Norm Inequalities for One Dimensional Sobolev Hilbert Spaces

Saburou Saitoh
Institute of Reproducing Kernels,
saburou.saitoh@gmail.com

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Abstract: In this paper, we shall consider norm inequalities for one dimensional Sobolev Hilbert spaces by using the theory of reproducing kernels as fundamental inequalities.

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1 Introduction

For the Bergman kernel and the Szegö kernel on a regular domain D on the complex $z = x + iy$ plane, we have the basic and deep relation

$$K(z, \overline{u}) >> 4\pi \hat{K}(z, \overline{u})^2$$

– the left minus the right is a positive definite quadratic form function – which was given by D. A. Hejhal [22]. This profound result was given on the long historical lines as in

G.F.B. Riemann (1826-1866); F. Klein (1849-1925); S. Bergman; G. Szegö; Z. Nehari; M.M. Schiffer; P.R. Garabedian (1949 published); D.A. Hejhal (1972 published).

It seems that any elementary proof is impossible, however, the result will, in particular, mean a fairly simple inequality:

For two functions φ and ψ of $H_2(D)$, analytic Hardy space, we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\int\int_D |\varphi(z)\psi(z)|^2 dxdy \leq \frac{1}{2\pi}\int_{\partial D} |\varphi(z)|^2|dz|\frac{1}{2\pi}\int_{\partial D} |\psi(z)|^2|dz|, \qquad (1.1)$$

and we can determine completely the case holding the equality here. In the thesis [42] of the author published in 1979 the result was given. The author realized the importance of the abstract and general theory of reproducing kernels by N. Aronszajn. In the paper, the core part was to determine the equality statement in the above inequality, surprisingly enough, some deep and general independing poof was appeared 26 years later in A. Yamada ([82]). A. Yamada was developed deeply equality problems for some general norm inequalities derived by the theory of reproducing kernels and it was published in the book appendix of [10]. Very recently his theory is developing much more in [14].

Of course, in the thesis we can find some fundamental idea for nonlinear transforms. In particular, for the special case $\varphi \equiv \psi \equiv 1$, for the plane measure $m(D)$ of D and the length ℓ of the boundary we have the isoperimetric inequality

$$4\pi m(D) \leq \ell^2.$$

We have similar results and theory for the Sobolev spaces. For example, let ρ be a positive continuous function on (a, b) satisfying $\rho \in L_1(a, b)$. Let f_j be complex valued- functions on (a, b) satisfying $\lim_{x\to a-0} f_j(x) = 0$. Then, we have the inequality

$$\int_a^b |(f_1(x)f_2(x))'|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)\rho(x)}$$

$$\leq 2\int_a^b |f_1'(x)|^2\frac{dx}{\rho(x)}\int_a^b |f_2'(x)|^2\frac{dx}{\rho(x)},$$

when the integrals in the last parts are finite. Equality holds here if and only if each f_j is expressible in the form $C_jK_\rho(x, x_2)$ for some constants C_j and for some point $x_2 \in [a, b]$ which is independent of j. Here, $K_\rho(x, \cdot)$ is the reproducing kernel of the Sobolev space with the norm

$$\int_a^b |f_1'(x)|^2\frac{dx}{\rho(x)} < \infty$$

([6]).

One basic meaning of the norm inequalities

Now, we note an important meaning or application of the inequality (1.1); that is, when we fix any member ψ of $H_2(D)$, the multiplication operator

$$\varphi \longmapsto \varphi(z)\psi(z), \tag{1.2}$$

on $H_2(D)$ to the Bergman space is bounded. Therefore, by the general theory for general fractional functions, we can consider the generalized fractional functions: for any Bergman function $f(z)$ on the domain D

$$\frac{f(z)}{\psi(z)}, \tag{1.3}$$

at least in the sense of Tikhonov; that is, we can consider the best approximation problem for the functions $\psi(z)^{-1}f(z)$ by the functions $H_2(D)$. See [2, 3] for more detailed results. See also [3] for applications.

As a very special fraction, we can consider the division by zero and division by zero calculus. See [11, 12] for the details.

As an important contribution of the theory of reproducing kernel is on the following fact:

For bounded linear operators on some reproducing kernel Hilbert spaces, we can give analytical and numerical solutions for the operator equations. See [9, 10].

For the recent similar type norm inequalities on Hilbert Sobolev spaces of one dimensional by A. Yamada [15], we refer to the corresponding norm inequalities as basic results applying the theory of reproducing kernels, directly.

There exist some interesting differences in nature with his concrete and deep results.

2 One dimensional Sobolev spaces

We will consider one dimensional Sobolev Hilbert spaces $H(a,b;\mathbf{R})$, $(a,b>0)$, as the basic reproducing kernel Hilbert space with finite norms

$$\int_{\mathbf{R}} \left(a^2|f'(x)|^2 + b^2|f(x)|^2\right) dx$$

admitting the reproducing kernel $K_{H(a,b;\mathbf{R})}(x,x_1)$

$$K_{H(a,b;\mathbf{R})}(x,x_1) = \frac{1}{2ab}\exp\left(-\frac{b}{a}|x-x_1|\right).$$

See [10], pages10-18 for the related basic materials.

We will consider this space as in the Szegö space in (1.1). Note the identity

$$K_{H(a,b;\mathbf{R})}(x,x_1)^2 = \frac{1}{ab}\frac{1}{2a(2b)}\exp\left(-\frac{(2b)}{a}|x-x_1|\right)$$
$$= \frac{1}{ab}K_{H(a,2b;\mathbf{R})}(x,x_1).$$

From the construction of the norms admitting the reproducing kernels corresponding to the product and multiplication of a positive number for reproducing kernels, we obtain the norm inequality as in (1.1).

(A) For any $f,g\in H(a,b;\mathbf{R})$, we have the norm inequality

$$\int_{\mathbf{R}} \left(a^2|(f(x)g(x))'|^2 + 4b^2|f(x)g(x)|^2\right) dx$$
$$\leq \frac{1}{ab}\int_{\mathbf{R}} \left(a^2|f'(x)|^2 + b^2|f(x)|^2\right) dx \int_{\mathbf{R}} \left(a^2|g'(x)|^2 + b^2|g(x)|^2\right) dx.$$

Of course, we have

(A') For any $f,g\in H(a,b/2;\mathbf{R})$, we have the norm inequality

$$\int_{\mathbf{R}} \left(a^2|(f(x)g(x))'|^2 + b^2|f(x)g(x)|^2\right) dx$$
$$\leq \frac{2}{ab}\int_{\mathbf{R}} \left(a^2|f'(x)|^2 + \frac{b^2}{4}|f(x)|^2\right) dx \int_{\mathbf{R}} \left(a^2|g'(x)|^2 + \frac{b^2}{4}|g(x)|^2\right) dx.$$

3 Finite interval cases

If we note that the kernel on an interval $[c, d], -\infty \leq c < d \leq +\infty$

$$K_{H(a,b;[c,d])}(x, x_1) = \frac{1}{2ab} \exp\left(-\frac{b}{a}|x - x_1|\right)$$

is the reproducing kernel on the Hilbert space $H(a, b; [c, d])$ with finite norms

$$\int_{[c,d]} \left(a^2|f'(x)|^2 + b^2|f(x)|^2\right) dx + ab(|f(c)|^2 + |f(d)|^2) < \infty$$

as in the whole space case, the results in Section 2 are valid in the corresponding way. This fact may be confirmed directly by checking the reproducing property of the kernel as in [10], pages 11-12. Meanwhile, the kernel $K_{H(a,b;[c,d])}(x, x_1)$ is the restriction to the interval $[c, d]$ of the kernel $K_{H(a,b;\mathbf{R})}(x, x_1)$ and so by the general property of reproducing kernels, we see that any member $f(x)$ of $H(a, b; [c, d])$ is the restriction of a function $h(x)$ in $H(a, b; \mathbf{R})$ and its norm is given by

$$||f||_{H(a,b;[c,d])} = \min ||h||_{H(a,b;\mathbf{R})},$$

where the minimum is taken over all functions h in $H(a, b; \mathbf{R})$ satisfying

$$f(x) = h(x) \quad on \quad [c, d].$$

See [10], pages 78-80. In particular, note that any member $f(x)$ of $H(a, b; [c, d])$ has a good property on the interval $[c, d]$.

4 Division by zero calculus

If $b = 0$, then, by the division by zero calculus

$$K_{H(a,0;\mathbf{R})}(x, x_1) = -\frac{1}{2a^2}|x - x_1|$$

and this is the reproducing kernel for the corresponding space $H(a, 0; \mathbf{R})$ equipped with the norm

$$\|f\|^2_{H(a,0;\mathbf{R})} = a^2 \int_0^a (f'(x)^2 dx.$$

See [11, 12] for the division by zero calculus. Note that it is the Green's function in one dimensional space on the whole space and the Green's function may be related to the reproducing kernel. See [10], pages 62-63.

Meanwhile, if $a = 0$, $K_{H(0,b;\mathbf{R})}(x, x_1) = 0$, then it is the trivial reproducing kernel for the zero function space.

However, from the representation

$$\frac{1}{2ab} \exp\left(-\frac{b}{a}|x-y|\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)} d\xi}{a^2\xi^2 + b^2},$$

for $a = 0$, we have the reasonable result

$$\frac{1}{b^2}\delta(x-y)$$

that may be considered as the reproducing kernel for the L_2 space. See Section 8.8 in [10].

5 Open problems

Our norm inequalities will be very beautiful and fundamental, so we wonder their direct derivations apart from the theory of reproducing kernels. It seems that Yamada's results [15] and our results are different in nature for the similar norm inequalities for one dimensional Sobolev Hilbert spaces. What are the relations between our results?

Professor Yamada made a deep research for the equality problem for some general norm inequalities derived from the theory of reproducing kernels, however, he stated that the new equality problem in (A) is still an open problem on 23 August, 2022.

Our norm inequalities here have a similar form as in the Schwarz inequality, so we wonder does there exist p, q $(1/p + 1/q = 1)$ version as in the Hölder inequality.

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