\varepsilon (D_q^\varepsilon)^* Z\| \\
&\overset{\substack{\text{diff.}\\(5.72)}}{\le} \tfrac{1}{m_H}\left\|dH|_q X\right\| + \varepsilon^\alpha \mu_\infty^2 \left\|P\tan X\right\| + \varepsilon^2\mu_\infty \left\|\ell\right\| + c_0\|\pi_\varepsilon (D_q^\varepsilon)^* Z\| \\
&\quad + c_0 c_d \left(\left\|dH|_q X\right\| + \varepsilon^\alpha \left\|\tan X\right\| + \varepsilon^2 \left\|\ell\right\|\right) \\
&\le \varepsilon\left(\tfrac{1}{m_H} + c_0 c_d + \mu_\infty^2\right)\left(\tfrac{1}{\varepsilon}\left\|dH|_q X\right\| + \varepsilon\left\|\ell\right\| + \varepsilon^{\alpha-1}\left\|X\right\|\right) \\
&\quad + c_0\|\pi_\varepsilon (D_q^\varepsilon)^* Z\| \\
&\overset{\substack{\text{amb.}\\(4.60)}}{\le} \varepsilon(c_a+1)\left(\tfrac{1}{m_H} + c_0 c_d + \mu_\infty^2\right)\left(\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + \|X\|\right) \\
&\quad + c_0\|\pi_\varepsilon (D_q^\varepsilon)^* Z\|.
\end{aligned}$$

Here (5.68) requires $\alpha \in [1,2]$, the last step $\alpha \ge 1$. Choose $\varepsilon_0 > 0$ so small that

$$\varepsilon_0 C := \varepsilon_0(c_a+1)\left(\tfrac{1}{m_H} + c_0 c_d + \mu_\infty^2\right) \le \tfrac{1}{2}.$$

Then we can incorporate the term $\|X\|$ into the left-hand side and get that

$$\|X\| \le 2C\varepsilon\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + 2c_0\|\pi_\varepsilon (D_q^\varepsilon)^* Z\|. \tag{5.80}$$

Multiply by ε the ambient estimate (4.60) for $(D_q^\varepsilon)^*$ with constant c_a to obtain

$$\begin{aligned}
\|dH(u)X\| + \varepsilon\|\ell\| &\overset{\substack{\text{amb.}\\(4.60)}}{\le} \varepsilon c_a \Big(\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + \|X\|\Big) \\
&\overset{(5.80)}{\le} \varepsilon c_a \Big((1+2\varepsilon C)\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + 2c_0\|\pi_\varepsilon (D_q^\varepsilon)^* Z\|\Big).
\end{aligned}$$

The previous two estimates provide inequality two in the following

$$\begin{aligned}
\|Z\|_{0,2,\varepsilon} &\overset{(4.55)}{\le} \|X\| + \varepsilon\|\ell\| \\
&\le \varepsilon(2C + c_a(1+2\varepsilon C))\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + 2c_0(1+c_a\varepsilon)\|\pi_\varepsilon (D_q^\varepsilon)^* Z\| \\
&\overset{(5.68)}{\le} \varepsilon(2C + c_a(1+2\varepsilon C))\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon} + 4c_0(1+\varepsilon c_a)\|(D_q^\varepsilon)^* Z\|_{0,2,\varepsilon}
\end{aligned}$$

where the last step uses the last estimate in (5.68). This proves the final assertions three and four of Proposition 5.7 whose proof is thereby complete. □

5.3.3 Surjectivity of D_q^ε and key estimate

Theorem 5.8 (Surjectivity and key estimates for D_q^ε on image of $(D_q^\varepsilon)^*$)**.** *In π_ε let $\alpha\in[1,2]$ and $\beta=2$. Let $x^\mp\in\mathrm{Crit} f$ be non-degenerate and $q\in\mathcal{M}^0_{x^-,x^+}$ a connecting base trajectory such that $D_q^0\colon W^{1,2}\to L^2$ is surjective. Then there are positive constants c and ε_0 (invariant under s-shifts of q) such that, for every $\varepsilon\in(0,\varepsilon_0]$, the following is true. The operator $D_q^\varepsilon\colon W^{1,2}\to L^2$ is onto and along the image of the to $W^{2,2}$ restricted adjoint, that is for every pair*

$$Z^*:=(X^*,\ell^*)\in\mathrm{im}\,(D_q^\varepsilon)^*|_{W^{2,2}}\subset W^{1,2}(\mathbb{R},q^*TM\oplus\mathbb{R}),$$

there are the key estimates

$$\begin{aligned}
\|X^*\|\le\|Z^*\|_{1,2,\varepsilon}&\le c\left(\varepsilon\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon}+\|\pi_\varepsilon(D_q^\varepsilon Z^*)\|\right)\\
\varepsilon^{1/2}\|Z^*\|_{0,\infty,\varepsilon}+\|Z^*\|_{1,2,\varepsilon}&\le c\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon}\\
\|dH|_q X^*\|+\varepsilon\|\ell^*\|+\varepsilon\|\overline{\nabla}_s X^*\|&+\varepsilon^2\|(\ell^*)'\|\\
&\le c\left(\varepsilon\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon}+\varepsilon\|\pi_\varepsilon(D_q^\varepsilon Z^*)\|\right)\\
&\le 3c\varepsilon\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon}.
\end{aligned}\tag{5.81}$$

Proof. A base connecting trajectory $q\in\mathcal{M}^0_{x^-,x^+}$ is smooth, by Lemma 4.1, and $\|\partial_s q\|\le\mathrm{osc} f$ is finite, by the energy identity (3.26). So we are in position to apply the difference Proposition 5.5 with constant c_d and the weak injectivity Proposition 5.7 which provides a constant $\varepsilon_0\in(0,1]$. Let $\varepsilon\in(0,\varepsilon_0]$.

To see surjectivity of the Fredholm operator D_q^ε or, equivalently, injectivity of $(D_q^\varepsilon)^*$, pick $Z=(X,\ell)\in W^{1,2}(\mathbb{R},q^*TM\oplus\mathbb{R})$. Use consequence (4.61) of the ambient linear estimate with constant C_a (shrink $\varepsilon_0>0$ if necessary) to obtain

$$\begin{aligned}
\|X\|\le\|Z\|_{1,2,\varepsilon}&\le\varepsilon C_a\|(D_q^\varepsilon)^*Z\|_{0,2,\varepsilon}+\|\tan X\|\\
&\le(\varepsilon C_a+c_w)\|(D_q^\varepsilon)^*Z\|_{0,2,\varepsilon}.
\end{aligned}\tag{5.82}$$

In the second step we used $\|\tan X\|\le\|X\|\le\|Z\|_{0,2,\varepsilon}$, then we applied the weak injectivity estimate (5.79) with constant c_w. Thus $(D_q^\varepsilon)^*$ is injective.

Now pick $Z=(X,\ell)\in W^{2,2}(\mathbb{R},q^*TM\oplus\mathbb{R})$ and set $Z^*:=(D_q^\varepsilon)^*Z$. To prove the first two lines in (5.81) let c_F be the constant of the Fredholm interchange Lemma 5.6. By (5.78) in Lemma 5.6, with $\xi=\pi_\varepsilon Z^*$ and $\eta=\pi_\varepsilon Z$, we have

$$\begin{aligned}
\|\pi_\varepsilon Z^*\|&\overset{(5.78)}{\le}c_F\|\pi_\varepsilon Z^*-(D_q^0)^*\pi_\varepsilon Z\|+c_F\|D_q^0\pi_\varepsilon Z^*\|\\
&\overset{\text{add }0}{\le}c_F\left(\|\pi_\varepsilon(D_q^\varepsilon)^*Z-(D_q^0)^*\pi_\varepsilon Z\|+\|D_q^0\pi_\varepsilon Z^*-\pi_\varepsilon D_q^\varepsilon Z^*\|+\|\pi_\varepsilon D_q^\varepsilon Z^*\|\right)\\
&\overset{\substack{\text{diff.}\\(5.72)}}{\le}c_F\,c_d\varepsilon\left(\tfrac{1}{\varepsilon}\|dH|_qX\|+\varepsilon^{\alpha-1}\|\tan X\|+\varepsilon\|\ell\|\right)+c_F\|\pi_\varepsilon D_q^\varepsilon Z^*\|\\
&\quad+c_F\,c_d\varepsilon\left(\tfrac{1}{\varepsilon}\|dH|_qX^*\|+\varepsilon^{\alpha-1}\|\tan X^*\|+\varepsilon\|\ell^*\|\right)\\
&\overset{\alpha\in[1,2]}{\le}c_Fc_d\varepsilon c_a\left(\underline{\|Z^*\|_{0,2,\varepsilon}+\|X\|}+\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon}+\underline{\|X^*\|}\right)+c_F\|\pi_\varepsilon D_q^\varepsilon Z^*\|\\
&\overset{(5.82)}{\le}c_1\varepsilon\underline{\|Z^*\|_{0,2,\varepsilon}}+c_Fc_dc_a\varepsilon\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon}+c_F\|\pi_\varepsilon D_q^\varepsilon Z^*\|
\end{aligned}$$

where $c_1 = c_F c_d c_a(2 + \varepsilon C_a + c_w)$. In step 4 we used twice the ambient linear estimate (4.60) with constant c_a, once for $(D_q^\varepsilon)^*$ and once for D_q^ε. In the final step (underlined terms) we estimate $\|X\|$ by (5.82) and $\|X^*\|$ by $\|Z^*\|_{0,2,\varepsilon}$.

Now add zero and use the formula for the linearized injection I_q prior to Definition 5.1, then apply estimate three of the component Lemma 5.4 to get

$$
\begin{aligned}
&\|Z^*\|_{0,2,\varepsilon}\\
&\le \|Z^* - I_q\pi_\varepsilon Z^*\|_{0,2,\varepsilon} + \|(\pi_\varepsilon Z^*, d\chi|_q \pi_\varepsilon Z^*)\|_{0,2,\varepsilon}\\
&\overset{\substack{\text{comps.}\\(5.68)}}{\le} 3\mu_\infty^2 \varepsilon\left(\tfrac{\varepsilon^{-1}}{m_H}\|dH|_q X^*\| + \|\tan X^*\| + \|\ell^*\|\right) + \|\pi_\varepsilon Z^*\| + \varepsilon\|d\chi|_q \pi_\varepsilon Z^*\|\\
&\overset{\substack{\text{amb.}\\(4.60)}}{\le} \varepsilon c_2\left(\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon} + \|X^*\|\right) + (1+\mu_\infty\varepsilon)\|\pi_\varepsilon Z^*\|\\
&\le (c_2 + c_3 c_F c_d c_a)\varepsilon\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon} + (c_2 + c_3 c_1)\varepsilon\|Z^*\|_{0,2,\varepsilon} + c_3 c_F\|\pi_\varepsilon D_q^\varepsilon Z^*\|
\end{aligned}
$$

where $c_2 = \frac{3\mu_\infty^2 \max\{1,m_H\}}{m_H} c_a$ and $c_3 = (1+\mu_\infty\varepsilon)$. Inequality three uses the ambient linear estimate (4.60) and definition (5.69) of the constant $\mu_\infty \ge 1$. The final inequality four uses that $\|X^*\| \le \|Z^*\|_{0,2,\varepsilon}$ and the previously established estimate for $\|\pi_\varepsilon Z^*\|$. Choosing $\varepsilon_0 > 0$ sufficiently small, we obtain

$$
\|\tan X^*\| \le \|X^*\| \le \|Z^*\|_{0,2,\varepsilon} \le c_4\varepsilon\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon} + 2c_3 c_F\|\pi_\varepsilon D_q^\varepsilon Z^*\|. \tag{5.83}
$$

By the ambient linear estimate consequence (4.61) for D_q^ε, constant C_a, we have

$$
\|Z^*\|_{1,2,\varepsilon} \le \varepsilon C_a\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon} + \|\tan X^*\|.
$$

Combining this with (5.83) proves inequality one in (5.81). Inequality two, second summand $\|Z^*\|_{1,2,\varepsilon}$, follows from line one via the last estimate in (5.68) with constant 2. To incorporate the first summand $\varepsilon^{1/2}\|Z^*\|_{0,\infty,\varepsilon}$ simply use (4.56).

To prove inequality three in (5.81) multiply the ambient linear estimate (4.60), for D_q^ε, by ε to obtain that

$$
\|dH|_q X^*\| + \varepsilon\|\ell^*\| + \varepsilon\|\overline{\nabla}_s X^*\| + \varepsilon^2\|(\ell^*)'\| \overset{(4.60)}{\le} \varepsilon c_a\|D_q^\varepsilon Z^*\|_{0,2,\varepsilon} + \varepsilon c_a\|X^*\|.
$$

Combining this with (5.83) proves inequality three in (5.81). Inequality four holds by estimate four in (5.68). This concludes the proof of Theorem 5.8. □

6 Implicit function theorem I – Ambience

Theorem 6.1 (IFT I – Existence)**.** *Assume (f,g) is Morse-Smale. Then there are constants $c > 0$ and $\varepsilon_0 \in (0,1]$ such that the following holds. For every $\varepsilon \in (0,\varepsilon_0]$, every pair $x^\mp \in \mathrm{Crit} f$ of index difference one, and every $q \in \mathcal{M}^0_{x^-,x^+}$, there exists a pair $(u^\varepsilon,\tau^\varepsilon) \in \mathcal{M}^\varepsilon_{x^-,x^+}$ of the form*

$$
u^\varepsilon = \mathrm{Exp}_q X, \qquad \tau^\varepsilon = \chi(q) + \ell, \qquad (X,\ell) \in \mathrm{im}\,(D_q^\varepsilon)^*,
$$

where the difference $Z = (X, \ell) \in C^\infty(\mathbb{R}, q^*TM \oplus \mathbb{R})$ is smooth and bounded by

$$\|Z\|_{1,2,\varepsilon} \le \|X\| + \varepsilon\|\ell\| + \varepsilon\|\overline{\nabla}_s X\| + \varepsilon^2\|\ell'\| \le c\varepsilon^2 \tag{6.84}$$

and by

$$\|X\|_\infty \le c\varepsilon^{3/2}, \qquad \|\ell\|_\infty \le c\varepsilon^{1/2}. \tag{6.85}$$

Theorem 6.2 (IFT I – Uniqueness)**.** *Assume (f, g) is Morse-Smale. Then there are constants $\delta_0, \varepsilon_0 \in (0, 1]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, any pair $x^\mp \in \mathrm{Crit} f$ of index difference one, and any $q \in \mathcal{M}^0_{x^-,x^+}$ the following holds. If*

$$(X_i, \ell_i) \in \mathrm{im}\,(D^\varepsilon_q)^*, \qquad \|X_i\|_\infty \le \delta_0\sqrt{\varepsilon},$$

for $i = 1, 2$ and both pairs of maps $(u^\varepsilon_1, \tau^\varepsilon_1)$ and $(u^\varepsilon_2, \tau^\varepsilon_2)$ defined by

$$u^\varepsilon_i := \mathrm{Exp}_q X_i, \qquad \tau^\varepsilon_i := \chi(q) + \ell_i, \tag{6.86}$$

belong to the moduli space $\mathcal{M}^\varepsilon_{x^-,x^+}$, then they are equal $(u^\varepsilon_1, \tau^\varepsilon_1) = (u^\varepsilon_2, \tau^\varepsilon_2)$.

Observe that each pair (X_i, ℓ_i) is smooth by hypothesis (6.86). Hence, by exponential decay of the derivatives of $(u^\varepsilon_i, \tau^\varepsilon_i)$, each pair (X_i, ℓ_i) belongs to $W^{k,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$ for every integer $k \ge 0$.

Definition 6.3. Assume (f, g) is Morse-Smale. Choose constants $\varepsilon_0, \delta_0 \in (0, 1]$ and $c > 0$ such that the assertions of Theorem 6.1 and 6.2 hold with these constants. Shrink ε_0 so that $c\varepsilon_0 < \delta_0$. Given a pair $x^\mp \in \mathrm{Crit} f$ of index difference one, define for $\varepsilon \in (0, \varepsilon_0)$ the map

$$\mathcal{T}^\varepsilon \colon \mathcal{M}^0_{x^-,x^+} \to \mathcal{M}^\varepsilon_{x^-,x^+}, \quad q \mapsto (u^\varepsilon, \tau^\varepsilon) := \big(\mathrm{Exp}_q X, \chi(q) + \ell\big), \tag{6.87}$$

where the pair $(X, \ell) \in \mathrm{im}\,(D^\varepsilon_q)^*$ is chosen such that (6.84) and (6.85) are satisfied and $(\mathrm{Exp}_q X, \chi(q) + \ell) \in \mathcal{M}^\varepsilon_{x^-,x^+}$. Such a pair exists, by Theorem 6.1, and is unique, by Theorem 6.2. The map $\mathcal{T}^\varepsilon$ is time shift equivariant.

Lemma 6.4 (Injectivity)**.** *Assume (f, g) is Morse-Smale. Then there is a constant $\varepsilon_0 \in (0, 1]$, such that for every $\varepsilon \in (0, \varepsilon_0]$ and every pair $x^\mp \in \mathrm{Crit} f$ of index difference one, the map $\mathcal{T}^\varepsilon \colon \mathcal{M}^0_{x^-,x^+} \to \mathcal{M}^\varepsilon_{x^-,x^+}$ is injective.*

Proof. As Σ is compact, the index difference is 1, and the metric is Morse-Smale, the moduli space $\widetilde{\mathcal{M}}^0_\mp := \mathcal{M}^0_{x^-,x^+}/\mathbb{R}$ is a finite set. So the smallest distance

$$d_{\min} := \min_{[q_1]\ne[q_2]\in\widetilde{\mathcal{M}}^0_\mp} \sup_{s\in\mathbb{R}} \inf_{t\in\mathbb{R}} \mathrm{dist}(q_1(s), q_2(t))) > 0$$

is positive. Choose the constant $\varepsilon_0 > 0$ in Theorem 6.1 smaller if necessary such that $2c{\varepsilon_0}^{3/2} < d_{\min}$. By construction of $\mathcal{T}^\varepsilon$, for $\varepsilon \in (0, \varepsilon_0)$, an element $\mathcal{T}^\varepsilon(q_1) = \mathcal{T}^\varepsilon(q_2)$ lies in both radius $c\varepsilon^{3/2}$ balls, the one about q_1 and the one about q_2. Thus we must have $[q_1] = [q_2]$ since otherwise these two balls, by definition of $d_{\min}$, would be disjoint. But $[q_1] = [q_2]$ means that there exists $\sigma \in \mathbb{R}$ such that $q_1 = \sigma_* q_2 := q_2(\cdot + \sigma)$. Since $\mathcal{T}^\varepsilon$ is time shift invariant we have $\mathcal{T}^\varepsilon(q_1) = \sigma_*\mathcal{T}^\varepsilon(q_2) = \sigma_*\mathcal{T}^\varepsilon(q_1)$. This implies $\sigma = 0$, hence $q_1 = q_2$. □

To prove Theorem 6.1 we carry out a modified Newton iteration to detect a zero of $\mathcal{F}^\varepsilon$ near an approximate zero for which we choose the pair $(q,\chi(q))$ with $q\in\mathcal{M}^0_{x^-,x^+}$. The first step is to define a suitable map between Banach spaces for which we choose the local trivialization $\mathcal{F}^\varepsilon_q:=\mathcal{F}^\varepsilon_{q,\chi(q)}$, see (4.48). In this model the origin corresponds to our approximate zero. One finds a true zero nearby if three conditions are satisfied. Firstly, a small initial value $\mathcal{F}^\varepsilon_q(0)$ where smallness will be taken care of by the weights in the $(0,2,\varepsilon)$ norm. Secondly, a uniformly bounded right inverse R^ε_q of $D^\varepsilon_q=d\mathcal{F}^\varepsilon_q(0)$ which holds due to the key estimate (5.81). Thirdly, we need quadratic estimates to gain control on the variation of the derivative $d\mathcal{F}^\varepsilon_q(Z)$ for Z near the origin.

6.1 Quadratic estimates

Pick a map $q\in W^{1,2}(\mathbb{R},\Sigma)$. Consider the map $z=(q,\chi(q))\in W^{1,2}(\mathbb{R},M\times\mathbb{R})$ and let $Z=(X,\ell)\in W^{1,2}(\mathbb{R},q^*\mathcal{O}\oplus\mathbb{R})$ be a vector field along it.[10] Denote parallel transport in (M,G) along the geodesic $r\mapsto \mathrm{Exp}_{q(s)}(rX(s))$ by

$$\Phi=\Phi_q(X)\colon T_qM\supset\mathcal{O}_q\to T_{E(q,X)}M,\qquad \Gamma_0=\mathrm{Exp}_q(X),\tag{6.88}$$

pointwise for $s\in\mathbb{R}$. A trivialization of the ambient section $\mathcal{F}^\varepsilon$ is defined by

$$\mathcal{F}^\varepsilon_q(X,\ell)=\begin{pmatrix}\Phi_q^{-1}(X)\left(\partial_s\Gamma_0+\overline{\nabla}F|_{\Gamma_0}+(\chi(q)+\ell)\overline{\nabla}H|_{\Gamma_0}\right)\\ (\chi(q)+\ell)'+\varepsilon^{-2}H|_{\Gamma_0}\end{pmatrix}\tag{6.89}$$

for every vector field $(X,\ell)\in W^{1,2}(\mathbb{R},q^*\mathcal{O}\oplus\mathbb{R})$. To compute the derivative of the trivialization $\mathcal{F}^\varepsilon_q$ at a point $Z=(X,\ell)$ in direction $\zeta=(\hat{X},\hat{\ell})$ abbreviate

$$\Phi_r:=\Phi_q(X+r\hat{X}),\qquad \Gamma_r:=E(q,X+r\hat{X}).$$

Then $\frac{d}{dr}\big|_0\Gamma_r=E_2(q,X)\hat{X}$ and the derivative is given by

$$\begin{aligned}
&d\mathcal{F}^\varepsilon_q(X,\ell)\begin{pmatrix}\hat{X}\\ \hat{\ell}\end{pmatrix}:=\tfrac{d}{dr}\big|_0\mathcal{F}^\varepsilon_q(X+r\hat{X},\ell+r\hat{\ell})\\
&\stackrel{1}{=}\frac{d}{dr}\Big|_0\begin{pmatrix}\Phi_r^{-1}\left(\partial_s\Gamma_r+\overline{\nabla}F|_{\Gamma_r}\right)+(\chi(q)+\ell+r\hat{\ell})\Phi_r^{-1}\overline{\nabla}H|_{\Gamma_r}\\ (\chi(q)+\ell+r\hat{\ell})'+\varepsilon^{-2}H|_{\Gamma_r}\end{pmatrix}\\
&\stackrel{2}{=}\begin{pmatrix}\tfrac{d}{dr}\big|_0\left(\Phi_r^{-1}\left(\partial_s\Gamma_r+\overline{\nabla}F|_{\Gamma_r}\right)\right)+\hat{\ell}\Phi_0^{-1}\overline{\nabla}H|_{\Gamma_0}+(\chi(q)+\ell)\,\tfrac{d}{dr}\big|_0\left(\Phi_r^{-1}\overline{\nabla}H|_{\Gamma_r}\right)\\ \hat{\ell}'+\varepsilon^{-2}dH|_{\Gamma_0}E_2(q,X)\hat{X}\end{pmatrix}\\
&\stackrel{3}{=}\begin{pmatrix}\tfrac{d}{dr}\big|_0\Phi_r^{-1}\partial_s\Gamma_r+\tfrac{d}{dr}\big|_0\Phi_r^{-1}\overline{\nabla}F|_{\Gamma_r}+(\chi(q)+\ell)\,\tfrac{d}{dr}\big|_0\Phi_r^{-1}\overline{\nabla}H|_{\Gamma_r}+\hat{\ell}\Phi_0^{-1}\overline{\nabla}H|_{\Gamma_0}\\ \hat{\ell}'+\varepsilon^{-2}dH|_{\Gamma_0}E_2(q,X)\hat{X}\end{pmatrix}
\end{aligned}$$

where step 1 is by definition of $\mathcal{F}^\varepsilon_q$ and step 3 by linearity of parallel transport.

[10] For $q\in\Sigma$ let $\mathcal{O}_q$ be the maximal domain of the exponential map $\mathrm{Exp}_q\colon T_qM\to M$. The subset $\mathcal{O}_q$ is open and star-shaped about 0; see e.g. [O'N83, §5 4. Cor.]. The maximal domain of $\mathrm{Exp}\colon T_\Sigma M\to M$ is an open neighborhood $\mathcal{O}\subset T_\Sigma M$ of the zero section with $\mathcal{O}\cap T_qM=\mathcal{O}_q$.

Proposition 6.5 (Quadratic estimate I)**.** *There is a constant $\delta \in (0,1]$ with the following significance. For every $c_0 > 0$ there is a constant $c > 0$ such that the following is true. Let $q \in W^{1,2}(\mathbb{R}, \Sigma)$ be a map and $Z = (X, \ell)$, $\zeta = (\hat{X}, \hat{\ell}) \in W^{1,2}(\mathbb{R}, q^*TM \times \mathbb{R})$ be two vector fields along $z = (q, \chi(q))$ such that*

$$\|\partial_s q\|_\infty + \|\chi(q)\|_\infty \le c_0, \qquad \|X\|_\infty + \|\hat{X}\|_\infty \le \delta.$$

Then the components F and f of the vector field along z, defined by

$$\mathcal{F}_q^\varepsilon(Z+\zeta) - \mathcal{F}_q^\varepsilon(Z) - d\mathcal{F}_q^\varepsilon(Z)\zeta =: \begin{pmatrix} F \\ f \end{pmatrix}, \tag{6.90}$$

satisfy the inequalities

$$\begin{aligned} \|F\| &\le c\|\hat{X}\|_\infty \left(\|\hat{X}\| + \|\hat{\ell}\| + \|\overline{\nabla}_s \hat{X}\| \cdot \|\hat{X}\|_\infty \right) \\ &\quad + c\|X\|_\infty \left(\|\hat{X}\| + \|\overline{\nabla}_s \hat{X}\| \cdot \|X\|_\infty \right) + c\|\ell\|_\infty \|\hat{X}\|_\infty \|\hat{X}\| \\ &\quad + c\|\hat{X}\|_\infty \|\overline{\nabla}_s X\| \left(\|\hat{X}\|_\infty + \|X\|_\infty \right) \\ \varepsilon\|f\| &\le c\varepsilon^{-1} \|\hat{X}\|_\infty \|\hat{X}\| \end{aligned} \tag{6.91}$$

whenever $\varepsilon > 0$.

By compactness of Σ the injectivity radius of the Riemannian vector bundle $(T_\Sigma M, G)$ is positive. The choice $\delta = \iota(T_\Sigma M)/2 > 0$ takes care that X and $\hat{X}$ are in the domain of Exp.

Proposition 6.6 (Quadratic estimate II)**.** *There is a constant $\delta \in (0,1]$ with the following significance. For any $c_0 > 0$ there is a constant $c > 0$ such that the following is true. Let $q \in W^{1,2}(\mathbb{R}, \Sigma)$ be a map and $Z = (X, \ell)$, $\zeta = (\hat{X}, \hat{\ell}) \in W^{1,2}(\mathbb{R}, q^*TM \times \mathbb{R})$ be two vector fields along $z = (q, \chi(q))$ such that*

$$\|\partial_s q\|_\infty + \|\chi(q)\|_\infty \le c_0, \qquad \|X\|_\infty \le \delta.$$

Then the components F and f of the vector field along z, defined by

$$d\mathcal{F}_q^\varepsilon(Z)\zeta - d\mathcal{F}_q^\varepsilon(0)\zeta =: \begin{pmatrix} \mathfrak{F} \\ \mathfrak{f} \end{pmatrix}, \tag{6.92}$$

satisfy the inequalities

$$\begin{aligned} \|\mathfrak{F}\| &\le c\|X\|_\infty \left(\|\hat{X}\| + \|\hat{\ell}\| + \|\overline{\nabla}_s \hat{X}\| \cdot \|X\|_\infty \right) \\ &\quad + c\|\ell\|_\infty \|\hat{X}\| + c\|X\|_\infty \|\hat{X}\|_\infty \|\overline{\nabla}_s X\| \\ \varepsilon\|\mathfrak{f}\| &\le c\varepsilon^{-1} \|X\|_\infty \|\hat{X}\| \end{aligned} \tag{6.93}$$

whenever $\varepsilon > 0$.

Theorem 6.7 (Exponential map – derivatives)**.** *Let u be a point in a Riemannian manifold M and $X \in \mathcal{O}_u$ a tangent vector. Then there are linear maps*

$$E_i(u,X) : T_uM \to T_{\mathrm{Exp}_u X}M, \qquad E_{ij}(u,X) : T_uM \times T_uM \to T_{\mathrm{Exp}_u X}M$$

for $i,j \in \{1,2\}$ such that the following is true. If $u : \mathbb{R} \to M$ is a smooth curve and X, Y are smooth vector fields along u with $X(s) \in \mathcal{O}_{u(s)}$ for every s, then the maps E_i and E_{ij} are characterized (uniquely determined) by the identities

$$\begin{aligned}
\frac{d}{ds}\mathrm{Exp}_u(X) &= E_1(u,X)\partial_s u + E_2(u,X)\overline{\nabla}_s X \\
\overline{\nabla}_s\left(E_1(u,X)Y\right) &= E_{11}(u,X)\left(Y,\partial_s u\right) + E_{12}(u,X)\left(Y,\overline{\nabla}_s X\right) + E_1(u,X)\overline{\nabla}_s Y \\
\overline{\nabla}_s\left(E_2(u,X)Y\right) &= E_{21}(u,X)\left(Y,\partial_s u\right) + E_{22}(u,X)\left(Y,\overline{\nabla}_s X\right) + E_2(u,X)\overline{\nabla}_s Y.
\end{aligned}$$

Here $\overline{\nabla}$ is the Levi-Civita connection.[11] *Furthermore, there are the identities*

$$E_1(u,0) = E_2(u,0) = \mathbb{1}, \quad E_{11}(u,0) = E_{21}(u,0) = E_{22}(u,0) = 0. \tag{6.94}$$

For all $u \in M$, $X \in \mathcal{O}_u$, and $Y, Z \in T_uM$ there are the symmetry properties

$$E_{12}(u,X)\left(Y,Z\right) = E_{21}(u,X)\left(Z,Y\right) \quad E_{22}(u,X)\left(Y,Z\right) = E_{22}(u,X)\left(Z,Y\right)$$

and the identity $E_{11}(u,X)\left(Y,Z\right) - E_{11}(u,X)\left(Z,Y\right) = E_2(u,X)\overline{R}(Y,Z)X$ where $\overline{R}$ is the Riemannian curvature operator.

Proof. Elĭasson [Elĭ67]. For details see also [Gai99, sec. 3.1.1] or [Web22]. □

The following lemma is a major technical tool in the proof of the pointwise quadratic estimates. The proof is standard, for details see e.g. [Web99, Le. 5.0.9]. Note that the lemma remains valid for covariant derivatives $\overline{D} = d + \Gamma\,\hat{X}$ since the Christoffel symbol Γ arrives together with the direction $\hat{X}$.

Lemma 6.8. *Let $m, n \in \mathbb{N}$ and $h \in C^2(\mathbb{R}^m, \mathbb{R}^n)$. Then for any $\delta > 0$ there exists a continuous function $c_\delta \in C^0(\mathbb{R}^m, \mathbb{R}^+)$ such that*

i) $|h(X+\hat{X}) - h(X)| \le c_\delta(\hat{X})|\hat{X}|$
ii) $|h(X+\hat{X}) - h(X) - dh(X)\,\hat{X}| \le c_\delta(\hat{X})|\hat{X}|^2$

for all $X \in \mathbb{R}^m$ with $|X| \le \delta$ and all $\hat{X} \in \mathbb{R}^m$.

[11] Our convention for derivatives, example $\partial_j E_i$, is to put both, the derivative index j and the arising new linear factor to the right. This way index order and linear factor order coincide, example $\partial_j(E_i(x_i,x_j)X_i) = E_{ij}(x_i,x_j)\,(X_i,X_j)$.

Proofs

Proof of Proposition 6.5. Write $F = F_1 + F_2 + F_3 + F_4$ and $f = f_1 + f_2$ where the summands F_i and f_j are defined now. The summand F_1 is defined by

$$\begin{aligned}
F_1 :&= \Phi_q^{-1}(X+\hat{X})\tfrac{d}{ds}E(q,X+\hat{X}) - \Phi_q^{-1}(X)\tfrac{d}{ds}E(q,X)\\
&\quad - \left(\tfrac{\overline{D}}{dr}\Big|_0 \Phi_q^{-1}(X+r\hat{X})\right)\tfrac{d}{ds}E(q,X) - \Phi_q^{-1}(X)\ \tfrac{\overline{D}}{dr}\Big|_0 \tfrac{d}{ds}E(q,X+r\hat{X})\\
&\overset{2}{=} \Phi_q^{-1}(X+\hat{X})\left(E_1(q,X+\hat{X})\partial_s q + E_2(q,X+\hat{X})\big(\overline{\nabla}_s X + \overline{\nabla}_s\hat{X}\big)\right)\\
&\quad - \Phi_q^{-1}(X)\left(E_1(q,X)\partial_s q + E_2(q,X)\overline{\nabla}_s X\right)\\
&\quad - \overline{D}\Phi_q^{-1}|_X\left(E_1(q,X)\partial_s q,\hat{X}\right) - \overline{D}\Phi_q^{-1}|_X\left(E_2(q,X)\overline{\nabla}_s X,\hat{X}\right)\\
&\quad - \Phi_q^{-1}(X)\left(E_{12}(q,X)\big(\partial_s q,\hat{X}\big) + E_{22}(q,X)\big(\overline{\nabla}_s X,\hat{X}\big) + E_2(q,X)\overline{\nabla}_s\hat{X}\right)\\
&\overset{3}{=} \Phi_q^{-1}(X+\hat{X})E_1(q,X+\hat{X})\partial_s q - \Phi_q^{-1}(X)E_1(q,X)\partial_s q\\
&\qquad - \overline{D}\Phi_q^{-1}|_X\left(E_1(q,X)\partial_s q,\hat{X}\right)\\
&\quad + \Phi_q^{-1}(X+\hat{X})E_2(q,X+\hat{X})\overline{\nabla}_s X - \Phi_q^{-1}(X)E_2(q,X)\overline{\nabla}_s X\\
&\qquad - \overline{D}\Phi_q^{-1}|_X\left(E_2(q,X)\overline{\nabla}_s X,\hat{X}\right)\\
&\quad - \Phi_q^{-1}(X)E_{12}(q,X)\big(\partial_s q,\hat{X}\big) - \Phi_q^{-1}(X)E_{22}(q,X)\big(\overline{\nabla}_s X,\hat{X}\big)\\
&\quad + \left(\Phi_q^{-1}(X+\hat{X})E_2(q,X+\hat{X}) - \mathbb{1}\ + \mathbb{1} - \Phi_q^{-1}(X)E_2(q,X)\right)\overline{\nabla}_s\hat{X}.
\end{aligned}$$

To get identity 2 we carried out the derivatives with respect to s and r using the characterizing identities from Theorem 6.7. In identity 3 we only reordered the summands. The estimate for $\|F_1\|$ is obtained by applying pointwise Lemma 6.8 followed by integration. More precisely, for the first triple of summands one applies part ii) of the lemma, same for the second triple. To the next two summands apply part i) individually. For example define and note that

$$h(X) := \Phi_q^{-1}(X)E_{22}(q,X), \qquad h(0) \overset{(6.94)}{=} 0.$$

Part ii) also applies to the final line where we added $-\mathbb{1}+\mathbb{1}=0$. To deal with the second part of the final line (analogously part one) define and note that

$$h(X) := \Phi_q^{-1}(X)E_2(q,X) - \mathbb{1}, \qquad h(0) \overset{(6.94)}{=} 0, \qquad \overline{D}h(0)X = 0. \tag{6.95}$$

It remains to show that the derivative vanishes, indeed

$$\begin{aligned}
\overline{D}h(0)X &= \tfrac{\overline{D}}{dr}\Big|_0 h(rX)\\
&= \overline{D}\Phi_q^{-1}|_0\left(E_2(q,0)\cdot,X\right) + \Phi_q^{-1}(0)E_{22}(q,0)\left(\cdot,X\right)\\
&= \left(\overline{D}\Phi_q^{-1}|_0 + E_{22}(q,0)\right)(\cdot,X)\\
&= 0.
\end{aligned}$$

The last step holds since both summands vanish individually, namely $E_{22}(q,0) = 0$ and a short calculation in local coordinates shows that

$$\left(\frac{\overline{D}}{dr}\Big|_0 \Phi_q^{-1}(r\hat{X})\right)_j^k = \left(\overline{D}\Phi_q^{-1}|_0(\cdot,\hat{X})\right)_j^k = \underbrace{\tfrac{d}{dr}\big|_0 \Phi_q^{-1}(r\hat{X})_j^k}_{=-\Gamma_{ij}^k \hat{X}^i} + \Gamma_{ij}^k \hat{X}^i = 0 \quad (6.96)$$

where the under-braced identity is Lemma A.1.3 in [Web99]. Recall from the primer article (remark in quadratic estimate section) that L^∞ norms should go preferably on the base point $Z = (X,\ell)$, but never on derivatives. As pointwise estimate for F_1, written in the same order as above, we obtain

$$\begin{aligned}
|F_1| &\le c_{\delta,\hat{X}} \|\partial_s q\|_\infty |\hat{X}|^2 + c_{\delta,\hat{X}} |\hat{X}|^2 |\overline{\nabla}_s X| + c_{\delta,X} \|\partial_s q\|_\infty |X| \cdot |\hat{X}| \\
&\quad + c_{\delta,X} |\hat{X}| \cdot |X| \cdot |\overline{\nabla}_s X| + c_{\delta,X+\hat{X}} |X+\hat{X}|^2 |\overline{\nabla}_s \hat{X}| + c_{\delta,X} |X|^2 |\overline{\nabla}_s \hat{X}| \\
&\le \tilde{c}_1 \left(|\hat{X}|^2 (1 + |\overline{\nabla}_s X|) + |X| \cdot |\hat{X}| (1 + |\overline{\nabla}_s X|) + (|X|^2 + |\hat{X}|^2 |\overline{\nabla}_s \hat{X}|)\right) \\
\|F_1\| &\le c_1 \|\hat{X}\|_\infty \left(\|\hat{X}\| + \|\overline{\nabla}_s \hat{X}\| \cdot \|\hat{X}\|_\infty\right) + c_1 \|X\|_\infty \left(\|\hat{X}\| + \|\overline{\nabla}_s \hat{X}\| \cdot \|X\|_\infty\right) \\
&\quad + c_1 \|\hat{X}\|_\infty \left(\|\hat{X}\|_\infty + \|X\|_\infty\right) \|\overline{\nabla}_s X\|
\end{aligned}$$

for suitable positive constants $\tilde{c}_1$ and c_1. In step 2 of the pointwise estimate we used that $|X+\hat{X}|^2 \le 2|X|^2 + 2|\hat{X}|^2$. The L^2 estimate for F_1 follows by squaring the estimate for $|F_1|$, integrate the result, and pull out L^∞ norms. The summand F_2 is defined and then, via Lemma 6.8 ii), estimated by

$$\begin{aligned}
F_2 :&= \Phi_q^{-1}(X+\hat{X}) \overline{\nabla} F|_{E(q,X+\hat{X})} - \Phi_q^{-1}(X) \overline{\nabla} F|_{E(q,X)} - \tfrac{d}{dr}\big|_0 \left(\Phi_r^{-1} \overline{\nabla} F|_{\Gamma_r}\right) \\
&= h(\hat{X}) - h(0) - dh(0)\hat{X}, \quad h(\hat{X}) := \Phi_q^{-1}(X+\hat{X}) \overline{\nabla} F|_{E(q,X+\hat{X})} \\
\|F_2\| &\le c_2 \|\hat{X}\|_\infty \|\hat{X}\|
\end{aligned}$$

for suitable $c_2 > 0$. Analogous to F_2 we define and treat the summand F_3 by

$$\begin{aligned}
F_3 :&= (\chi(q) + \ell)\Big(\Phi_q^{-1}(X+\hat{X})^{-1} \overline{\nabla} H|_{E(q,X+\hat{X})} - \Phi_q^{-1}(X)^{-1} \overline{\nabla} H|_{E(q,X)} \\
&\quad - \tfrac{d}{dr}\big|_0 \left(\Phi_q^{-1}(X + r\hat{X}) \overline{\nabla} H|_{E(q,X+r\hat{X})}\right)\Big) \\
\|F_3\| &\le c_3 \left(\|\hat{X}\|_\infty \|\hat{X}\| + \|\ell\|_\infty \|\hat{X}\| \cdot \|\hat{X}\|_\infty\right)
\end{aligned}$$

for suitable $c_3 > 0$. For suitable $c_4 > 0$ we define and treat summand F_4 by

$$\begin{aligned}
F_4 :&= \hat{\ell} \left(\Phi_q^{-1}(X+\hat{X}) \overline{\nabla} H|_{E(q,X+\hat{X})} - \Phi_q^{-1}(X)^{-1} \overline{\nabla} H|_{E(q,X)}\right), \\
\|F_4\| &\le c_4 \|\hat{X}\|_\infty \|\hat{\ell}\|.
\end{aligned}$$

Summand f_1 is defined by $f_1 := (\chi(q) + \ell + \hat{\ell})' - (\chi(q) + \ell)' - \hat{\ell}' = 0$ and f_2 by

$$\begin{aligned}
f_2 :&= \varepsilon^{-2} \left(H|_{E(q,X+\hat{X})} - H|_{E(q,X)} - dH|_{E(q,X)} E_2(q,X)\hat{X}\right), \\
\|f_2\| &\le \varepsilon^{-2} c_5 \|\hat{X}\|_\infty \|\hat{X}\|.
\end{aligned}$$

This concludes the proof of Proposition 6.5 (Quadratic Estimate I). □

Proof of Proposition 6.6. The derivative of $\mathcal{F}_q^\varepsilon$ at 0 in direction $\zeta = (\hat X, \hat\ell)$ is

$$d\mathcal{F}_q^\varepsilon(0,0)\begin{pmatrix}\hat X\\ \hat\ell\end{pmatrix}$$
$$\overset{(4.49)}{=}\begin{pmatrix}\frac{d}{dr}\big|_0\Phi_q^{-1}(r\hat X)\left(\partial_s E(q,r\hat X)+\overline{\nabla}F|_{E(q,r\hat X)}+\chi(q)\overline{\nabla}H|_{E(q,r\hat X)}\right)+\hat\ell\overline{\nabla}H|_q\\ \hat\ell'+\varepsilon^{-2}dH|_q\hat X\end{pmatrix}.$$

Write $F = F_1 + F_2 + F_3 + F_4$ and $f = f_1 + f_2$ where the summands F_i and f_j are defined in what follows. The summand F_1 is defined by

$$\begin{aligned}
F_1 :&= \left(\tfrac{d}{dr}\big|_0\Phi_q^{-1}(X+r\hat X)\right)\tfrac{d}{ds}E(q,X)-\left(\tfrac{d}{dr}\big|_0\Phi_q^{-1}(r\hat X)\right)\tfrac{d}{ds}E(q,0)\\
&\quad+\Phi_q^{-1}(X)\,\tfrac{\overline{D}}{dr}\Big|_0\tfrac{d}{ds}E(q,X+r\hat X)-\underline{\Phi_q^{-1}(0)\,\tfrac{\overline{D}}{dr}\Big|_0\tfrac{d}{ds}E(q,r\hat X)}\\
&\overset{2}{=}\overline{D}\Phi_q^{-1}|_X\left(E_1(q,X)\partial_s q+E_2(q,X)\overline{\nabla}_s X,\hat X\right)-\overline{D}\Phi_q^{-1}|_0\left(\partial_s q,\hat X\right)-\underline{\overline{\nabla}_s\hat X}\\
&\quad+\Phi_q^{-1}(X)\left(E_{12}(q,X)\big(\partial_s q,\hat X\big)+E_{22}(q,X)\big(\overline{\nabla}_s X,\hat X\big)+E_2(q,X)\overline{\nabla}_s\hat X\right)\\
&\overset{3}{=}\overline{D}\Phi_q^{-1}|_X\left(E_1(q,X)\partial_s q,\hat X\right)-\overline{D}\Phi_q^{-1}|_0\left(\partial_s q,\hat X\right)\\
&\quad+\Phi_q^{-1}(X)E_{12}(q,X)\big(\partial_s q,\hat X\big)+\left(\Phi_q^{-1}(X)E_2(q,X)-\mathbb{1}\right)\overline{\nabla}_s\hat X\\
&\quad+\overline{D}\Phi_q^{-1}|_X\left(E_2(q,X)\overline{\nabla}_s X,\hat X\right)+\Phi_q^{-1}(X)E_{22}(q,X)\big(\overline{\nabla}_s X,\hat X\big).
\end{aligned}$$

To get identity 2 we carried out the derivatives with respect to s and r using the characterizing identities from Theorem 6.7. In identity 3 we only reordered the summands. The estimate for $\|F_1\|$ is obtained by applying pointwise Lemma 6.8 followed by integration. One uses the same techniques as for term F_1 in quadratic estimate I, in particular (6.95) and the identities $E_1(q,0) = \mathbb{1} = E_2(q,0)$ and $E_{12}(q,0) = 0 = E_{22}(q,0)$. Note that the last but one term

$$g(X) := \overline{D}\Phi_q^{-1}|_X\left(E_2(q,X)\overline{\nabla}_s X,\hat X\right),\qquad g(0)=0,$$

vanishes at the origin as we saw earlier in (6.96). Recall from the primer article (remark in quadratic estimate section) that L^∞ norms should go preferably on the base point $Z = (X, \ell)$, but not on derivatives. We get the estimate

$$\|F_1\|\le c_1\left(\|\partial_s q\|_\infty\|X\|_\infty\|\hat X\|+\|X\|_\infty^2\|\overline{\nabla}_s\hat X\|+\|\hat X\|_\infty\|X\|_\infty\|\overline{\nabla}_s X\|\right).$$

The summand F_2 is defined, and then estimated, by

$$\begin{aligned}
F_2 :&= \tfrac{d}{dr}\big|_0\left(\Phi_q^{-1}(X+r\hat X)\overline{\nabla}F|_{E(q,X+r\hat X)}\right)-\tfrac{d}{dr}\big|_0\left(\Phi_q^{-1}(r\hat X)\overline{\nabla}F|_{E(q,r\hat X)}\right)\\
&=\overline{D}\Phi_q^{-1}|_X\left(\overline{\nabla}F|_{E(q,X)},\hat X\right)-\overline{D}\Phi_q^{-1}|_0\left(\overline{\nabla}F|_q,\hat X\right)\\
&\quad+\Phi_q^{-1}(X)\overline{D}\overline{\nabla}F|_{E(q,X)}\,E_2(q,X)\hat X-\overline{D}\overline{\nabla}F|_q\,\hat X,
\end{aligned}$$
$$\|F_2\|\le c_2\|X\|_\infty\|\hat X\|.$$

Summand F_3 is defined, and then estimated, by

$$\begin{aligned}
F_3 : &= \ell \tfrac{d}{dr}\big|_0 \left(\Phi_q^{-1}(X+r\hat{X})\overline{\nabla} H|_{E(q,X+r\hat{X})}\right) \\
&\quad + \chi(q) \tfrac{d}{dr}\big|_0 \left(\Phi_q^{-1}(X+r\hat{X})\overline{\nabla} H|_{E(q,X+r\hat{X})} - \Phi_q^{-1}(r\hat{X})\overline{\nabla} H|_{E(q,r\hat{X})}\right), \\
&= \ell \overline{D}\Phi_q^{-1}|_X \left(\overline{\nabla} H|_{E(q,X)}, \hat{X}\right) + \ell\, \Phi_q^{-1}(X)\overline{D}\overline{\nabla} H|_{E(q,X)}\, E_2(q,X)\, \hat{X} \\
&\quad + \chi(q)\overline{D}\Phi_q^{-1}|_X \left(\overline{\nabla} H|_{E(q,X)}, \hat{X}\right) - \chi(q)\overline{D}\Phi_q^{-1}|_0 \left(\overline{\nabla} H|_q, \hat{X}\right) \\
&\quad + \chi(q)\Phi_q^{-1}(X)\overline{D}\overline{\nabla} H|_{E(q,X)}\, E_2(q,X)\hat{X} - \chi(q)\overline{D}\overline{\nabla} H|_q\, \hat{X}, \\
\|F_3\| &\le c_3\Big(\|X\|_\infty \|\ell\|_\infty + \|\ell\|_\infty + \|\chi(q)\|_\infty \|X\|_\infty\Big)\|\hat{X}\|.
\end{aligned}$$

Summand F_4 is defined by

$$\begin{aligned}
F_4 : &= \hat{\ell}\left(\Phi_q^{-1}(X)\overline{\nabla} H|_{E(q,X)} - \overline{\nabla} H|_q\right), \\
\|F_4\| &\le c_4 \|X\|_\infty \|\hat{\ell}\|.
\end{aligned}$$

Summand f_1 is defined by $f_1 := \hat{\ell}' - \hat{\ell}' = 0$ and f_2 by

$$\begin{aligned}
f_2 : &= \varepsilon^{-2}\left(dH|_{E(q,X)}E_2(q,X) - dH|_q\right)\hat{X}, \\
\|f_2\| &\le \varepsilon^{-2} c_5 \|X\|_\infty \|\hat{X}\|.
\end{aligned}$$

This concludes the proof of Proposition 6.6 (Quadratic Estimate II). $\square$

6.2 Existence – definition of $\mathcal{T}^\varepsilon$

We prove Theorem 6.1. Assume the Morse-Smale condition holds true. Up to time-shift there are only finitely many elements q of $\mathcal{M}^0_{x^-,x^+}$, that is base solutions q between critical points of f of Morse index difference 1. The constant

$$c_0 := \max\left\{\|\partial_s q\|_\infty \mid q \in \mathcal{M}^0_{x^-,x^+}\right\} + \|\chi\|_{L^\infty(\Sigma)} < \infty$$

is finite since the function χ is bounded along the compact Σ and since $\|\partial_s q\|_\infty$ is finite due to exponential decay and since, by index difference one, there are only finitely many q's up to time shift. Fix $\varepsilon_0 > 0$ sufficiently small such that the key estimate, Theorem 5.8, applies to all $q \in \mathcal{M}^0_{x^-,x^+}$ and $\varepsilon \in (0, \varepsilon_0]$.

Pick $q \in \mathcal{M}^0_{x^-,x^+}$. Recall that χ is defined by (2.8). The trivialized section along the canonical embedding $i(q) = (q, \chi(q))$, namely $\mathcal{F}^\varepsilon_q(X, \ell)$ defined by (6.89), acts on the elements $Z = (X, \ell)$ of the Banach space $W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R})$. At the origin the first component vanishes

$$\mathcal{F}^\varepsilon_q \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_s q + \overline{\nabla} F(q) + \chi(q)\overline{\nabla} H(q) \\ (\chi(q))' + \varepsilon^{-2} H(q) \end{pmatrix} = \begin{pmatrix} 0 \\ d\chi|_q \partial_s q \end{pmatrix} \tag{6.97}$$

since $H(q) \equiv 0$. Therefore for the initial point

$$Z_0 := (0, 0)$$

we have

$$\left\|\mathcal{F}_q^\varepsilon(Z_0)\right\|_{0,2,\varepsilon} = \|\mathcal{F}^\varepsilon(q,\chi(q))\|_{0,2,\varepsilon} = \|(0, d\chi|_q \partial_s q)\|_{0,2,\varepsilon} \le \varepsilon\mu_\infty\sqrt{c^*}$$

where μ_∞ is defined by (5.69) and

$$\|\partial_s q\| \overset{(3.27)}{=} \sqrt{f(x^-) - f(x^+)} =: \sqrt{c^*}.$$

Now define the initial correction term $\zeta_0 = (\hat{X}_0, \hat{\ell}_0)$ by

$$\zeta_0 := -D_q^{\varepsilon *}\left(D_q^\varepsilon D_q^{\varepsilon *}\right)^{-1}\mathcal{F}_q^\varepsilon(0)$$

where $D_q^\varepsilon = d\mathcal{F}_q^\varepsilon(0,0)$. Recursively, for $\nu \in \mathbb{N}$, define the sequence $\zeta_\nu = (\hat{X}_\nu, \hat{\ell}_\nu)$ of correction terms by

$$\begin{aligned} \zeta_\nu = (\hat{X}_\nu, \hat{\ell}_\nu) &:= -D_q^{\varepsilon *}\left(D_q^\varepsilon D_q^{\varepsilon *}\right)^{-1}\mathcal{F}_q^\varepsilon(Z_\nu), \\ Z_\nu = (X_\nu, \ell_\nu) &:= \sum_{k=0}^{\nu-1}\zeta_k = Z_{\nu-1} + \zeta_{\nu-1}. \end{aligned} \tag{6.98}$$

We prove by induction that there is a constant $c > 0$ such that

$$\begin{aligned} \varepsilon^{1/2}\|\zeta_\nu\|_{0,\infty,\varepsilon} + \|\zeta_\nu\|_{1,2,\varepsilon} &\le \frac{c}{2^\nu}\varepsilon^2 \\ \|\mathcal{F}_q^\varepsilon(Z_{\nu+1})\|_{0,2,\varepsilon} &\le \frac{c}{2^\nu}\varepsilon^{5/2} \end{aligned} \tag{H_ν}$$

for every $\nu \in \mathbb{N}_0$. The $(1,2,\varepsilon)$ and $(0,\infty,\varepsilon)$ norms were defined in (4.55).

Initial step: $\nu = 0$. By definition of ζ_0 we have

$$D_q^\varepsilon\zeta_0 = -\mathcal{F}_q^\varepsilon(0) = \begin{pmatrix} 0 \\ -d\chi|_q\partial_s q \end{pmatrix}. \tag{6.99}$$

Thus, by the key estimate, Theorem 5.8, (with constant $c_1 > 0$) we get

$$\begin{aligned} \|\zeta_0\|_{1,2,\varepsilon} &\overset{(5.81)}{\le} c_1\left(\varepsilon\|(0, d\chi|_q\partial_s q)\|_{0,2,\varepsilon} + \|\pi_\varepsilon(0, d\chi|_q\partial_s q)\|\right) \\ &\overset{(5.63)}{\le} c_1\left(\varepsilon^2\mu_\infty\|\partial_s q\| + \|(\mathbb{1} + \varepsilon^2\mu^2 P)^{-1}\varepsilon^2(d\chi|_q\partial_s q)\nabla\chi\|\right) \\ &\overset{(5.66)}{\le} 2c_1\mu_\infty^2\sqrt{c^*}\varepsilon^2 \\ \|\zeta_0\|_{0,\infty,\varepsilon} &\overset{(4.56)}{\le} 3\varepsilon^{-1/2}\|\zeta_0\|_{1,2,\varepsilon} \\ &\le 6c_1\mu_\infty^2\sqrt{c^*}\varepsilon^{3/2} \le \delta. \end{aligned} \tag{6.100}$$

To get the bound δ (needed by the quadratic estimates Proposition 6.5 and 6.6) choose $\varepsilon_0 > 0$ smaller if necessary. This proves estimate one in (H_ν) for $\nu = 0$ and with a suitable constant $c > 0$ depending only on c_1 and the L^∞-norms of $\nabla\chi\colon \Sigma \to T\Sigma$ and $\partial_s q$. To prove estimate two we observe that $Z_1 = \zeta_0$ and hence, by Proposition 6.5 (with constant $c_2 > 0$), we get

$$
\begin{aligned}
\|\mathcal{F}_q^\varepsilon(Z_1)\|_{0,2,\varepsilon} &\stackrel{(6.99)}{=} \|\mathcal{F}_q^\varepsilon(\zeta_0) \overbrace{-\mathcal{F}_q^\varepsilon(0) - D_u^\varepsilon\zeta_0}^{=0}\|_{0,2,\varepsilon} \\
&\stackrel{(6.91)}{\le} \frac{c_2}{\varepsilon}\Bigl(\|\hat{X}_0\|_\infty \bigl(\|\hat{X}_0\| + \varepsilon\|\hat{\ell}_0\| + \varepsilon\|\overline{\nabla}_s\hat{X}_0\| \cdot \underline{\|\hat{X}_0\|_\infty}\bigr)\Bigr) \\
&\stackrel{(6.91)}{\le} \frac{2c_2}{\varepsilon}\|\zeta_0\|_{0,\infty,\varepsilon}\, \|\zeta_0\|_{1,2,\varepsilon} \\
&\stackrel{(6.100)}{\le} 48c_1^2c_2\mu_\infty^4 c^*\, \varepsilon^{5/2}.
\end{aligned}
\tag{6.101}
$$

In step 3 we discarded the underlined term $\|\hat{X}_0\|_\infty \le 1$. Then, up to a factor 2, see (4.59), the $(1,2,\varepsilon)$ norm (4.55) appears. This proves (H_ν) for $\nu = 0$. From now on we fix the constant c for which the estimate (H_0) has been established.

Induction step: $\boldsymbol{\nu - 1 \Rightarrow \nu}$. Let $\nu \ge 1$ and assume that the hypotheses $(H_0), \dots, (H_{\nu-1})$ are true. Then we obtain that

$$
\begin{aligned}
\varepsilon^{1/2}\|Z_\nu\|_{0,\infty,\varepsilon} + \|Z_\nu\|_{1,2,\varepsilon} \quad &\le \quad \sum_{k=0}^{\nu-1}\Bigl(\varepsilon^{1/2}\|\zeta_k\|_{0,\infty,\varepsilon} + \|\zeta_k\|_{1,2,\varepsilon}\Bigr) \\
&\stackrel{(H_{0\dots\nu-1})}{\le} c\varepsilon^2\sum_{k=0}^{\nu-1} 2^{-k} \le 2c\varepsilon^2 \le \delta
\end{aligned}
\tag{6.102}
$$

(for the bound δ choose $\varepsilon_0 > 0$ smaller if necessary) and we also obtain that

$$
\|\mathcal{F}_q^\varepsilon(Z_\nu)\|_{0,2,\varepsilon} \stackrel{(H_{\nu-1})}{\le} \frac{c}{2^{\nu-1}}\varepsilon^{5/2}. \tag{6.103}
$$

By (6.98), using the property of a right inverse, we have

$$
D_q^\varepsilon\zeta_\nu = -\mathcal{F}_q^\varepsilon(Z_\nu), \quad \zeta_\nu \in \operatorname{im}(D_q^\varepsilon)^*.
$$

Hence, together with the key estimate (5.81), (with constant $c_1 > 0$), we get

$$
\begin{aligned}
\varepsilon^{1/2}\|\zeta_\nu\|_{0,\infty,\varepsilon} + \|\zeta_\nu\|_{1,2,\varepsilon} &\stackrel{(5.81)}{\le} c_1\|\mathcal{F}_q^\varepsilon(Z_\nu)\|_{0,2,\varepsilon} \\
&\stackrel{(6.103)}{\le} c_1\varepsilon^{1/2}\frac{c}{2^{\nu-1}}\varepsilon^2 \le \frac{c}{2^\nu}\varepsilon^2 \le \delta.
\end{aligned}
\tag{6.104}
$$

The last but one inequality holds if $9c_1\sqrt{\varepsilon_0} \le \frac{1}{2}$. The last inequality holds by the last inequality in (6.102). This proves the first estimate in (H_ν).

In what follows in step 1 add twice zero and in step 2 apply the quadratic estimates, Proposition 6.5 and 6.6 (with constant $c_2 > 0$), in order to obtain

$$\begin{aligned}
&\|\mathcal{F}_q^\varepsilon(Z_{\nu+1})\|_{0,2,\varepsilon} \\
&\le \|\mathcal{F}_q^\varepsilon(Z_\nu+\zeta_\nu) - \mathcal{F}_q^\varepsilon(Z_\nu) - d\mathcal{F}_q^\varepsilon(Z_\nu)\zeta_\nu\|_{0,2,\varepsilon} + \|d\mathcal{F}_q^\varepsilon(Z_\nu)\zeta_\nu - D_q^\varepsilon\zeta_\nu\|_{0,2,\varepsilon} \\
&\le \frac{c_2}{\varepsilon}\|\hat{X}_\nu\|_\infty \left(\|\hat{X}_\nu\| + \varepsilon\|\hat{\ell}_\nu\| + \varepsilon\|\overline{\nabla}_s\hat{X}_\nu\|\right) + \underline{c_2\|\overline{\nabla}_s X_\nu\|\cdot\|\hat{X}_\nu\|_\infty} \\
&\quad + \frac{c_2}{\varepsilon}\|X_\nu\|_\infty \left(\|\hat{X}_\nu\| + \varepsilon\|\hat{\ell}_\nu\| + \varepsilon\|\overline{\nabla}_s\hat{X}_\nu\|\right) + c_2\|\ell_\nu\|_\infty\|\hat{X}_\nu\| \\
&\le \frac{c_2}{\varepsilon}\left(\|\zeta_\nu\|_{0,\infty,\varepsilon} + \|Z_\nu\|_{0,\infty,\varepsilon}\right)\|\zeta_\nu\|_{1,2,\varepsilon} + \underline{c_2\varepsilon^{-1}\|Z_\nu\|_{1,2,\varepsilon}\|\zeta_\nu\|_{0,\infty,\varepsilon}} \\
&\overset{(6.104)}{\le} \underbrace{c_2\varepsilon^{-1}\left(c\varepsilon^{3/2} + 2c\varepsilon^{3/2}\right)c_1}_{\le 1/4}\frac{c}{2^{\nu-1}}\varepsilon^{5/2} + \underbrace{c_2 2c\varepsilon^{1/2}c_1}_{\le 1/4}\frac{c}{2^{\nu-1}}\varepsilon^{5/2} \\
&\le \frac{c}{2^\nu}\varepsilon^{5/2}.
\end{aligned}$$

In inequality two we already estimated some factors $\|\hat{X}\|_\infty \le 1$ and $\|X\|_\infty \le 1$ in triple products. The last inequality holds by choosing $\varepsilon_0 > 0$ sufficiently small. This completes the induction and proves (H_ν) for every $\nu \in \mathbb{N}_0$.

Conclusion. It follows from (H_ν) that Z_ν is a Cauchy sequence with respect to $\|\cdot\|_{1,2,\varepsilon}$. We denote its limit by

$$Z^\varepsilon := \lim_{\nu\to\infty} Z_\nu = \sum_{\nu=0}^\infty \zeta_\nu \in W^{1,2}(\mathbb{R}, q^*TM \oplus \mathbb{R}).$$

By construction, and since the image of $(D_q^\varepsilon)^*$ is closed, the limit satisfies

$$\varepsilon^{1/2}\|Z^\varepsilon\|_{1,\infty,\varepsilon} + \|Z^\varepsilon\|_{1,2,\varepsilon} \overset{(6.102)}{\le} 2c\varepsilon^2, \qquad \mathcal{F}_q^\varepsilon(Z^\varepsilon) = 0, \qquad Z^\varepsilon \in \operatorname{im}(D_q^\varepsilon)^*.$$

This concludes the proof of Theorem 6.1.

6.3 Uniqueness – injectivity of $\mathcal{T}^\varepsilon$

We prove Theorem 6.2 under the conventions and notations of Section 6.2, in particular Section 6.2 provides $\varepsilon_0 \in (0,1]$, whereas $\delta \in (0,1]$ is the constant that appears in the quadratic estimates. Shrink $\delta_0 > 0$ such that $\delta_0\sqrt{\varepsilon_0} \le \delta/4$. Pick $q \in \mathcal{M}^0_{x^-,x^+}$ and $\varepsilon \in (0,\varepsilon_0]$. Let the base point $Z = (X,\ell) := \mathcal{T}^\varepsilon(q)$ be the zero of the trivialized section $\mathcal{F}_q^\varepsilon$ provided by the existence Theorem 6.1. Then

$$Z \in \operatorname{im}(D_q^\varepsilon)^*, \qquad \mathcal{F}_q^\varepsilon(Z) = 0, \qquad \varepsilon^{1/2}\|Z\|_{0,\infty,\varepsilon} + \|Z\|_{1,2,\varepsilon} \le c\varepsilon^2 \le \delta/4.$$

for a suitable constant $c > 0$ and where the norms are defined by (4.55) and the δ estimate holds by choosing $\varepsilon_0 > 0$ smaller, if necessary. Shrink $\varepsilon_0 > 0$

further such that $c\varepsilon_0 < \delta_0$. Now assume $\zeta = (\hat{X}, \hat{\ell})$ satisfies the hypotheses of the present Theorem 6.2, that is

$$\zeta = (\hat{X}, \hat{\ell}) \in \operatorname{im}(D_q^\varepsilon)^*, \qquad \mathcal{F}_q^\varepsilon(\zeta) = 0, \qquad \|\hat{X}\|_\infty \le \delta_0 \varepsilon^{1/2}.$$

The difference

$$(X^*, \ell^*) = \zeta^* := \zeta - Z = (\hat{X} - X, \hat{\ell} - \ell) \in \operatorname{im}(D_q^\varepsilon)^*$$

then satisfies the inequalities[12]

$$\|X^*\|_\infty \le (\delta_0 + c\varepsilon)\, \varepsilon^{1/2} \le 2\delta_0 \varepsilon^{1/2} \le \delta/2, \qquad \|\ell^*\|_\infty < \infty.$$

With the difference abbreviations (6.90) and (6.92) and since both $\zeta = Z + \zeta^*$ and Z are zeroes of $\mathcal{F}_q^\varepsilon$ we get the first identity in the following

$$\begin{aligned}
&\|D_q^\varepsilon \zeta^*\|_{0,2,\varepsilon} \\
&= \Big\| \underbrace{\big(\mathcal{F}_q^\varepsilon(Z+\zeta^*) - \mathcal{F}_q^\varepsilon(Z) - d\mathcal{F}_q^\varepsilon(Z)\zeta^*\big)}_{=:(F,f)} + \underbrace{\big(d\mathcal{F}_q^\varepsilon(Z)\zeta^* - d\mathcal{F}_q^\varepsilon(0)\zeta^*\big)}_{=:(\mathfrak{F},\mathfrak{f})} \Big\|_{0,2,\varepsilon} \\
&= \|(F + \mathfrak{F}, f + \mathfrak{f})\|_{0,2,\varepsilon} \\
&\le \|F\| + \|\mathfrak{F}\| + \varepsilon\|f\| + \varepsilon\|\mathfrak{f}\|.
\end{aligned}$$

By definition (5.65) of π_ε with $\beta = 2$ and $\alpha \in [1,2]$ and by Lemma 5.3 we obtain

$$\begin{aligned}
\|\pi_\varepsilon D_q^\varepsilon \zeta^*\| &= \|\pi_\varepsilon (F + \mathfrak{F}, f + \mathfrak{f})\| \\
&= \|(\mathbb{1} + \varepsilon^\alpha \mu^2 P)^{-1} (\tan(F + \mathfrak{F}) + \varepsilon^2 (f + \mathfrak{f}) \nabla\chi)\| \\
&\le \|F\| + \|\mathfrak{F}\| + \mu_\infty \varepsilon^2 \|f\| + \mu_\infty \varepsilon^2 \|\mathfrak{f}\|
\end{aligned} \tag{6.105}$$

where we also used that $\|\tan\| \le 1$ since the projection tan is orthogonal. The choice $\beta = 2$ neutralizes the toxic factor ε^{-2} that comes with the f and $\mathfrak{f}$ terms.

Thus, by estimate four in the key estimate (5.81), with a constant $c_1 > 0$, by the quadratic estimates (6.91) and (6.93), with a constant $c_2 \ge 2$, we obtain

$$\begin{aligned}
&\|\ell^*\| \cdot \|X^*\|_\infty \\
&\le c_1 \|D_q^\varepsilon \zeta^*\|_{0,2,\varepsilon} \|X^*\|_\infty \\
&\le c_1 \left(\|F\| + \|\mathfrak{F}\| + \varepsilon\|f\| + \varepsilon\|\mathfrak{f}\|\right) \|X^*\|_\infty \\
&\le c_1 c_2 \|X^*\|_\infty \Big(\tfrac{1}{\varepsilon} \underline{\|X^*\|_\infty \|X^*\|} + \|\ell^*\| \cdot \|X^*\|_\infty + \|\overline{\nabla}_s X^*\| \cdot \|X^*\|_\infty^2 \\
&\qquad + \|X\|_\infty \left(\tfrac{1}{\varepsilon}\|X^*\| + \|\ell^*\| + \|\overline{\nabla}_s X^*\| \right) + \|\ell\|_\infty \|X^*\| + \underline{\underline{\|X^*\|_\infty}} \|\overline{\nabla}_s X\| \Big) \\
&\le c_1 c_2 \Big(\underline{4\delta_0^2} + 8\delta_0^3 \sqrt{\varepsilon} + 2c\delta_0\varepsilon + 2c\delta_0\varepsilon + 2c\delta_0\varepsilon + 2c\delta_0\varepsilon \Big) \|\zeta^*\|_{1,2,\varepsilon} \\
&\quad + c_1 c_2 2\delta_0 \sqrt{\varepsilon} \underline{\underline{\Big(\tfrac{1}{\sqrt{\varepsilon}} \|X^*\| + \sqrt{\varepsilon} \|\overline{\nabla}_s X^*\| \Big)}} c\varepsilon + c_1 c_2 2\delta_0 \sqrt{\varepsilon}\, \|\ell^*\| \cdot \|X^*\|_\infty \\
&\le \tfrac{1}{8 \cdot 2\mu_\infty c_1 c_2} \|\zeta^*\|_{1,2,\varepsilon} + \tfrac{1}{2} \|\ell^*\| \cdot \|X^*\|_\infty.
\end{aligned}$$

[12] a numerical bound $\|\ell^*\|_\infty < C$ is irrelevant in the proof, only finiteness ($< \infty$) matters

In inequality three we already discarded in a few triple products some factors $\|X^*\|_\infty \le 1$ or $\|X\|_\infty \le 1$. The once underlined term enforces the smallness assumption in Theorem 6.2. The doubly underlined estimate in inequality three is by (4.58) with $\beta = 1/2$. The final inequality holds by choosing δ_0 and ε_0 sufficiently small. We summarize the estimate, which comes in handy below, by

$$2\mu_\infty c_1 c_2 \|\ell^*\| \cdot \|X^*\|_\infty \le \tfrac{1}{4}\|\zeta^*\|_{1,2,\varepsilon}.$$

Similarly, by estimate one in the key estimate (5.81), with a constant $c_1 > 0$, by the quadratic estimates (6.91) and (6.93), with a constant $c_2 \ge 2$, and with the constant μ_∞ defined by (5.69), we obtain

$$\begin{aligned}
\|\zeta^*\|_{1,2,\varepsilon} &\le c_1 \left(\varepsilon\|D_q^\varepsilon \zeta^*\|_{0,2,\varepsilon} + \|\pi_\varepsilon(D_q^\varepsilon\zeta^*)\|\right)\\
&\le 2\mu_\infty c_1 \left(\|F\| + \|\mathfrak{F}\| + \varepsilon^2\|f\| + \varepsilon^2\|\mathfrak{f}\|\right)\\
&\le 2\mu_\infty c_1 c_2 \Bigl(\|X^*\|\cdot\|X^*\|_\infty + \|\ell^*\|\cdot\|X^*\|_\infty + \|\overline{\nabla}_s X^*\|\cdot\underline{\|X^*\|_\infty^2}\\
&\quad + \|X\|_\infty\left(\|X^*\| + \|\ell^*\| + \|\overline{\nabla}_s X^*\|\right) + \|\ell\|_\infty\|X^*\| + \underline{\underline{\|X^*\|_\infty}}\|\overline{\nabla}_s X\|\Bigr)\\
&\le 2\mu_\infty c_1 c_2 \Bigl(\delta_0\sqrt{\varepsilon} + \underline{\delta_0^2} + c\varepsilon^{3/2} + c\varepsilon^{1/2} + c\varepsilon^{3/2}\varepsilon^{-1} + c\sqrt{\varepsilon}\Bigr)\|\zeta^*\|_{1,2,\varepsilon}\\
&\quad + 2\mu_\infty c_1 c_2\left(\underline{\underline{\varepsilon^{-1/2}\|X^*\| + \varepsilon^{1/2}\|\overline{\nabla}_s X^*\|}}\right) c\varepsilon + \tfrac{1}{4}\|\zeta^*\|_{1,2,\varepsilon}\\
&\le \frac{1}{2}\|\zeta^*\|_{1,2,\varepsilon}.
\end{aligned}$$

In inequality three we discarded in a few triple products some factors $\|X^*\|_\infty \le 1$ or $\|X\|_\infty \le 1$. The once underlined term enforces the smallness assumption in Theorem 6.2. The doubly underlined estimate in inequality three is by (4.58) with $\beta = 1/2$. The final inequality holds by choosing δ_0 and ε_0 sufficiently small. Thus the element $\zeta^* = \zeta - Z$ is zero in $W^{1,2}$. This proves Theorem 6.2.

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