Applied Phase Modulation (for Astronomers)

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Phase modulation of gravitational waves often occupies a large bandwidth and degrades the S/N. This paper explains the basics, properties and implications of PM with examples from astronomy as well as how to eliminate PM. The focus is on the search for gravitational waves.

1 Introduction

The detection of continuous gravitational waves (CGW) would open another window for astronomy to observe distant celestial bodies. In the records of previous interferometers (LIGO), no signs of CGW have been detected so far, despite an intensive search. Many previous publications give the impression that the laws of phase modulation (PM) are not always observed, maybe even unknown.

PM is a standard method in communications engineering for transmitting signals. PM is closely related to frequency modulation and the corresponding formula sets can be converted into each other. All types of modulation have in common that they require a certain signal bandwidth.

GW sources outside the solar system do not generate phase modulated signals, these are only generated by the motion of the earth. GW in the frequency range around 1 Hz are characterized by a very high bandwidth, which is not easy to process. The assumption that one can identify a CGW as a single spectral line without thorough preprocessing is absurd.

2 Properties of phase modulation

The properties of the modulation type PM can be understood more easily, if they are explained not with formulas but with realistic examples of astronomy. Therefore, it is assumed below that a single constant frequency $f_{year} = 31.688$ μHz phase modulates the constant carrier frequency $f_{GW} \geq 10^{-3}$ Hz. The effects of other modulation frequencies are discussed below (Sektion 5).

1. In a phase-modulated oscillation, the *instantaneous frequency* f_{inst} oscillates sinusoidally around the long-term average f_{GW}. To measure f_{inst}, the recording period must be considerably shorter than 365 days. A short period of, say, five days reduces the spectral resolution and broadens the linewidth to at least 1200 nHz (section 6). This exceeds the actual value of f_{inst} (section 4, formula 2) and means that f_{inst} is difficult to measure accurately. Also in communications engineering f_{inst} has more theoretical than practical value.

2. In astronomy, the orbital period of the Earth's orbit and the Doppler effect define the frequency deviation Δf of a GW. Δf is the maximum value of $|f_{inst} - f_{GW}|$. The modulation index $\eta = \frac{\Delta f}{f_{year}}$ calculated from this is key to understanding the PM.

3. Processing a PM signal claims the Carson bandwidth $2{\cdot}(\Delta f{+}f_{year}) = 2{\cdot}f_{year}(\eta{+}1)$, which contains at least three spectral lines with mutual separation f_{year}. As η increases, the number of spectral lines increases (Fig 1). Remarkably, the separation of spectral lines is f_{year} and *not* Δf, as is often mistakenly assumed.

4. With increasing η, the already low energy of a GW is distributed over more and more spectral lines of low amplitude. At the same time, the bandwidth of the signal processing must increase in order to be able to process the PM without distortion. Both degrade the signal-to-noise (S/N) ratio.

5. Without eliminating this spectral fragmentation, it will not be possible to detect GW. This is especially true for searches for the GW of rapidly rotating neutron stars, because in these cases η takes on enormous values around 10^5. This is the subject of the sections 9 and 7.

3 Important signatures of a CGW

Below we discuss properties of GWs in the 10^{-4} Hz $< f_{GW} <$ 1 Hz range, such as those produced by binary star systems.

The drift of typical GWs is about 5 nHz per year, exceeding the target bandwidth (BW) of $\sim$ 2 nHz. Such a narrow BW is necessary to ensure good S/N. This point is further explained in section 6.

Each received BW is doubly phase modulated because the earth moves in the wave field of the BW:

- The high orbital velocity of 30 km/s of the orbit (frequency f_{year} = 1/365 days) produces a frequency deviation of about 500 nHz, which exceeds the targeted bandwidth by orders of magnitude. Such a slow phase modulation of weak signals can be detected only with several years of measurements. This modulation is the main feature of a GW (see section 4).

- The PM as a result of the earth's rotation (f_{day}=1/24 h) generates two spectral lines at a distance of f_{day} = 11.574 μHz left and right of the central frequency f_{GW}. Their amplitude is negligible, which is why they disappear in the noise even with very good S/N. In section 5 this problem is discussed in more detail.

The gravitational influences of the planets produce spectral lines in the range below 80 μHz, which are tabulated [1] and do not affect the measurement of higher frequency GW.

4 Phase modulation in annual rhythm

In section 3 it was mentioned that the detection of a PM with $f_{year} = 31.688$ nHz is the primary characteristic of a GW. It would be surprising if any signal generated in our solar system has this signature. Any modulation produces sidebands that occupy what is called the channel bandwidth (Fig 1). If one switches off the modulation, the sidebands also disappear. In PM with a *single* frequency, these appear in the spectrum as additional spectral lines symmetrical to the central frequency f_{GW}. The mutual frequency spacing is as large as the modulation frequency f_{year} and the modulation index η and the Bessel functions of the first kind ($J_n(\eta)$) define the amplitudes. The definition of the modulation index η is:

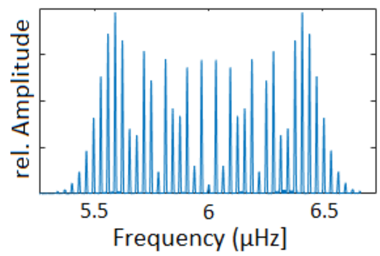

Figure 1): *The spectrum of a phase-modulated oscillation with $\eta = 10$ fills a wide band. The modulation frequency determines the distance between the lines.*

$$\eta = \frac{\textit{max. Frequency deviation from the average frequency}}{\textit{Modulation frequency}} = \frac{\Delta f}{f_{year}} \tag{1}$$

The maximum frequency deviation Δf results from the relativistic Doppler effect due to the orbit around the sun. In the case of HM Cancri, the GW source is nearly in the plane of the ecliptic and the maximum frequency deviation Δf can be calculated from the orbital velocity of the Earth ($v_{Earth} \approx 30$ km/s).

$$\Delta f = f_{GW} \cdot \left(\sqrt{\frac{c + v_{orbit}}{c - v_{orbit}}} - 1 \right) \approx 622 nHz \tag{2}$$

The modulation index $\eta_{year} = 20$ requires the Carson bandwidth of 1.4 μHz for the associated spectrum. The entire range is filled with $\frac{1400\ nHz}{31.69\ nHz} \approx 44$ closely neighboring spectral lines and looks like a frequency comb. It does not seem very promising to search in the noise for a set of 44 spectral lines with unknown amplitude distribution (section 9). Isolated spectral lines are not enough to reconstruct the original PM signal.

5 Phase modulation in the daily rhythm

Every terrestrial sensor orbits the earth's axis daily, which is why the receiving frequency is phase modulated with the rotation frequency of the earth. The small peripheral speed at the equator of only 463 $\frac{m}{s}$ causes a tiny frequency deviation of only $f_{GW} = 6220$ μHz

$$\Delta f = f_{GW} \cdot \left(\sqrt{\frac{c + v_{equator}}{c - v_{equator}}} - 1 \right) = 9.6\ nHz \tag{3}$$

This value is considerably smaller than $f_{day} = 11.574$ μHz and ensures the extremely small modulation index

$$\eta = \frac{\Delta f}{f_{day}} = 8.3 \times 10^{-4} \tag{4}$$

In communications engineering, such a small value is called phase noise and is neglected. This is also true when searching for GW signals, because the amplitude of the two sideband frequencies are smaller than the amplitude of f_{GW} by a factor

$$\frac{A_{sideband}}{A_{carrier}} = \frac{J_1(\eta)}{J_0(\eta)} = 0.0004 \tag{5}$$

and can be detected in the spectrum only with extremely good S/N $> 6 \times 10^6$. The sideband frequencies $f_{GW} \pm f_{day}$ do not depend on η. It is a mistake to assume that frequency of these spectral lines can be changed by choosing the integration time or the spectral resolution.

6 Noise, signal bandwidth and the recording period

This section is not directly related to the PM. It is necessary to show that observation periods of days or weeks are not sufficient to detect GW because the S/N and the spectral frequency resolution are too low.

GW is discovered with sensors that react to the smallest changes in length (interferometer) or acceleration (gravimeter) in the frequency range between 10^{-8} Hz and about 1000 Hz. In order to reduce disturbing influences, the bandwidth of the signal processing has to be adjusted: Too narrow a bandwidth distorts the signal and generates errors. Too much bandwidth allows unwanted frequencies and unnecessary noise to pass through, degrading the S/N.

In addition to external sources of interference, the inherent noise of the sensor must be taken into account, which is characterized by the key figure *Power Spectral Density* (PSD) [2].

- The PSD of the widely used gravimeter STS1 is 10^{-19} $\frac{m^2}{s^4 Hz}$ [3]. It should be possible to improve this value significantly if the gravimeters are not mounted directly on the ground, as has been the case up to now.
- In the successful LISA Pathfinder model experiment, small-scale PSD values around 10^{-29} $\frac{m^2}{s^4 Hz}$ [4] were achieved, which are also hoped for in the future GW observatory LISA.
- The LIGO interferometers have PSD values of around 10^{-46} $\frac{m^2}{s^4 Hz}$ [5] [6]. So far, despite a lot of manpower, no continuous GW have been found.

The PSD and the choice of bandwidth BW determine the average amplitude A_{noise} of the interfering noise after the filter:

$$A_{noise} = \sqrt{PSD \cdot BW} \tag{6}$$

One cannot narrow the bandwidth of the signal processing arbitrarily in order to eliminate the disturbing noise. Because then the necessary recording period T_{min}, which the filter needs to settle down, increases.

This relationship was first formulated by Küpfmüller and is reminiscent of the Heisenberg uncertainty principle.

$$T_{min} \cdot BW \geq 0.5 \tag{7}$$

If one looks for weak GW signals in the recordings of gravimeters, this means: If one wants to reduce the amplitude of the interfering noise to $10^{-14}\ \frac{m}{s^2}$, the bandwidth of the filters must not exceed 1 nHz (formula (6)). Because of (7), gravimeters must be operated for at least 15 years and the frequency of the GW must not vary by more than 0.5 nHz during the entire period to keep the signal within the filter range. Some gravimeters have been recording data for more than 20 years.

7 Some remarks on LIGO

Presumably continuous GW are hidden in the records of LIGO, which are searched in vain so far. How well do the chosen bandwidth and observation duration fit the modulation index η of the GW we are looking for?

At $f_{GW} = 300$ Hz the Doppler effect of the Earth's orbit produces the frequency deviation $\Delta f = 0.03$ Hz (formula (2)) and $\eta = 10^6$. No technical application uses a comparably high modulation index. If the signal is processed with the necessary Carson-BW = 0.06 Hz, we measure at the output of the filter the noise amplitude

$$A_{noise} = \sqrt{PSD \cdot BW} = \sqrt{10^{-46} \cdot 0.06}\ \frac{m}{s^2} = 2.5 \times 10^{-24}\ \frac{m}{s^2} \tag{8}$$

This fantastically high sensitivity should be sufficient to detect continuous GW. The records of LIGO are analyzed on a day-by-day basis using their spectra [7]. One problem is the puzzling frequency combs (the tooth spacing is often 1 Hz), which are searched at 10^{-5} Hz resolution. Although that could be used to discover a PM signal in the diurnal rhythm - in section 5 it was justified that this can hardly succeed. It is impossible to identify the important annual rhythm ($f_{year} = 31.688$ nHz) with this coarse resolution. According to formula (7), the necessary spectral resolution can only be achieved with integration times on the order of 365 days. Formula (7) cannot be outwitted by any *comb finding algorithm*.

8 The reception of an idealized GW

Let us assume that a pulsar generates a GW of constant $f_{GW} = 300$ Hz and the distance to the Earth remains constant. When the GW passes the Earth, LIGO's beam lines with $L = 4000$ m oscillate in the same rhythm with the maximum amplitude ΔL. The strain h is calculated with the ansatz

$$h = \Delta L/L = h_0 \cdot sin(\omega t) \tag{9}$$

Previous estimates give values of $h_0 \approx 10^{-25}$ for fast rotating pulsars. Thus the change of the local gravity $\ddot{L}$ near the Earth surface is

$$\ddot{L} = L \cdot \omega^2 \cdot h_0 = 4000\ m \cdot (2\pi \cdot 300\ \frac{1}{s})^2 \cdot 10^{-25} = 1.4 \times 10^{-15}\ \frac{m}{s^2} \tag{10}$$

This value exceeds the noise level A_{noise} of LIGO by a factor of 6×10^8 and actually cannot be overlooked in the spectrum. Why has nothing been found so far? The complete spectrum of the PM signal consists of

$$\frac{BW_{Carson}}{f_{year}} = \frac{0.06\ Hz}{31.688 \times 10^{-9}\ Hz} \approx 2 \times 10^6 \tag{11}$$

spectral lines with defined phase relationships. The original signal cannot be reconstructed with arbitrary subsets of the spectrum. Data processing with too small a bandwidth distorts the phase-modulated signal so much that it is no longer decipherable.

9 What brings a compensation of the phase modulation?

The Earth orbits the Sun and therefore we will never measure GW without PM. Unless we succeed in removing the PM, the prospects of ever detecting GW at frequencies above about 10^{-3} Hz are very slim.

An example illuminates the problem: The source of a GW lies in the plane of the ecliptic and generates the frequency $f_{GW} = 100$ mHz. The frequency deviation of 10 μHz leads to the very high modulation index $\eta = 315$. Approximately 630 closely spaced spectral lines with mutual separation $f_{year} = 31.69$ nHz fill the Carson bandwidth (compare Fig 1). The GW transports a certain amount of energy, which is distributed over 630 spectral lines. This reduces the amplitudes of these lines and many of them will disappear in the noise.

When PM is completely removed ($\eta = 0$), the GW reappears in the spectrum as a single, strong line because the total energy is again concentrated in a narrow frequency range around this line. A calculation with MATLAB shows empirically that the amplitude increases by a factor of

$$k_1 = (2.22 \cdot \eta + 1)^{0.351} \approx 10 \tag{12}$$

when PM with $\eta = 315$ is compensated. Because all sidebands disappear, no Carson bandwidth is needed anymore and one can filter the only remaining spectral line of the GW with the lowest possible bandwidth. This is $BW_{min} \approx 2$ nHz for a recording duration of ten years (formula 7). The low value reduces the noise within the channel bandwidth (formula 6) by a factor of

$$k_2 = \sqrt{\frac{PSD \cdot BW_{min}}{PSD \cdot BW_{Carson}}} = \sqrt{\frac{BW_{min}}{2 \cdot f_{year}(\eta + 1)}} \approx 0.01 \tag{13}$$

So compensating for phase modulation (and frequency drift) has a twofold effect and dramatically improves the S/N:

$$\frac{SNR_{unmodulated}}{SNR_{PM+drift}} = \left(\frac{k_1}{k_2}\right)^2 = 10^6 \tag{14}$$

It will hardly be possible to detect contiuous GW without first eliminating the phase modulation completely. This also concerns the data analysis of the future LISA telescope and even more the search for the GW of fast rotating pulsars in the frequency range around 100 Hz. The occasionally expressed assumption that PM only causes a rearrangement of spectral lines is wrong.

The broad spectrum of a PM causes another problem: Probably the Milky Way hosts many thousands of GW sources of similar frequency and all of them fill similarly wide Carson bandwidths with their individual bundle of spectral lines. It is not easy to separate these overlapping spectra - especially if the spectral resolution is unreasonably large.

The *modified superhet* (MSH) method treats the entire spectrum as a unit and can concentrate all the energy that the PM distributes among very many separate spectral lines back into the narrow frequency range around the central frequency f_{GW}. This procedure was explained in another paper [8].

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