A Formula of Zhi-Wei Sun

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Abstract

In this note we discuss various equivalent formulations for the sum of an infinite series considered by Zhi-Wei Sun

Keywords: Number Pi, harmonic numbers, series, hypergeometric functions.

Introduction

In Reference [1], Sun derived the following expression

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{2^n H_{n-1}^{(2)}}{n \binom{2n}{n}}$$

where $\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ and $H_n^{(2)}$ denotes, for $n \in \mathbb{N}$, the Harmonic number

$$H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}$$

In this note we give some series related to Sun formula.

Series

Entry 1.

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{2^n H_n^{(2)}}{\binom{2n}{n}(2n+1)}$$

Entry 2.

$$\frac{\pi^3}{16} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \frac{2^k}{\binom{2k}{k}(2k+1)}$$

Entry 3.

$$\frac{\pi^3}{48}=\sum_{n=1}^{\infty}\frac{2^n n}{\binom{2n}{n}(2n+1)}\sum_{k=0}^{n}\frac{H_k^{(2)}}{2k+1}$$

Entry 4.

$$\frac{\pi^3}{24}=\sum_{n=1}^{\infty}\frac{3n+1}{\binom{2n}{n}(2n+1)}\sum_{k=0}^{n}\frac{2^k H_k^{(2)}}{2k+1}$$

Entry 5.

$$\frac{\pi^3}{24}=\sum_{n=1}^{\infty}\frac{2^{2n}\,H_{2n}^{(2)}\,(6n+1)}{n\binom{4n}{2n}(4n+1)}-\sum_{n=1}^{\infty}\frac{2^{2n-2}}{n^3\binom{4n}{2n}}$$

$$\frac{\pi^3}{24}=\sum_{n=1}^{\infty}\frac{2^{2n}\,H_{2n}^{(2)}\,(6n+1)}{n\binom{4n}{2n}(4n+1)}-\frac{1}{6}\,F\left(\begin{matrix}1,1,1,1,\frac{3}{2}\\ \frac{5}{4},\frac{7}{4},2,2\end{matrix}\middle|\,\frac{1}{4}\right)$$

Remark: $F\equiv 5F4$ is the generalized hypergeometric function.

Entry 6.

$$\frac{\pi^3}{48}=\sum_{n=2}^{\infty}\frac{2^n\,n\,H_n^{(2)}\,(H_n-1)}{\binom{2n}{n}(2n+1)}-\sum_{n=2}^{\infty}\frac{2^n(H_n-1)}{n^2\binom{2n}{n}}$$

Remark: $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is the harmonic number.

Entry 7.

$$\frac{\pi^3}{96}=\sum_{n=1}^{\infty}\frac{(2^n-1)\left((n+1)(3n+5)H_n^{(2)}-1\right)}{\binom{2n}{n}(2n+1)(2n+2)(2n+3)}$$

Entry 8.

$$\frac{\pi^3}{96}=\sum_{n=1}^{\infty}\frac{2^n(2n^2+5n+2)}{(2n+3)!}\sum_{k=1}^{n}H_k^{(2)}(k!)^2$$

Entry 9.

$$\frac{\pi^3}{24}=\sum_{n=1}^{\infty}\frac{3n+5}{\binom{2n}{n}(2n+1)(2n+3)}\sum_{k=1}^{n}H_k^{(2)}2^k$$

Entry 10.

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{2^n}{n^2 \binom{2n}{n}(2n+1)}\ F\left(\begin{matrix}1, n+1\\ n+\frac{3}{2}\end{matrix}\middle|\frac{1}{2}\right)$$

$$\frac{\pi^3}{96} = \sum_{n=1}^{\infty} \frac{2^n}{n^2 \binom{2n}{n}(2n+1)}\ F\left(\begin{matrix}1, \frac{1}{2}\\ n+\frac{3}{2}\end{matrix}\middle|-1\right)$$

$$\frac{\pi^3}{96} = \sum_{n=1}^{\infty} \frac{2^{2n}}{n^2 \binom{2n}{n}(2n+1)}\ F\left(\begin{matrix}n+1, n+\frac{1}{2}\\ n+\frac{3}{2}\end{matrix}\middle|-1\right)$$

$$\frac{\pi^3}{48\sqrt{2}} = \sum_{n=1}^{\infty} \frac{2^n}{n^2 \binom{2n}{n}(2n+1)}\ F\left(\begin{matrix}\frac{1}{2}, n+\frac{1}{2}\\ n+\frac{3}{2}\end{matrix}\middle|\frac{1}{2}\right)$$

Remark: $F \equiv 2F1$ is the Gauss hypergeometric function.

Entry 11.

$$\frac{\pi^3}{24} = \sum_{n=1}^{\infty} \frac{n!\ 2^{-n}\ H_n^{(2)}}{(1/2)_{n+1}}$$

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{n!\ 2^{-n}\ H_n^{(2)}}{(3/2)_n}$$

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{2n+1}{(3/2)_n(2n+3)} \sum_{k=1}^{n} H_k^{(2)}\ 2^{-k}\ k!$$

Remark: $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ is the Pochhammer symbol.

Entry 12.

$$\frac{\pi^3}{48\sqrt{2}} = \sum_{n=1}^{\infty} \frac{2^{-3n}}{2n+1} \sum_{k=1}^{n} \frac{2^{4k}\binom{2n-2k}{n-k}}{k^2\binom{2k}{k}}$$

$$\frac{\pi^3}{48\sqrt{2}} = \sum_{n=0}^{\infty} \frac{2^{-3n}}{2n+3} \binom{2n}{n} F\left(\left. \begin{matrix} 1,1,1,-n \\ \frac{3}{2}, 1, \frac{1}{2}-n \end{matrix} \right| 1 \right)$$

Remark: $F \equiv 4F3$ is the generalized hypergeometric function.

Entry 13.

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{n!\ H_n^{(2)}}{(2n+1)!!}$$

Remark: $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$

Entry 14.

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{(2n)!!\ H_n^{(2)}}{(n+1)_{n+1}}$$

Remark: $(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n)$
Remark: $(a)_n = a(a+1)(a+2) \ldots (a+n-1)$ is the Pochhammer symbol.

Entry 15.

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{2^n\ n!\ H_n^{(2)}}{(n+1)_{n+1}}$$

Remark: $(a)_n = a(a+1)(a+2) \ldots (a+n-1)$ is the Pochhammer symbol.

Entry 16.

$$\frac{\pi^3}{96} = \sum_{n=1}^{\infty} \frac{n+1}{(2n+3)!!} \sum_{k=1}^{n} k!\ H_k^{(2)}$$

Remark: $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$.

Entry 17.

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{2^n\ H_n^{(2)}\ n\ (n+2)}{\binom{2n}{n}\ (2n+1)(2n+3)} - \sum_{n=1}^{\infty} \frac{2^n\ n}{(n+1)\binom{2n}{n}(2n+1)(2n+3)}$$

Remark: $\sum_{n=1}^{\infty} \frac{2^n\ n}{(n+1)\binom{2n}{n}(2n+1)(2n+3)} = 6 - \frac{\pi}{8}(12+\pi)$.

Entry 18.

$$\pi^3 = 16 + 48\sum_{n=1}^{\infty}\frac{2^n\, H_n^{(2)}\,(n+1)}{\binom{2n}{n}\,(2n+1)(2n+3)} + 48\sum_{n=1}^{\infty}\frac{2^n}{(n+1)\binom{2n}{n}(2n+1)(2n+3)}$$

Remark: $\sum_{n=1}^{\infty}\frac{2^n}{(n+1)\binom{2n}{n}(2n+1)(2n+3)} = \pi + \frac{\pi^2}{8} - \frac{13}{3}$.

Endnote and Future Research

Entry 19.

$$\frac{\pi^3}{24} = \sum_{n=0}^{\infty}\frac{2^{-n}}{(1/2)_{n+2}}\sum_{k=0}^{n}\frac{(-1)^k 2^{-k}(n-k)!\,(1/2)_{k+1}}{(2k+1)(n-k+1)}\left(1 + F\left(\left.\begin{matrix} k+\frac{3}{2}, 2k+1 \\ n+\frac{5}{2}\end{matrix}\right| -\frac{1}{2}\right)\right)$$

$$\frac{\pi^3}{24} = \sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}\frac{(-1)^k 2^{-k}(n-k)!}{(2k+1)(n-k+1)\left(k+\frac{3}{2}\right)_{n-k+1}}\left(1 + F\left(\left.\begin{matrix} k+\frac{3}{2}, 2k+1 \\ n+\frac{5}{2}\end{matrix}\right| -\frac{1}{2}\right)\right)$$

Remark: $F \equiv 2F1$ is the Gauss hypergeometric function.
Remark: $(a)_n = a(a+1)(a+2)\dots(a+n-1)$ is the Pochhammer symbol.

Entry 20.

$$\frac{\pi^3}{48} = \sum_{n=0}^{\infty}\frac{2^{-n}}{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\left(\frac{\pi(1+(-1)^k)}{4} - \sum_{m=0}^{k}\frac{(-1)^m}{2m+1}\right)$$

References

[1] ZHI-WEI SUN, A new series for π^3 and related congruences, Internat. J. Math., 26 (2015), no. 8., arXiv:1009.5375v8 [math.NT] 20 Oct 2015.

[2] S. RAMANUJAN, Collected papers, Cambridge University Press, Cambridge, 1927, reprinted by Chelsea, New York, 1962, reprinted by Amer. Math. Soc., Providence, RI, 2000.